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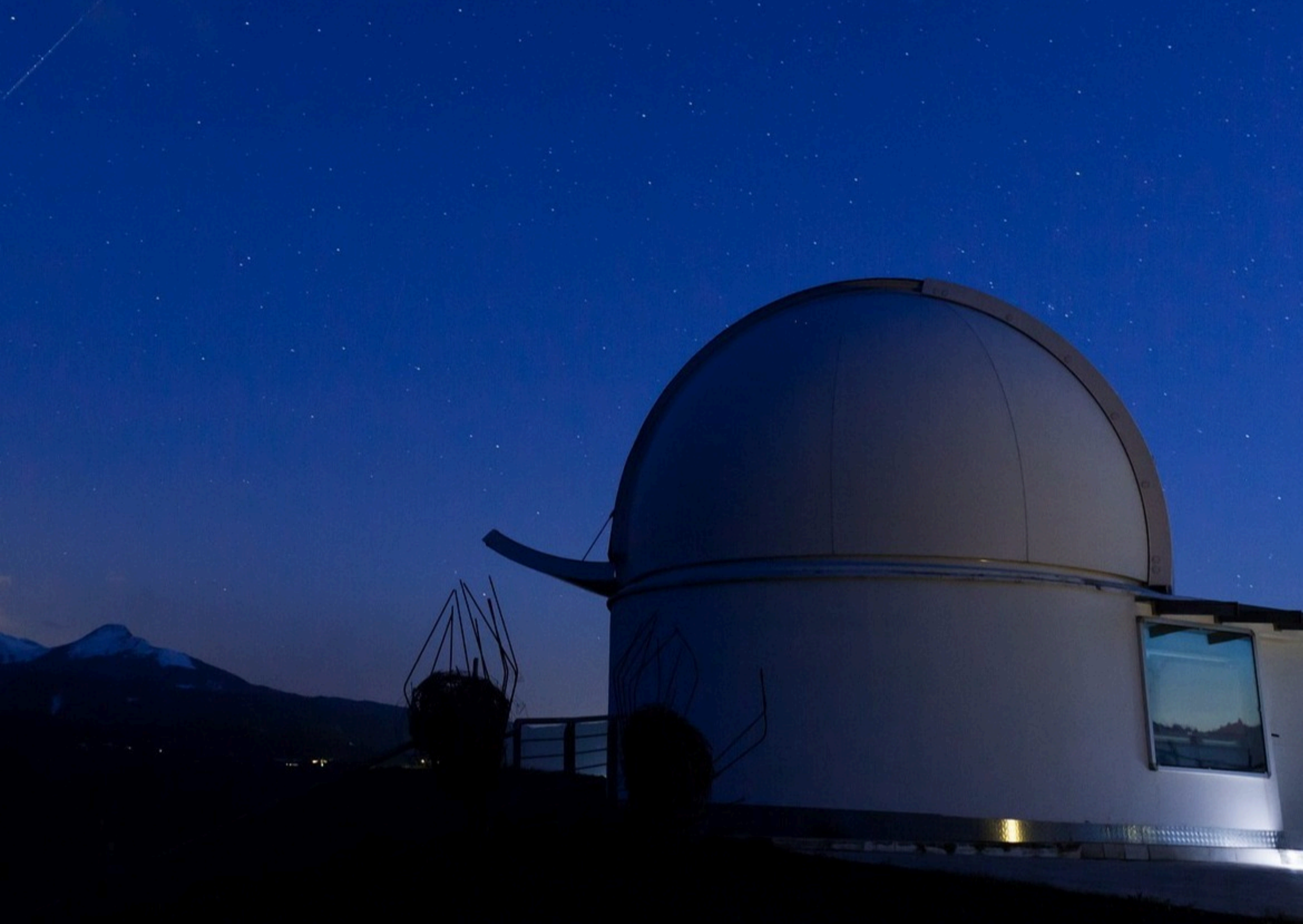

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London Journal of Research in Science: Natural & Formal

Print ISSN: 2631-8490
Online ISSN: 2631-8504
DOI: 10.17472/LJRS

Volume 24 | Issue 10 | Compilation 1.0

COMPILED IN UNITED KINGDOM

ENGLISH

Great Britain Journals Press

London Journal of Research in Science: Natural and Formal

Volume 24 | Issue 10 | Compilation 1.0

PUBLISHER

Great Britain Journals Press
1210th, Waterside Dr,Opposite Arlington Building, Theale, Reading
Phone:+444 0118 965 4033 Pin: RG7-4TY United Kingdom

SUBSCRIPTION

Frequency: Quarterly

Print subscription
$280USD for 1 year
$500USD for 2 year
(*color copies including taxes and international shipping with TSA approved)*
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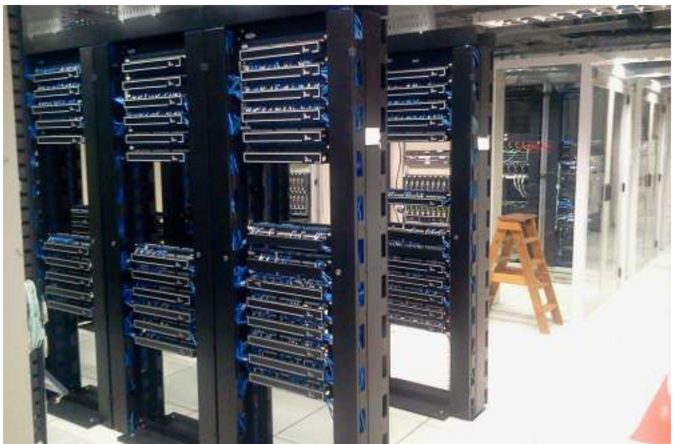

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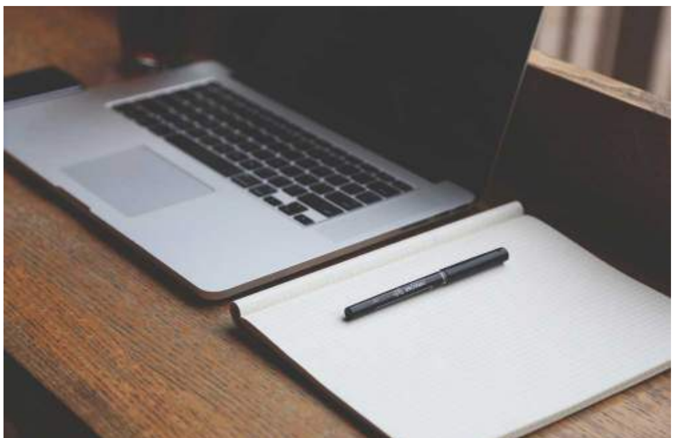

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Classification: LCC Code: QA241

Language: English

Great Britain Journals Press

LJP Copyright ID: 925610
Print ISSN: 2631-8490
Online ISSN: 2631-8504

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An Investigation of the Order of Integral Powers of Set of all Natural Numbers

Ladan, Umaru Ibrahim[α], Emmanuel, J. D. Garba[σ] & Tanko Ishaya[ρ]

ABSTRACT

This article established a fact on the order of difference of integral powers of all sets of natural numbers. The analysis was proof by use of established property of difference operator and principle of mathematical induction. The result proved conclusively that "if the elements of an arithmetic progression of set of natural numbers with positive common difference are raised to positive power k, then the k^{th} difference is equal to the product of the common difference raised to power k (d^K) and k factorial (k!).

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Author α: Department of Computer Science, Faculty of Natural Sciences, University of Jos.
σ: Department of Mathematics, Faculty of Natural Sciences, University of Jos.
ρ: Department of Computer science, Faculty of Natural Sciences, University of Jos.

I. INTRODUCTION

A set of positive integers is a set of natural numbers. A set is a collection of well-defined elements. A set of natural number is a sequence, defined with respect to a constant value called a common difference (d). The terms of a sequence are defined with respect to three parameters, the first term (a), the numbers of terms in the sequence (n) and the common difference (d). The sequence of natural numbers expressed mathematically as:

$$T_n = a + (n - 1)\, d \text{ .... I}$$

For all $a \geq 1, d > 1, n \in (o, \infty)$

Any sequence generated with the above formular is called an ***arithmetic progression.***

A perfect power is a rational root chase.[1] An integral perfect power is the irrational root that is an integer. A finite difference is a mathematical expression of the form, f(x + b) – f(x + a) and a forward difference is of the form $\Delta h[f](x) = f(x + h) = f(x)$[2]

This article investigates all Arithmetic Progression of set of natural numbers with positive common difference. Further investigation reveals that there is a significant relationship between the difference of powers of set of natural numbers and factorial, which is represented symbolically as:

$$\Delta^k\ [a + (n-1)d]^k = d^k K! \text{ .... II}$$

For all $a \geq 1, d \geq 1, k > 1$ and

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For some $\cap \epsilon (o, \infty)$

The implication of equation (II) is on the coefficient of the kth factorial, which sows the value of k is the power of the common difference and at the same time is the power of the sequence under consideration for which the k^{th} difference of the set is equal to the product of the common difference raised to power k and k!.

An examination of any set of natural numbers having common difference greater than 2 e.g. {2, 5, 8, 11, 14, 17, 20, 23, 26, 29} reveals that the pattern is sustained (See Table I and Appendix A).

The set of perfect squares of the set is {4, 25, 64, 121, 196, 289, 400, 529, 676, 841}. The actual set of the first difference is {21, 39, 57, 75, 93, 111, 129, 147, 165}. Furthermore, the set of the second difference is {18, 18, 18, 18, 18, 18, 18, 18} = {18}, a singleton set. Clearly, the set of differences of the above second difference set is {0}. The emerging pattern motives the investigation of whether the pattern will persist for all set of natural numbers with positive common difference d > 2. The set of the second difference of the set of perfect squares clearly shows that the element of the singleton set 18 = $3^2(2!)$.

This shows that for d = 3, k = 2.

$$\Delta^2\, [a + (n-1)d)]^2 = d^k.K!, \text{ where}$$

$$a = 2, d = 3, k = 2, \text{ and } n \,\epsilon\, (0,9)$$

$$\Rightarrow \Delta^2\, [a + (n-1)d)]^2 = 18 = 3^2(2!)$$

In their contribution in Exponential Diophantine Equations, Shorey and Tijdeman investigated perfect powers at integral values of a polynomial with rational integer coefficients and obtain in particular the following. Let f(x) be a polynomial with rational coefficients and with at least two simple rational zeros.

Suppose b ≠ 0, m ≥ 0, x and y with /Y/ > 1 are rational integers. Then the equation $f(x) = by^m$ implies that m is bounded by a computable number depending only on b and f. Also in their contribution Ladan, Tanko, Aliyu, Ahmad and Kabiru[4], they investigated integral powers of polynomials with Binomial coefficients. The result of their investigation shows that "the disposition of powers of polynomials with binomial coefficients generates even positive factorial (2k)!

Also in a research work by Ladan, Aliyu, Tanko, Ahmad and Kabiru[5], on the location of points on the plane and the order of disposition of sum of powers of cardinal coordinates. The result of their work proved conclusively that "the sum of the powers of cardinal points is equal to the coefficients of the Binomial expansion with respect to the Pascal triangle pattern and entries". In their contribution in the difference of perfect powers of integers, Ladan, Ukwu and Apine[6] investigated order of integral perfect powers and proved that "if any number of consecutive integers are raised to a positive integral power k, then the k^{th} difference is equal to k! Based on the generalization of the theorem in this article, it shows that Ladan, Ukwu and Apine work has common difference d = 1, for all k > 1.

Similarly, in a research conducted by Ladan, Emmanuel and Tanko[7], investigated order of product of perfect powers and proved conclusively that if any number of consecutive integer are raised to a positive power k, then the (2k)th difference of the product of the kth power of two consecutive integers is equal to (2k)!. This established a relation between difference of powers of natural numbers and factorial. For more definitions, see [8, 9, 10 ……].

There was no discussion on the relationship between the difference of powers of set of all natural numbers and factorial with respect to the common difference which is pertinent to this article. Review of literature shows that no such investigation has been undertaken. Thus, this article adds to the existing body of knowledge.

II. METHODS

2.1 *Preliminary Definitions*

In what follows, the difference of finite order will be defined.

2.1.1 *Differences of Order One (1)*

Given a sequence $\{f_j\}_1^{\infty}$, defined the difference of order one at j with respect to the sequence by:

$\Delta(f_j) = f_{j+1} - f_j$, for every j

2.1.2 *Higher Order Differences*

Higher order differences can be defined recursively by:

$\Delta^k(f_j) = \Delta(\Delta^{k-1}(f_j)) = \Delta^{k-1}(\Delta(f_j)) = \Delta^{k-1}[f_{j+1} - f_j]$ for ≥ 2

III. RESULTS AND DISCUSSION

3.1 *Preliminary Theorem*

Suppose $f_j = j$ for all j belong to set of natural numbers.

Then:

(i) Δf_j^2 is natural number

(ii) $\Delta^2(f_j^2) = 18 = 3^2.(2!)$

(iii) $\Delta^k(f_j^2) = 0,\ k \epsilon \{3, 4 \ldots..\}$

Proof

Let j = {2, 5, 8, 11, 14, 17, 20, 23, 26, 29}

Then Table I below yields value of $\Delta^k(f_j^p)$ for selected values of k and p in the set {0, 1, 2, 3} and {1, 2} respectively, when $\Delta^0(f_j^p) = f_j^p$

Table I: Difference Order Table for Selected Elements of Set of Natural Numbers

f_j	f_j^2	$\Delta(f_j^2)$	$\Delta^2(f_j^2)$	$\Delta^3(f_j^2)$
2	4	21	18	0
5	25	39	18	0
8	64	57	18	0
11	121	75	18	0
14	196	93	18	0
17	289	111	18	0
20	400	129	18	0
23	529	147	18	0
26	676	165	18	
29	841			

It is clear that $\Delta^k(f_j^k) = d^k K!$ Where d = 3, k = 2, and j ϵ {2, 5, . . . 26, 29}.

That is $\Delta^2(f_j^2) = 3^2(2!) = 18$

Obviously, the theorem is valid for j ϵ {2, 5, 8, 11, 14, 17, 20, 23, 26, 29} as observed from column 4. The proofs of (i), (ii) and (iii) are direct.

(i) $\Delta(f_j^2) = f_{j+1}^2 - f_j^2$ = {j + 1)² – j2 = 2j + 1, which is natural number for all j, proving (i).

(ii) $\Delta(\Delta f_j^2) = 3^2\,\Delta(2j + 1) = 3^2(\Delta 2j + \Delta 1) = 3^2(2\Delta j) = 3^2[2(j + 1 - j)] = 3^2 2[1] = 3^2(2)$

$= 3^2.2 = 18$, for d = 3, proving (ii).

The principle of mathematical induction is needed in the proof of (iii).

For k = 3, $\Delta^3(f_j^2) = \Delta(\Delta^2(f_j^2)) = \Delta\,(18) = 18 - 18 = 0$

Consequently $\Delta^k(f_j^2) = \Delta^{k-3}(\Delta^3(f_j^2)) = \Delta^{k-3}(0) = 0$

For all k ≥ 4, proving $\Delta^k(f_j^2) = 0$, for all positive integer k ≥ 3. This established the proof of (iii).

3.2 Main Theorem

Let $f_j = j$

Let $\{j\}_1^\infty$ be a sequence defined by $j_n = a + (n-1)d$, for all $n \geq 1$, $d \geq 1$ and $a \geq 1$.

Let $[j_n]^k = [a + (n-1)d]^k$, for $k = \{1, 2, 3, \ldots\}$.

We shall consider the case for the power of k = {1, 2, 3} to establish the proof of the theorem.

Case (1), k = 1, n = {1, 2, 3, ...}

$[j_n]^k = [a + (n-1)d]^k$, k = 1 are constant.

$\Rightarrow [j_n]^1 = [a + (n-1)d]^1$ is the sequence.

The first difference is given as:

$\Delta^1[j_n]^1 \quad = \Delta^1[a + (n-1)d]^1,$

$\Delta[j_n] \quad = \Delta(a + dn - d)$

$= \Delta a + \Delta dn - \Delta d$

$= \Delta dn$

$= d[(n+1) - n]$

$= d(1)$

$= d$

$= d^1(1!)$

Therefore, for k = 1

$\Delta^1[j_n]^1 = \Delta^1[a + (n-1)d]^1 = d^1(1!)$

Case (2), k = 2, n = {1, 2, 3, ...}

$[j_n]^2 \quad = [a + (n-1)d]^2$ is set of perfect squares.

$\Delta^2[j_n]^2 \quad = \Delta^2[a + (n-1)d]^2,$

$\Delta[\Delta j_n^2] = \Delta^2[a + dn - d]$

$= \Delta^2[(a + dn - d)(a + dn - d)]$

$= \Delta^2[a + adn - ad + adn + d^2n^2 - d^2n - ad - d^2n + d^2]$

$= \Delta^2[a^2 + 2adn - 2ad + d^2n^2 - 2d^2n + d^2]$

$= \Delta[\Delta(a^2 + 2adn - 2ad + d^2n^2 - 2d^2n + d^2)]$

$= \Delta[\Delta a^2 + 2ad\Delta n - \Delta 2ad + d^2\Delta(n^2) - 2d^2\Delta(n) - \Delta d^2]$

$= \Delta[2ad((n+1) - (n) + d^2((n+1)^2 - (n)^2) - 2d^2((n+1) - (n)]$

$= \Delta[2ad(1) - d^2((n^2 + 2n - 1) - (n)^2 - 2d^2(1)]$

$= \Delta[2ad + d^2(2n - 1) - 2d^2)$

$= \Delta 2ad + \Delta 2d^2n - \Delta 3d^2$

$= \Delta 2d^2n$

$= 2d^2\Delta(n)$

$= 2d^2(n + 1 - n)$

$= 2d^2(1) = d^2 2!$

Therefore,

$\Delta^2[j_n]^2 \quad = \Delta^2[a + (n-1)d]^2 = d^2(2!)$, satisfied.

Case (3), k = 3, n = {1, 2, 3, ...}

$\Delta^3[j_n]^3 \quad = \Delta^3[a + (n-1)d]^3$

$= \Delta^3[(a + dn - d)^3]$

$= \Delta^3[(a + dn - d)(a + dn - d)^2]$

$= \Delta^3[(a + dn - d)(a^2 + 2adn - 2ad - 2d^2n + d^2n^2 + d^2)]$

$= \Delta^3[a^3 + 2a^2dn - 2a^2d - 2ad^2n + ad^2n^2 + ad^2 + a^2dn + 2ad^2n^2 - 2ad^2n^2 - 2d^3n^2 + d^3n^3 + d^3n - a^2d - 2ad^2n + 2ad^2 + 2d^3n - d^3n^2 - d^3]$

$= \Delta^3[a^3 + 3a^2dn - 3a^2d - 6ad^2n + 3ad^2n^2 + 3ad^2 - 3d^3n^2 + d^3n^3 + 3d^3n - d^3]$

$= \Delta^2[\Delta(a^3 + 3a^2dn - 3a^2d - 6ad^2n + 3ad^2n^2 + 3ad^2 - 3d^3n^2 + d^3n^3 + 3d^3n - d^3)]$

$= \Delta^2[\Delta(a^3 + \Delta 3a^2dn - \Delta 3a^2d - \Delta 6ad^2n + \Delta ad^2n^2 + \Delta 3ad^2 - \Delta 3d^3n^2 + \Delta d^3n^3 + \Delta 3d^3n - \Delta d^3)]$

$= \Delta^2[3a^2d\Delta n - 6ad^2\Delta n + ad^2\Delta n^2 + 3d^3\Delta n^2 + d^3\Delta n^3 + 3d^3\Delta n]$

$= \Delta^2[3a^2d(n + 1 - n) - 6ad^2(n + 1 - n) + ad^2((n+1)^2 - n^2) - 3d^3((n+1)^2 - n^2) + d^3((n+1)^3 - n^3) + 3d^3(n + 1 - n)\ ]$

$= \Delta^2[3a^2d - 6ad^2(n^2 + 2n + 1 - n^2) - 3d^3n^2 + 2n + 1 - n^2 + d^3([n+1)(n^2 + 2n + 1 - n^3 + 3d^3]$

$= \Delta^2[3a^2d - 6ad^2 + ad^2(2n+1) - 3d^3(2n+1) + d^3(n^3 + 3n^2 + 3n + 1) - n^3 + 3d^3]$

$= \Delta^2[3a^2d - 5ad^2 + 2ad^2n - 3d^3n - 2d^3 + 3d^3n^2 + 3d^3]$

$= \Delta[\Delta(3a^2d - 5ad^2 + 2ad^2n - 3d^3n - 2d^3 + 3d^3n^2 + d^3]$

$= \Delta[\Delta 3a^2d - \Delta 5ad^2 + \Delta 2ad^2n - \Delta 3d^3n + \Delta 3d^3n^2 + \Delta d^3]$

$= \Delta[2ad^2\Delta n - 3d^3\Delta n + 3d^3\Delta n^2]$

$= \Delta[2ad^2(n + 1 - n) - 3d^3(n + 1 - n) + 3d^3(n^2 + 2n + 1 - n^2)]$

$= \Delta[2ad^3 - 3d^3 + 3d^3(2n + 1)]$

$= \Delta[2ad^3 - 3d^3 + 6d^2 + 3d^3]$

$= \Delta 2ad^3 - \Delta 3d^3 + \Delta 6d^3 + \Delta 3d^3]$

$= \Delta 6d^3n$

$= 6d^3\Delta n$

$= 6d^3$

$= d^3(6)$

$= d^3 3!$

Therefore,

$\Delta^3[\mathrm{j_n}]^3 \quad = \Delta^3[a + (n - 1)d]^3 = d^3(3!)$

Thus, the proof of the theorem is established for the values integral power of k = {1, 2, 3}. This clearly reveals that the pattern is sustained for the values of k belong to natural numbers for all d > 2 and n ϵ $(1, \infty)$.

Assume the validity of the theorem of k = {4, 5, ... m} for some natural number m ≥ 4.

Inductively, the theorem holds for k = 1, see case (1) $\Delta[\mathrm{j_n}] = \Delta[a + (n - 1)d] = d.1!$

The truth of theorem for k = 1, implies is true for all positive k > 1, that is $\Delta^{\mathrm{k}}[\mathrm{j_n}]^{\mathrm{k}} = \Delta^{\mathrm{k}}[a + (n - 1)d]^k = d^k.k!$, for some k > 1. (See Case 2 and 3).

Thus by induction hypothesis, the truth of the theorem for k implies the validity of the theorem for k + 1.

Therefore, $\quad \Delta^{\mathrm{k+1}}[\mathrm{j_n}]^{\mathrm{k+1}} = \Delta^{\mathrm{k+1}}[a + (n - 1)d]^{k+1}$

$$\Delta^{\mathrm{k+1}}[a + (n - 1)d]^{k+1} = \mathrm{d}^{\mathrm{k+1}}(k + 1)!$$

This completes the proof, that is the pattern is sustained for all powers of set of natural numbers, for all k ≥ 1 and d ≥ 1, a ≥ 1, n ϵ $\{1, 2, 3, \dots m\}, m \ \geq 1$.

The theorem is established.

3.3 Corollary

$\Delta^k[j_n]^k = d^k.k!$ for every arithmetic progression of set of natural numbers j_n and for any positive integer k, for all d ≥ 2. In other words, for any positive integer k, the k^{th} order difference of the k^{th} power of a sequence of natural numbers is equal to $d^k.k!$ where d is the common difference, d ≥ 1.

By implication, the theorem states that "if the elements of an arithmetic progression of the set of natural numbers with positive common difference are raised to a positive power k, then the k^{th} difference is equal to the product of the common difference and k factorial $[d^k.k!]$.

3.4 Remarks

The following strong relationship exist between the difference operator Δ and D operator (differential operator).

$D^k(K^p) = 0 \; if \; k > p$

$D^k x^k = k!$

The coefficient of x^{p-k}

$D^k(K^p) = \frac{P!}{(p-k)} x^{p-k}$ for k ≥ 2, k < p is even.

IV. CONCLUSION

This article established the structures of finite difference orders with respect to the powers of set of natural numbers. The result reveals a startling relationship between the common difference of the sequence and k!, as reflected in the theorem.

Competing Interests: Authors have declared that no competing interest exist.

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APPENDIX A

Table 1: d = 3, k = 2

j_n	j_n^2	Δj_n^2	$\Delta^2 j_n^2$	$\Delta^3 j_n^2$
2	4	21	18	0
5	25	39	18	0
8	64	57	18	0
11	121	75	18	0
14	196	93	18	0
17	289	111	18	0
20	400	129	18	0
23	529	147	18	0
26	676	165	18	
29	841			

We have that: $\Delta^2 j_n^2 = 18 = 3^2(2!)$

Table 2: d = 3, k = 2

j_n	j_n^3	Δj_n^3	$\Delta^2 j_n^3$	$\Delta^2 j_n^3$	$\Delta^4 j_n^3$
2	8	117	270	162	0
5	125	387	432	162	0
8	512	819	594	162	0
11	1331	1413	756	162	0
14	2744	2169	918	162	
17	4913	3087	1080		
20	8000	4167			
23	12167				

We have that: $\Delta^3 j_n^3 = 162 = 3^3(3!)$

Table 3: d = 3, k = 2

j_n	j_n^2	Δj_n^2	$\Delta^2 j_n^2$	$\Delta^3 j_n^2$
1	1	15	18	0
4	16	33	18	0
7	49	51	18	0
10	100	69	18	0
13	169	87	18	
16	256	105		
19	361			

We have that: $\Delta^2 j_n^2 = 18 = 3^2(2!)$

Table 4: d = 3, k = 4

j_n	j_n^4	Δj_n^4	$\Delta^2 j_n^4$	$\Delta^3 j_n^4$	$\Delta^4 j_n^4$	$\Delta^5 j_n^4$
2	16	609	2862	4212	1944	0
5	625	3471	7074	6156	1944	0
8	4096	10545	13230	8100	1944	0
11	14641	23775	21330	10044	1944	0
14	38416	45105	31374	11988	1944	
17	83521	76479	43362	13932	1944	
20	16000	119841	57294	15876		
23	279841	177135	73170			
26	456976	250305				
29	707281					

We have that: $\Delta^4 j_n^4 = 1944 = 3^4(4!)$

Table 5: d = 3, k += 5

j_n	j_n^5	Δj_n^5	$\Delta^2 j_n^5$	$\Delta^3 j_n^5$	$\Delta^4 j_n^5$	$\Delta^5 j_n^5$	$\Delta^6 j_n^5$
2	32	3093	26550	72090	77760	29160	0
5	3125	29643	98640	149850	106920	29160	0
8	32768	128283	248490	256770	136080	29160	0
11	161051	376773	505260	392850	165240	29160	0
14	537824	882033	898110	558090	194400	29160	
17	1419857	1780143	1456200	752490	223560		
20	3200000	3236343	2208690	976050			
23	6436343	5445033	3184740				
26	11881376	8629773					
29	20511149						

We have that: $\Delta^5 j_n^5 = 29160 = 3^5(5!)$

Table 6: d = 4, k = 2

j_n	j_n^2	Δj_n^2	$\Delta^2 j_n^2$	$\Delta^3 j_n^2$
2	4	32	32	0
6	36	64	32	0
10	100	96	32	0
14	196	128	32	
18	324	160		
22	484			

We have that: $\Delta^2 j_n^2 = 32 = 4^2(2!)$

Table 7: d = 4, k = 3

j_n	j_n^3	Δj_n^3	$\Delta^2 j_n^3$	$\Delta^3 j_n^3$	$\Delta^4 j_n^3$
2	8	208	576	384	0
6	216	784	960	384	0
10	1000	1744	1344	384	0
14	2744	3088	1728	384	0
18	5832	4816	2112	384	0
22	10648	6928	2496	384	
26	17576	9424	2880		
30	27000	2304			
38	39304				

We have that: $\Delta^4 j_n^4 = 4^3(3!)$

Table 8: d = 4, k = 4

j_n	j_n^4	Δj_n^4	$\Delta^2 j_n^4$	$\Delta^3 j_n^4$	$\Delta^4 j_n^4$	$\Delta^5 j_n^5$
2	16	1280	7424	12288	6144	0
6	1296	8704	19712	18432	6144	0
10	10000	28416	38144	24576	6144	0
14	38416	66560	62720	30720	6144	0
18	104976	129280	93440	36864	6144	0
22	234256	222720	130304	43008	6144	
26	456976	353024	173312	49152		
30	810000	526336	222464			
34	1336336	748800				
38	2085136					

We have that: $\Delta^4 j_n^4 = 6144 = 4^4(4!)$

Table 9: d = 4, k = 5

j_n	j_n^5	Δj_n^5	$\Delta^2 j_n^5$	$\Delta^3 j_n^5$	$\Delta^4 j_n^5$	$\Delta^5 j_n^5$	$\Delta^6 j_n^5$
1	1	3124	52800	203520	276480	122880	0
5	3125	55924	256320	480000	399360	122880	0
9	59049	312244	736320	879360	522240	122880	0
13	371293	1048654	1615680	1401600	645120	122880	
17	1419857	2664244	3017280	2046720	768000		
21	4084101	5681524	5064000	2814720			
25	9765625	10745524	7878720				
29	20511149	18624244					
33	39135393						

We have that: $\Delta^5 j_n^5 = 122880 = 4^5(5!)$

Table 10: d = 7, k = 2

j_n	j_n^5	Δj_n^5	$\Delta^2 j_n^5$
2	4	77	98
9	81	175	98
16	256	273	98
23	527	271	
30	900		

We have that: $\Delta^2 j_n^2 = 98 = 7^2(2!)$

Table 11: d = 7, k = 3

j_n	j_n^3	Δj_n^3	$\Delta^2 j_n^3$	$\Delta^3 j_n^3$	$\Delta^4 j_n^4$
2	8	721	2646	2058	0
9	729	3367	4704	2058	0
16	4096	8071	6762	2058	0
23	12167	14833	8820	2058	0
30	27000	23653	10878	2058	
37	50653	34531	12936		
44	85184	47467			
51	132651				

We have that: $\Delta^3 j_n^3 = 2058 = 7^3(3!)$

Table 12: d = 7, k = 4

j_n	j_n^4	Δj_n^4	$\Delta^2 j_n^4$	$\Delta^3 j_n^4$	$\Delta^4 j_n^4$	$\Delta^5 j_n^4$
2	16	6545	52430	102900	57624	0
9	6561	58975	155330	160524	57624	0
16	65536	214305	315854	218148	57624	0
23	279841	530159	534002	275772	57624	0
30	810000	1064161	809774	333396		
37	1874161	1873935	1143170			
44	3748096	3017105				
51	6765201					

We have that: $\Delta^4 j_n^4 = 57624 = 7^4(4!)$

Table 13: d = 19, k = 3

j_n	j_n^3	Δj_n^3	$\Delta^2 j_n^3$	$\Delta^3 j_n^3$	$\Delta^4 j_n^3$
2	8	9253	45486	41154	0
21	9261	54739	86640	41154	0
40	64000	141379	127794	41154	0
59	205379	269173	168948	41154	0
78	474552	438121	210102	41154	0
97	912673	648223	251256	41154	0
116	1560896	899479	292410	41154	
135	2460375	1191889	333564		
154	3652264	1525453			
173	5177717				

We have that: $\Delta^3 j_n^3 = 41154 = 19^3(3!)$

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Sultan Moulay Slimane University

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Classification: MSC Code: 35Q35

Language: English

Great Britain Journals Press

LJP Copyright ID: 925612
Print ISSN: 2631-8490
Online ISSN: 2631-8504

London Journal of Research in Science: Natural & Formal

Volume 24 | Issue 10 | Compilation 1.0

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Fatima Ouidirne[α], Mohamed Oukessou[σ] & Jalila El ghordaf [ρ]

ABSTRACT

In the present paper, we study the Cauchy problem of the incompressible micropolar fluid system in $\mathbb{R}^3$. *We show that this problem is locally well-posed in Fourier-Besov-Morrey spaces* $\mathcal{F}\mathcal{N}_{1,\bar{\lambda},q}$ for $1 \leq q \leq \infty$, *and is globally well-posed in these spaces with small initial data.*

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Author α σ ρ: Laboratory LMACS, FST of Beni-Mellal, Sultan Moulay Slimane University, Morocco.

I. INTRODUCTION

In this paper, we are intersted the following initial value problem for the system of partial differential equations describing the motion of incompressible micropolar fluid:

$$\begin{cases} \partial_t u - (\chi + \nu)\Delta u + u \cdot \nabla u + \nabla \pi - 2\chi \nabla \times \omega = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ \partial_t \omega - \mu \Delta \omega + u \cdot \nabla \omega + 4\chi \omega - \kappa \nabla \operatorname{div} \omega - 2\chi \nabla \times u = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}+, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}+, \\ (u, \omega)|_{t=0} = (u_0, \omega_0) & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $u = u(x,t), \pi = \pi(x,t)$ and $\omega = \omega(x,t)$ are unknown functions representing the linear velocity field, the pressure field of the fluid and the micro-rotation velocity field, respectively. κ, μ, ν and χ are positive constants reflecting various viscosity of the fluid. Throughout this paper we only consider the situation with $\kappa = \mu = 1$ and $\chi = \nu = 1/2$.

Theory of micropolar fluid was proposed by Eringen [8] in 1996, his idea allows us to consider several physical phenomena which cannot be treated by the classical Navier-Stokes system for the viscous incompressible fluid, then the problem (1) was presented as a necessary modification to the traditional Navier-Stokes equations in order to better characterize the motion of real-world fluids consisting of rigid but randomly oriented particles (such as blood) by examining the influence of micro-rotation of the particles suspended in the fluid.

There are several results on the weak and strong solvency of the micropolar fluid system and some related topics. The weak solution of (1) was firstly considered by Galdi and Rionero [11]. The existence theorem of the micropolar fuid system with sufficiently regular initial data has been showed by Lukaszewicz [15]. Inoue et al. [12] proved similar result for the magneto-micropolar fluid system. Many authors obtained the well-posedness of the problem (1) in various function spaces.

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For instance in the Besov spaces $\dot{B}_{p,q}^{-1+\frac{3}{p}}$ for $p \in [1,6)$ and $q = \infty$, Chen and Miao [5] obtained global well-posedness of the problem (1) for small initial data. Zhu and Zhao [21] proved that the Cauchy problem (1) is locally well-posed in the Fourier-Besov spaces $\mathcal{F}\dot{B}_{p,q}^{-1+\frac{3}{p}}$ for $1 < p \leq \infty$ and $1 \leq q < \infty$ and globally well posed in these spaces with small initial data. Recently, Weipeng Zhu [22] considered a critical case $p = 1$ and showed that this problem is locally well-posed in $\mathcal{F}\dot{B}_{1,q}^{-1}$ for $1 \leq q \leq 2$, and is globally well-posed in these spaces with small initial data. Also, Zhu proved the ill-posedness of (1) in $\mathcal{F}\dot{B}_{1,q}^{-1}$ for $2 < q \leq \infty$. In addition, by using a similar argument Zhu established the ill-posedness of (1) in Besov spaces $\dot{B}_{\infty,q}^{-1}$ with $2 < q \leq \infty$. The well-posedness of a more general model than (1) is established by Ferreira and Villamizar-Roa [9] in pseudo-measure spaces. For the other studies of the problem (1), we refer to the monographs [6, 16, 17, 20] .

We remark that if $\chi = 0$ and $\omega = 0$, then we have the classical Navier-Stokes equations:

$$\begin{cases} u_t - \mu\Delta u + (u.\nabla)u + \nabla p = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\ \nabla \cdot u = 0, & \\ u(0,x) = u_0(x) & x \in \mathbb{R}^3. \end{cases}$$

The local and global well-posedness of the classical Navier-Stokes equations have been established by a lot of researches in various function spaces, we refer to [13, 14] and references cited therein.

In the present paper, we show that the problem (1) is locally well-posed in Fourier-Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-1}$ for $1 \leq q \leq \infty$, and globally well-posed in these spaces with small initial data. Before stating the main result of this paper, we first recall the definitions of Morrey spaces, Besov-Morrey spaces and Fourier-Besov-Morrey spaces and present some properties about these spaces. Our results on well-posedness of solutions are stated in Section 3. In Section 4, we obtain the needed linear and nonlinear estimates and we prove the well-posedness result.

II. GENERAL NOTATION

Before stating our main result, we shall introduce the notations used throughout this paper.

We denote by C a positive constant such that whose value may change with each appearance, $x \lesssim y$ means that there exists a positive constant such that $x \leq Cy$, we write $(a,b) \in X$ for $a \in X$ and $b \in X$ and $\|\cdot\|_{E\cap F} = \|\cdot\|_E + \|\cdot\|_F$. The symbol $\mathcal{S}\left(\mathbb{R}^3\right)$ is the usual Schwartz space of infinitely differentiable rapidly decreasing complex-valued functions on $\mathbb{R}^3$.

By $\hat{\varphi}$ we denote the Fourier transform of $\varphi \in \mathcal{S}\left(\mathbb{R}^3\right)$ in the version

$$\hat{\varphi}(x) := \mathcal{F}\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix\xi}\varphi(\xi)d\xi, \quad x \in \mathbb{R}^3.$$

and we define its inverse Fourier transform by

$$\check{\varphi}(\xi) = \mathcal{F}^{-1}\varphi(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{ix\cdot\xi}\varphi(x)dx.$$

For two complex or extended real-valued measurable functions f, g on $\mathbb{R}^3$, the convolution $f * g$ is given by

$$(f*g)(x) := \int_{\mathbb{R}^3} f(x-y)g(y)dy, \quad for\ x \in \mathbb{R}^3.$$

III. PRELIMINARIES AND MAIN RESULTS

Let us introduce some basic knowledge on the Littlewood-Paley theory and Fourier-Besov-Morrey spaces.

Let φ, ψ be two radial positive functions such that $\operatorname{supp}(\varphi) \subset \left\{\xi \in \mathbb{R}^3 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$ and $\operatorname{supp}(\psi) \subset \left\{\xi \in \mathbb{R}^3 : |\xi| \leq \frac{4}{3}\right\}$
and

$$\sum_{j\in\mathbb{Z}} \varphi\left(2^{-j}\xi\right) = 1, \qquad \text{for all } \xi \neq 0$$

and

$$\psi(\xi) + \sum_{j\geq 0} \varphi(2^{-j\xi}) = 1 \text{ for all } \xi \in \mathbb{R}^3.$$

We denote

$$\varphi_j(\xi) = \varphi\left(2^{-j}\xi\right), \quad \psi_j(\xi) = \sum_{k\leq j-1} \varphi_k(\xi)$$

and

$$h(x) = \mathcal{F}^{-1}\varphi(x), \quad g(x) = \mathcal{F}^{-1}\psi(x).$$

We define the homogeneous dyadic blocks $\dot{\Delta}_j$ and $\dot{S}_j$ for all $u \in S'(\mathbb{R}^3)$ as follows:

$$\dot{\Delta}_j u := \mathcal{F}^{-1}\left(\varphi(2^{-j}\xi)\mathcal{F}(u)\right) = 2^{3j}\int_{\mathbb{R}^3} h\left(2^j y\right) u(x-y)dy,$$
$$\dot{S}_j u := \sum_{k\leq j-1} \Delta_k f = \mathcal{F}^{-1}\left(\psi(2^{-j}\xi)\mathcal{F}(u)\right) = 2^{3j}\int_{\mathbb{R}^3} g\left(2^j y\right) u(x-y)dy,$$

where $\dot{\Delta}_j = \dot{S}_j - \dot{S}_{j-1}$ is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$ and S_j is a frequency to the ball $\{|\xi| \lesssim 2^j\}$.

Then for any $u \in S'(\mathbb{R}^3)/\mathcal{P}(\mathbb{R}^3)$ wehre $\mathcal{P}(\mathbb{R}^3)$ is the set of polynomials (See. [19]) we have the Littlewood-Paley decomposition:

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \;\text{ and }\; \dot{S}_j u = \sum_{k\leq j-1} \dot{\Delta}_k u.$$

By using the definition of $\dot{\Delta}_j$ and $\dot{S}_j$, one easily obtains that

$$\dot{\Delta}_j\dot{\Delta}_k u = 0, \quad \text{if } |j-k| \geq 2$$
$$\dot{\Delta}_j\left(\dot{S}_{k-1} f \dot{\Delta}_k u\right) = 0, \quad \text{if } |j-k| \geq 5.$$

Now, we define the Morrey spaces $\mathrm{M}_p^\lambda\left(\mathbb{R}^3\right)$.

Definition 3.1. ([22]) For $1 \leq p < \infty, 0 \leq \lambda < 3$, the Morrey space $\mathrm{M}_p^\lambda = \mathrm{M}_p^\lambda\left(\mathbb{R}^3\right)$ is defined by $\mathrm{M}_p^\lambda\left(\mathbb{R}^3\right) = \left\{f \in L^p_{loc}\left(\mathbb{R}^3\right); \|f\|_{\mathrm{M}_p^\lambda} < \infty\right\}$, where

$$\|f\|_{M_p^\lambda} = \sup_{x_0 \in \mathbb{R}^3} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0,r))}.$$

Remark 3.2. ([10])

1) The space M_p^λ equipped with the norm $\|\cdot\|_{M_p^\lambda}$ is a Banach space.

2) If $1 \le p_1, p_2, p_3 < \infty$, $0 \le \lambda_1, \lambda_2, \lambda_3 < 3$ with $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$, and $\frac{\lambda_3}{p_3} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$, then we have the Hölder inequality

$$\|fg\|_{M_{p_3}^{\lambda_3}} \le \|f\|_{M_{p_1}^{\lambda_1}} \|g\|_{M_{p_2}^{\lambda_2}}.$$

3) For $1 \le p < \infty$ and $0 \le \lambda < 3$,

$$\|\varphi * g\|_{M_p^\lambda} \le \|\varphi\|_{L^1} \|g\|_{M_p^\lambda}, \tag{2}$$

for all $\varphi \in L^1$ and $g \in M_p^\lambda$.

Lemma 3.3. *([10]) Let* $1 \le p_2 \le p_1 < \infty, 0 \le \lambda_1, \lambda_2 < 3$, $\frac{3-\lambda_1}{p_1} \le \frac{3-\lambda_2}{p_2}$ *and let* γ *be a multi-index. If* $\sup p(\widehat{f}) \subset \left\{|\xi| \le A2^j\right\}$, *then there is a constant* $C > 0$ *independent of* f *and* j *such that*

$$\left\|(i\xi)^\gamma \widehat{f}\right\|_{M_{p_2}^{\lambda_2}} \le C 2^{j|\gamma| + j\left(\frac{3-\lambda_2}{p_2} - \frac{3-\lambda_1}{p_1}\right)} \|\widehat{f}\|_{M_{p_1}^{\lambda_1}}. \tag{3}$$

Then, we define the function spaces $\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^3)$.

Definition 3.4. ([7]) (Homogeneous Besov-Morrey spaces) Let $s \in \mathbb{R}, 1 \le p < +\infty, 1 \le q \le +\infty$, and $0 \le \lambda < 3$. The space $\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^3)$ is defined by $\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^3) = \left\{u \in \mathcal{Z}'(\mathbb{R}^3); \quad \|u\|_{\dot{\mathcal{N}}_{p,\lambda,q}^s}(\mathbb{R}^3) < \infty\right\}$, where

$$\|u\|_{\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^3)} = \begin{cases} \left\{\sum_{j \in \mathbb{Z}} 2^{jqs} \left\|\dot{\Delta}_j u\right\|_{M_p^\lambda}^q\right\}^{1/q}, & \text{for } q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \left\|\dot{\Delta}_j u\right\|_{M_p^\lambda}, & \text{for } q = \infty, \end{cases}$$

with appropriate modifications made when $q = \infty$. The space $\mathcal{Z}'(\mathbb{R}^3)$ is the dual space of

$$\mathcal{Z}(\mathbb{R}^3) = \left\{f \in S(\mathbb{R}^3) : (\partial^\beta \hat{f})(0) = 0, \text{ for every multi-index } \beta\right\}.$$

Definition 3.5. ([7]) (Homogeneous Fourier-Besov-Morrey spaces) Let $s \in \mathbb{R}, 0 \le \lambda < 3, 1 \le p < +\infty$, and $1 \le q \le +\infty$. The space $\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^3)$ denotes the set of all $u \in \mathcal{Z}'(\mathbb{R}^3)$ such that

$$\|u\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^3)} = \left\{\sum_{j \in \mathbb{Z}} 2^{jqs} \left\|\widehat{\dot{\Delta}_j u}\right\|_{M_p^\lambda}^q\right\}^{1/q} < +\infty, \tag{4}$$

with appropriate modifications made when $q = \infty$.

Note that the space $\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^3)$ equipped with the norm (4) is a Banach space. Since $M_p^0 = L^p$, we have $\mathcal{F}\dot{\mathcal{N}}_{p,0,q}^s = \mathcal{F}\dot{B}_{p,q}^s$, $\mathcal{F}\dot{\mathcal{N}}_{1,0,q}^s = \mathcal{F}\dot{B}_{1,q}^s = \dot{B}_q^s$ and $\mathcal{F}\dot{\mathcal{N}}_{1,0,1}^{-1} = \chi^{-1}$, where $\dot{B}_q^s$ is the Fourier-Herz space, and χ^{-1} is the Lei-Lin space.

Now, we give the definition of the mixed space-time spaces.

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Definition 3.6. ([7]) Let $s \in \mathbb{R}, 1 \leq p < \infty, 1 \leq q, \rho \leq \infty, 0 \leq \lambda < 3$ and $T \in (0,\infty]$. The space-time norm is defined on $u(t,x)$ by

$$\|u(t,x)\|_{\mathcal{L}^{\rho}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}\right)} = \left\{\sum_{j\in\mathbb{Z}} 2^{jqs} \left\|\widehat{\dot{\Delta}_j u}\right\|^{q}_{L^{\rho}\left(0,T;M^{\lambda}_{p}\right)}\right\}^{1/q}.$$

We denote by $\mathcal{L}^{\rho}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}\right)$ the set of distributions in $S'\left(\mathbb{R}\times\mathbb{R}^3\right)/\mathcal{P}(\mathbb{R}^3)$ with finite $\|\cdot\|_{\mathcal{L}^{\rho}(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q})}$ norm.

We will use the next lemma to prove our main theorem.

Lemma 3.7. *([22]) Let $\mathcal{X}$ be a Banach space, $\mathfrak{B}$ a continuous bilinear map from $\mathcal{X}\times\mathcal{X}$ to $\mathcal{X}$, and ε a positive real number such that*

$$\varepsilon < \frac{1}{4\|\mathfrak{B}\|} \text{ with } \|\mathfrak{B}\| := \sup_{\|u\|\cdot\|v\|\leq 1} \|\mathfrak{B}(u,v)\|.$$

For any y in the ball $B(0,\varepsilon)$ (ie., with center 0 and radius ε) in $\mathcal{X}$, then there exists a unique x in $B(0,2\varepsilon)$ such that

$$x = y + \mathfrak{B}(x,x).$$

Below, we shall present our main result that establishes the local and global existence.

Theorem 3.8. *Let $q \in [1,+\infty]$, $\alpha \in (0,1)$ and $0 < \lambda < 3$.*

***(1)** For any initial data $(u_0,\omega_0) \in \mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}\left(\mathbb{R}^3\right)$ satisfying div $u_0 = 0$, there exists a positive T such that the system (1) has a unique mild solution such that*

$$(u,\omega) \in \mathcal{L}^{\frac{2}{1+\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}\left(\mathbb{R}^3\right)\right) \cap \mathcal{L}^{\frac{2}{1-\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}\left(\mathbb{R}^3\right)\right).$$

***(2)** There exists a positive constant ε such that for any initial data $(u_0,\omega_0) \in \mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}\left(\mathbb{R}^3\right)$ satisfying div $u_0 = 0$ and*

$$\|(u_0,\omega_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}(\mathbb{R}^3)} < \varepsilon,$$

the system (1) has a unique global mild solution such that

$$(u,\omega) \in \mathcal{L}^{\frac{2}{1+\alpha}}\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}\left(\mathbb{R}^3\right)\right) \cap \mathcal{L}^{\frac{2}{1-\alpha}}\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}\left(\mathbb{R}^3\right)\right).$$

Before proving our main result we will present the corresponding linear system of the nonlinear system (1).

$$\begin{cases} \partial_t u - \Delta u - \nabla\times\omega = 0 \\ \partial_t\omega - \Delta\omega + 2\omega - \nabla\operatorname{div}\omega - \nabla\times u = 0, \\ \operatorname{div} u = 0, \\ (u,\omega)|_{t=0} = (u_0,\omega_0). \end{cases} \tag{5}$$

The solution operator of the above problem is denoted by the notation $G(t)$, i.e., for specified initial data (u_0,ω_0) in suitable function space, $(u,\omega)^T = G(t)(u_0,\omega_0)^T$ is the unique solution of the problem (5). The operator $G(t)$ has the following expression, as shown by a simple calculation:

$$\widehat{(G(t)f)}(\xi) = e^{-\mathcal{A}(\xi)t}\hat{f}(\xi) \quad \text{for } f(x) = (f_1(x), f_2(x))^T,$$

where

$$\mathcal{A}(\xi) = \begin{bmatrix} |\xi|^2 I & \mathcal{B}(\xi) \\ \mathcal{B}(\xi) & (|\xi|^2+2)I + \mathcal{C}(\xi) \end{bmatrix},$$

with

$$\mathcal{B}(\xi) = i\begin{bmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{bmatrix} \text{ and } \mathcal{C}(\xi) = \begin{bmatrix} {\xi_1}^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & {\xi_2}^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & {\xi_3}^2 \end{bmatrix}.$$

On the other hand, by applying the Leray projection $\mathbf{P}$ to both sides of the first equations of (1), one can eliminate the pressure π and one check

$$\begin{cases} \partial_t u - \Delta u + \mathbf{P}(u\cdot\nabla u) - \nabla\times\omega = 0 \\ \partial_t\omega - \Delta\omega + u\cdot\nabla\omega + 2\omega - \nabla\operatorname{div}\omega - \nabla\times u = 0 \\ \operatorname{div} u = 0 \\ (u,\omega)|_{t=0} = (u_0,\omega_0), \end{cases} \tag{6}$$

where $\mathbf{P} = T + \nabla(-\Delta)^{-1}$ div is the 3×3 matrix pseudo-differential operator in $\mathbb{R}^3$ with the symbol $\left(\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2}\right)_{i,j=1}^{3}$. We denote

$$U(x,t) = \begin{pmatrix} u(x,t) \\ \omega(x,t) \end{pmatrix}, \quad U_0 = \begin{pmatrix} u(x,0) \\ \omega(x,0) \end{pmatrix} = \begin{pmatrix} u_0 \\ \omega_0 \end{pmatrix}, \quad U_i(x,t) = \begin{pmatrix} u_i(x,t) \\ \omega_i(x,t) \end{pmatrix}, \quad i = 1,2$$

and

$$U_1\tilde{\otimes}U_2 = \begin{pmatrix} u_1\otimes u_2 \\ u_1\otimes\omega_2 \end{pmatrix}, \quad \widetilde{\mathbf{P}}\nabla\cdot(U_1\tilde{\otimes}U_2) = \begin{pmatrix} \mathbf{P}\nabla\cdot(u_1\otimes u_2) \\ \nabla\cdot(u_1\otimes\omega_2) \end{pmatrix}.$$

To solve system (6) it suffices to find the solution U of the following integral equations:

$$U(t) = G(t)U_0 - \int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U\otimes U)(\tau)d\tau. \tag{7}$$

A solution of (7) is called a mild solution of (1).

IV. PROOF OF MAIN RESULT

In this section, we will establish the local and global existence and uniqueness of solution for the problem (1). For that, we prove some estimates for the semigroup $G(\cdot)$.

4.1 Linear estimates

Firstly we give the property of semigroup $G(\cdot)$.

Lemma 4.1. *[9] For $t\geq 0$ and $|\xi|\neq 0$. We have*

$$\|e^{-t\mathcal{A}(\xi)}\| \leq e^{-|\xi|^2 t} \ \textit{with} \ \left\|e^{-t\mathcal{A}(\xi)}\right\| = \sup_{\|f\|\leq 1}\left\|e^{-t\mathcal{A}(\xi)}f\right\|. \tag{8}$$

Here $\|f\| = \max_i |a_i|$ *with* $\|f\| = \sum_{i=1}^{6} a_i v_i, v_1, v_2, \ldots, v_6$ *are the eigenvectors for* $\mathcal{A}(\xi)$.

Next, we present the linear esimates for the semigroup $G(\cdot)$.

Lemma 4.2. *Let* $q \in [1, +\infty]$, $0 < \lambda < 3$. *Then there exists a positive constant* C *such that*

$$\|G(t)U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}} \leq C\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}} \tag{9}$$

for all $t \geq 0$ *and all* $U_0 \in \mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}$.

Proof. By Lemma 4.1, we have

$$\begin{aligned}
\|G(t)U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}} &= \sum_{i\in Z} 2^{(\lambda-1)jq} \left\|\mathcal{F}\left[G(t)\dot{\Delta}_j U_0\right]\right\|^q_{M^\lambda_1}\Bigg)^{\frac{1}{q}} \\
&= \sum_{\in Z} 2^{(\lambda-1)jq} \left\|e^{-t\mathcal{A}(\xi)}\mathcal{F}\left[\dot{\Delta}_j U_0\right]\right\|^q_{M^\lambda_1}\Bigg)^{\frac{1}{q}} \\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{(\lambda-1)q} e^{-|\xi|^2 qt} \left\|\mathcal{F}\left[\dot{\Delta}_j U_0\right]\right\|^q_{M^\lambda_1}\Bigg)^{\frac{1}{q}} \\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{(\lambda-1)q} \left\|\mathcal{F}\left[\dot{\Delta}_j U_0\right]\right\|^q_{M^\lambda_1}\Bigg)^{\frac{1}{q}} \\
&\lesssim \|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}}.
\end{aligned}$$

This completes the proof of Lemma 4.2 . $\square$

Lemma 4.3. *Let* $q \in [1, +\infty]$, $0 < \lambda < 3$, $\alpha \in (0;1)$ *and* $T \in (0, \infty]$. *Then there exists a positive constant* $C = C(\alpha)$ *such that*

$$\|G(t)U_0\|_{L^{\frac{2}{1\pm\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda\pm\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right)} \leq C\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}}, \tag{10}$$

for all $t \geq 0$ *and all* $U_0 \in \mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}$.

Proof. From Definition 3.6, it is easy to see that

$$\begin{aligned}
\|G(t)U_0\||_{L^{\frac{2}{1\pm\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda\pm\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right)} &= \sum_{j\in\mathbb{Z}} 2^{(\lambda\pm\alpha)jq} \left\|\mathcal{F}\left[G(t)\dot{\Delta}_j U_0\right]\right\|^q_{L^{\frac{2}{1\pm\alpha}}(0,T;M^\lambda_1)}\Bigg)^{\frac{1}{q}} \\
&= \sum_{j\in\mathbb{Z}} 2^{(\lambda\pm\alpha)jq} \left\|e^{-t\mathcal{A}(\xi)}\mathcal{F}\left[\dot{\Delta}_j U_0\right]\right\|^q_{L^{\frac{2}{1\pm\alpha}}(0,T;M^\lambda_1)}\Bigg)^{\frac{1}{q}} \\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{(\lambda\pm\alpha)jq} \left\| e^{-t2^{2j}} \left\|\mathcal{F}\left[\dot{\Delta}_j U_0\right]\right\|_{M^\lambda_1} \right\|^q_{L^{\frac{2}{1\pm\alpha}}(0,T)}\Bigg)^{\frac{1}{q}} \\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{(\lambda\pm\alpha)jq} 2^{-(1\pm\alpha)jq} \left\|\mathcal{F}\left[\dot{\Delta}_j U_0\right]\right\|^q_{M^\lambda_1}\Bigg)^{\frac{1}{q}} \\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{(\lambda-1)jq} \left\|\mathcal{F}\left[\dot{\Delta}_j U_0\right]\right\|^q_{M^\lambda_1}\Bigg)^{\frac{1}{q}} \\
&\lesssim \|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}}.
\end{aligned}$$

$\square$

This completes the proof of Lemma 4.3 .

4.2 Bilinear estimates and product laws

Lemma 4.4. *Let $T>0$, $s\in\mathbb{R}$ and $p,q,\gamma\in[1;+\infty]$ and $0<\lambda<3$. Then there exists a positive constant C such that*

$$\left\|\int_0^t G(t-\tau)f(\tau)d\tau\right\|_{\mathcal{L}^\gamma(0;T;\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q})}\leq C\|f\|_{\mathcal{L}^1(0;T;\mathcal{F}\dot{\mathcal{N}}^{s-\frac{2}{\gamma}}_{p,\lambda,q})}$$

for all $f\in\mathcal{L}^1(0;T;\mathcal{F}\dot{\mathcal{N}}^{s-\frac{2}{\gamma}}_{p,\lambda,q})$.

Proof. By Young's inequality, we obtain

$$\begin{aligned}
&\left\|\int_0^t G(t-\tau)f(\tau)d\tau\right\|_{\mathcal{L}^\gamma(0;T;\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q})}\\
&=\left(\sum_{j\in\mathbb{Z}}2^{jqs}\left\|\int_0^t e^{-(t-\tau)A(\xi)}\mathcal{F}\left[\dot{\Delta}_j f\right](\tau)d\tau\right\|^q_{L^\gamma\left(0,T;M^\lambda_p\right)}\right)^{\frac{1}{q}}\\
&\leq C\left(\sum_{j\in\mathbb{Z}}2^{jqs}\left\|\int_0^t e^{-(t-\tau)2^{2j}}\left\|\mathcal{F}\left[\dot{\Delta}_j f\right](\tau)\right\|_{M^\lambda_p}d\tau\right\|^q_{L^\gamma(0,T)}\right)^{\frac{1}{q}}\\
&\leq C\left(\sum_{j\in\mathbb{Z}}2^{jq\left(s-\frac{2}{\gamma}\right)}\left\|\mathcal{F}\left[\dot{\Delta}_j f\right](\tau)\right\|^q_{L^1\left(0,T;M^\lambda_p\right)}\right)^{\frac{1}{q}}\\
&\leq C\|f\|_{\mathcal{L}^1\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s-\frac{2}{\gamma}}_{p,\lambda,q}\right)}.
\end{aligned}$$

Which finish the proof. □

In the framework of homogeneous Fourier-Besov-Morrey spaces, we now gather an essential multiplication estimates.

Lemma 4.5. *Let $p,q\in[1.+\infty]$, $0<\lambda<3$, $T\in(0,+\infty]$ and $\alpha\in(0,1)$. Then there exists a positive constant C such that*

$$\begin{aligned}
\|uv\|_{\mathcal{L}^1(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda}_{1,\lambda,q})}\lesssim&\|u\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q})}\|v\|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q})}\\
&+\|v\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q})}\|u\|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q})}.
\end{aligned}$$

Proof. We introduce some notations about the standard localization operators. We set

$$u_j=\dot{\Delta}_j u,\quad \dot{S}_j u=\sum_{k\leq j-1}\dot{\Delta}_k u,\quad \tilde{\dot{\Delta}}_j u=\sum_{|k-j|\leq 1}\dot{\Delta}_k u,\quad j\in\mathbb{Z}.$$

Bony's decomposition for $\dot{\Delta}_j(uv)$ reads

$$\begin{aligned}
\dot{\Delta}_j(uv)=&\sum_{|k-j|\leq 4}\dot{\Delta}_j\left(\dot{S}_{k-1}u\dot{\Delta}_k v\right)+\sum_{|k-j|\leq 4}\dot{\Delta}_j\left(\dot{S}_{k-1}v\dot{\Delta}_k u\right)+\sum_{k\geq j-3}\dot{\Delta}_j(\dot{\Delta}_k u\tilde{\dot{\Delta}}_j v)\\
:=&I_1+I_2+I_3.
\end{aligned}$$

Then by the triangle inequalities, we have

$$\|uv\|_{L^1\left(0,T:\mathcal{F}\dot{\mathcal{N}}^{\lambda}_{1,\lambda,q}\right)} \leq \left(\sum_{j\in\mathbb{Z}} 2^{j\lambda q}\left\|\widehat{I_1}\right\|^q_{L^1\left(0,T:\mathrm{M}^{\lambda}_1\right)}\right)^{\frac{1}{q}} + \left(\sum_{j\in\mathbb{Z}} 2^{j\lambda q}\left\|\widehat{I_2}\right\|^q_{L^1\left(0,T:\mathrm{M}^{\lambda}_1\right)}\right)^{\frac{1}{q}} + \left(\sum_{j\in\mathbb{Z}} 2^{j\lambda q}\left\|\widehat{I_3}\right\|^q_{L^1\left(0,T:\mathrm{M}^{\lambda}_1\right)}\right)^{\frac{1}{q}} := J_1+J_2+J_3. \tag{11}$$

The terms I_1 and I_2 are symmetrical. Using Young's inequality and Hölder's inequality we have

$$\begin{aligned}
2^{j\lambda}\left\|\widehat{I_1}\right\|_{L^1\left(0,T,M^{\lambda}_1\right)} &\leq 2^{j\lambda}\sum_{|k-j|\leq 4}\left\|\left(\widehat{\dot{S}_{k-1}u\dot{\Delta}_k v}\right)\right\|_{L^1\left(0,T,M^{\lambda}_1\right)}\\
&\leq 2^{j\lambda}\sum_{|k-j|\leq 4}\|\widehat{v_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)}\sum_{l\leq k-2}\|\widehat{u_l}\|_{L^{\frac{2}{1-\alpha}}\left(0,T,L^1\right)}\\
&\leq 2^{j\lambda}\sum_{|k-j|\leq 4}\|\widehat{v_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)}\sum_{l\leq k-2}2^{\lambda l}\|\widehat{u_l}\|_{L^{\frac{2}{1-\alpha}}\left(0,T,M^{\lambda}_1\right)}
\end{aligned}$$

$$\begin{aligned}
&\leq 2^{j\lambda}\sum_{|k-j|\leq 4}\|\widehat{v_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)}\Big(\sum_{l\leq k-2}2^{(\lambda-\alpha)lq}\|\widehat{u_l}\|_{L^{\frac{2}{1-\alpha}}\left(0,T,M^{\lambda}_1\right)}\Big)^{\frac{1}{q}}\left(\sum_{l\leq k-2}2^{l\alpha q'}\right)^{\frac{1}{q'}}\\
&\leq 2^{j\lambda}\sum_{|k-j|\leq 4}2^{\alpha k}\|\widehat{v_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)}\|u\|_{L^{\frac{2}{1-\alpha}}\left(0;T;\mathcal{F}\dot{\mathcal{N}}^{-\alpha+\lambda}_{1,\lambda,q}\right)}\\
&\leq \sum_{|k-j|\leq 4}2^{(\alpha+\lambda)k}2^{(j-k)\lambda}\|\widehat{v_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)}\|u\|_{L^{\frac{2}{1-\alpha}}\left(0;T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}\right)}.
\end{aligned}$$

Taking ℓ^q-norm we get

$$J_1 \leq \| v \|_{L^{\frac{2}{1+\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}\right)}\| u \|_{L^{\frac{2}{1-\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}\right)},$$

and in a similar way we obtain

$$J_2 \leq \| u \|_{L^{\frac{2}{1+\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}\right)}\| v \|_{L^{\frac{2}{1-\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}\right)}.$$

To estimate I_3 , let

$$I^k_3 = \dot{\Delta}_j\Big(\sum_{|k'-k|}\dot{\Delta}_{k'}v\dot{\Delta}_k u\Big) = \sum_{k'=-1}^{1}\dot{\Delta}_k u\dot{\Delta}_{k+k'}v.$$

First we use Young's inequality (2) in Morrey spaces, and Lemma 3.3 with $|\gamma|=0$, to obtain

$$\begin{aligned}
2^{j\lambda}\left\|\widehat{I_3}\right\|_{L^1\left(0,T,M^{\lambda}_1\right)} &\leq 2^{j\lambda}\sum_{k\geq j-3}\left\|\widehat{I^k_3}\right\|_{L^1\left(0,T,M^{\lambda}_1\right)}\\
&\leq 2^{j\lambda}\sum_{k\geq j-3}\sum_{|k'-k|\leq 1}\|\widehat{u_k}\|_{L^{\frac{2}{1-\alpha}}\left(0,T,M^{\lambda}_1\right)}\|\widehat{v_{k'}}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,L^1\right)}\\
&\leq 2^{j\lambda}\sum_{k\geq j-3}\sum_{|k'-k|\leq 1}\|\widehat{u_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)}2^{\lambda k'}\|\widehat{v_{k'}}\|_{L^{\frac{2}{1-\alpha}}\left(0,T,M^{\lambda}_1\right)}\\
&\leq 2^{j\lambda}\sum_{k\geq j-3}\left(\sum_{|k'-k|\leq 1}2^{k'(\lambda-\alpha)q}\|\widehat{v_{k'}}\|_{L^{\frac{2}{1-\alpha}}\left(0,T,M^{\lambda}_1\right)}\right)^{\frac{1}{q}}2^{\alpha k}\|\widehat{u_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)}\\
&\leq C\|v\|_{L^{\frac{2}{1-\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}\right)}\sum_{k\geq j-3}2^{(\lambda+\alpha)k}2^{\lambda(j-k)}\|\widehat{u_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)},
\end{aligned} \tag{12}$$

taking ℓ^q-norm on both sides in the above estimate, we get

$$J_3 \leq \| v \|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q})} \| u \|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q})}.$$

Which gives the result. □

Below, we shall prove our result.

4.3 Proof of Theorem 3.8

1) Let $T > 0$ and $\alpha \in (0;1)$ we define the space X^α_T as

$$X^\alpha_T = \left\{ U : U \in \mathcal{L}^{\frac{2}{1+\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right) \cap \mathcal{L}^{\frac{2}{1-\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right)\right\}$$

and equipped with the following norm:

$$\| U \|_{X^\alpha_T} = \| U \|_{\mathcal{L}^{\frac{2}{1+\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right)} + \| U \|_{\mathcal{L}^{\frac{2}{1-\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right)}.$$

For all $U \in X^\alpha_T$ we define $\phi(U)$ as follows

$$\phi(U) = G(t)U_0 - \int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U\widetilde{\otimes}U)(\tau)d\tau. \tag{13}$$

Our goal is to show that U is a fixed point of ϕ.
Considering

$$B(U_1,U_2) = \int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U_1\widetilde{\otimes}U_2)(\tau)d\tau. \tag{14}$$

Then, by Lemma 4.4, Lemma 4.5 and the embedding $\mathcal{F}\dot{\mathcal{N}}^{\lambda}_{1,\lambda,q} \hookrightarrow \mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}$, we have

$$\begin{aligned}
\| B(U_1,U_2) \|_{X^\alpha_T} &= \| \int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U_1\widetilde{\otimes}U_2)(\tau)d\tau \|_{\mathcal{L}^{\frac{2}{1+\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right)} \\
&\quad + \| \int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U_1\widetilde{\otimes}U_2)(\tau)d\tau \|_{\mathcal{L}^{\frac{2}{1-\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right)} \\
&\lesssim \|\widetilde{\mathbf{P}}\nabla\cdot(U_1\widetilde{\otimes}U_2)\|_{\mathcal{L}^1(0;T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q})} \\
&\lesssim \|\widetilde{\mathbf{P}}\nabla\cdot(U_1\widetilde{\otimes}U_2)\|_{\mathcal{L}^1(0;T;\mathcal{F}\dot{\mathcal{N}}^{\lambda}_{1,\lambda,q})} \\
&\lesssim \| U_1 \|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q})} \| U_2 \|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q})} + \| U_2 \|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q})} \| U_1 \|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q})} \\
&\leq C_1 \|U_1\|_{X^\alpha_T} \|U_2\|_{X^\alpha_T}.
\end{aligned} \tag{15}$$

Then, by (13) and (15), one concludes

$$\|\phi(U)\|_{X^\alpha_T} \leq \|G(t)U_0\|_{X^\alpha_T} + C_1\|U_1\|_{X^\alpha_T}\|U_2\|_{X^\alpha_T}.$$

By Lemma 4.3 , we get

$$\|G(t)U_0\|_{X^\alpha_T} \leq C_2 \|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}}. \tag{16}$$

London Journal of Research in Science: Natural & Formal

Now, consider the norm $\|G(t)U_0\|_{X_T^\alpha}$. Using the given expression for $G(t)$ and the definition of X_T^α, we can write:

$$\|G(t)U_0\|_{X_T^\alpha} = \|G(t)U_0\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}(\mathbb{R}^3))} + \|G(t)U_0\|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}(\mathbb{R}^3))}$$

Now, let's analyze each term separately. First, note that $G(t)$ is a linear operator, so we can factor out U_0 from the norm:

$$\|G(t)U_0\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}(\mathbb{R}^3))} \leq \|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}(\mathbb{R}^3))} \|G(t)\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T)}.$$

Similarly for the second term. Now, for sufficiently small T, $e^{-\mathcal{A}(\xi)t}$ tends to I (the identity operator) as t approaches 0. Therefore, for small T, $\|G(t)\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T)}$ tends to 0.

Then $\|G(t)U_0\|_{X_T^\alpha} \to 0$ as $T \to 0$, hence $\alpha \neq 1$, and there exists $T > 0$ such that $\|G(t)U_0\|_{X_T^\alpha} < \frac{1}{4C_1}$. Using Lemma 3.7, system (1) admits a unique mild solution $U \in X_T^\alpha$ with $\|U\|_{X_T^\alpha} < \frac{1}{2C_1}$.

For **2)** we replace X_T^α by X_∞^α and we get

$$\| B(U_1,U_2) \|_{X_\infty^\alpha} \leq C_1 \|U_1\|_{X_\infty^\alpha} \|U_2\|_{X_\infty^\alpha}. \tag{17}$$

Then, by (16) and (17), one obtaines

$$\|\phi(U)\|_{X_\infty^\alpha} \leq C_2 \|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}} + C_1 \|U_1\|_{X_\infty^\alpha} \|U_2\|_{X_\infty^\alpha}.$$

Then, by applying Lemma 3.7 , with $\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}} < \frac{1}{4C_1C_2}$. Then, system (1) admits a unique global mild solution $U \in X_\infty^\alpha$ with $\|U\|_{X_\infty^\alpha} < \frac{1}{2C_1}$. This completes the proof of Theorem3.8 (2).

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University of Jos, Ahmadu Bello University Zaria

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Classification: LCC Code: QA214

Language: English

Great Britain Journals Press

LJP Copyright ID: 925613
Print ISSN: 2631-8490
Online ISSN: 2631-8504

London Journal of Research in Science: Natural & Formal

Volume 24 | Issue 10 | Compilation 1.0

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ABSTRACT

This article analysed the order of difference of integral perfect powers of the set of even and odd numbers. The analysis was proof by the use of combinatorial terminologies, established property of the difference operator and the principle of mathematical induction. The results proved conclusively that "if any number of consecutive odd or even integers are raised to a positive power k, then the kth difference is equal to $2^k k!$

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Author α: Department of Computer Science, Faculty of Natural Sciences, University of Jos.
σ: Department of Mathematics, Faculty of Natural Sciences, University of Jos.
ρ: Department of Computer science, Faculty of Natural Sciences, University of Jos.
Ѡ: Department of Political science, Faculty of Social Sciences, Ahmadu Bello University Zaria.

I. INTRODUCTION

A perfect power is a number which has a rational root. Chase [1]. An integral perfect power is the irrational root that is an integer. A finite difference is a mathematical expression of the form, $f(x + b) - f(x + a)$ and a forward difference is of the form $\Delta h\,[f]\,(x) = f(x + h) - f(x)$[2].

In this article, an examination on the set of even numbers: {0, 2, 4, 6, 8, 10, 12, 14, 16, 18} and the set of odd numbers: {1, 3, 5, 7, 9, 11, 13, 15, 17, 19} reveals that the first difference of the square of consecutive elements of both the set of even and odd numbers is even. The second difference of the two distinct sets is {8, 8, 8, 8, 8, 8, 8, 8, 8} = {8}, a singleton set.

Clearly, the third difference is {0}. The emerging pattern motivates the investigation of whether or not this pattern persists for all perfect squares of elements of even and odd sets, resulting from integers. In their contribution in Exponential Diophantine Equations, Shorey and Tijdeman[3] investigated perfect powers at integral values of a polynomial with rational integer coefficients and obtain in particular the following.

Let f(x) be a polynomial with rational coefficients and with at least two simple rational zeros.

Suppose $b \neq 0$, $m \geq 0$, x and y with $|y| > 1$ are rational integers. Then the equation $f(x) = by^m$ implies that m is bounded by a computable number depending only on b and f. Also, in their contribution in the difference of perfect powers of integers, Ladan, Ukwu and Apine[4] investigated order of integral perfect powers and proved that "if any number of consecutive integers are raised to a positive integral power k, then the k^{th} difference is equal to k!

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More generally, the question at the heart of the matter is the following. What is the computational disposition of the orders of difference of integral perfect powers of successive elements of set of even and odd numbers? The Review of Literature shows that no such investigation has been undertaken. Thus, this article adds to the existing body of knowledge, by providing answers to the above question.

II. METHODS

2.1 Preliminary Definitions

In what follows, the difference of finite order will be defined.

2.1.1 Difference of Order One (1)

Given a sequence $\{U_j\}_{j=0}^{\infty}$, define the difference of order one at j with respect to the sequence by:

$$\Delta(U_j) = U_{j+1} - U_j, \text{ for every integral j.}$$

2.1.2 Higher Order Difference

Higher difference can be defined recursively by:

$$\Delta^k(U_j) = \Delta \ \ \Delta^{k-1}(U_j) \ \ = \Delta^{k-1} \ \ \Delta(U_j) \ \ = \Delta^{k-1}[U_{j+1} - U_j] \text{ for } k \geq 2.$$

III. RESULTS AND DISCUSSION

3.1 Preliminary Theorem

Suppose that $U_j = j$ and $U_j = \hat{j}$, for integral j.

Let $\propto_j U_j = \propto_j j$ be the set of even numbers and $\propto_j \hat{U}_j + \beta_j = \propto_j j + \beta_j$ be the set of odd numbers.

Where $\propto_j$=2 fixed, j $\geq$0, $\beta = 1$, fixed.

Then,

(i)$_1$ $\Delta(\propto_j U_j)^2 = \propto_j^2(\text{odd})$ for even case

(i)$_2$ $\Delta(\propto_j \hat{U}_j + \beta)^2 = \propto_j^2(\text{even})$ for odd case

(ii) $\Delta^2(\propto_j U_j)^2 = \Delta^2(\propto_j \hat{U}_j + \beta)^2 = 8 = 2^2(2!)$

(iii) $\Delta^k(\propto_j U_j)^2 = \Delta^k(\propto_j \hat{U}_j + \beta_j)^2 = 0, k \in \{3, 4, \ldots\ldots\}$

Proof:

Case 1. (even)

Let $\langle \propto_j U_j \rangle_{j=0}^{9} = \langle \propto_j j \rangle_{j=0}^{9} = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$

Then, the table below display the values of $\Delta^k(\propto_j U)^p$ for selected values of k and P in the set {0, 1, 2, 3} and {1, 2} respectively when $\Delta^0(\propto_j U_j)^p = (\propto_j U_j)^p$.

Table I: Difference Order Table for Selected Consecutive Positive Integers.

$\propto_j j$	$(\propto_j j)^2$	$\Delta(\propto_j j)^2$	$\Delta^2(\propto_j j)^2$	$\Delta^3(\propto_j j)^2$
0	0	4	8	0
2	4	12	8	0
4	16	20	8	0
6	36	28	8	0
8	64	36	8	0
10	100	44	8	0
12	144	52	8	0
14	196	60	8	0
16	256	68	8	
18	324			

It is clear that $\Delta^k(\propto_j j)^2 = 0$, for all $k \geq 3$, $\propto_j j = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$. Obviously the theorem is valid for $j \geq 0$ as observed from columns 4 and 5.

Case II. (odd)

Let $\langle \propto_j + \beta \rangle_{j=0}^{9} = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$.

Then the table below displayed the values of $\Delta^k(\propto_j + \beta_j)^p$ for selected values of k and p in the set {0, 1, 2, 3} and {1, 2} respectively when $\Delta^0(\propto_j j + \beta_j)^p = (\propto_j j + \beta_j)^p$.

Table II: Difference Order table for selected Consecutive Positive Integers

$\propto_j + \beta$	$(\propto_j + \beta)^2$	$\Delta(\propto_j + \beta)^2$	$\Delta^2(\propto_j + \beta)^2$	$\Delta^3(\propto_j + \beta)^2$
1	1	8	8	0
3	9	16	8	0
5	25	24	8	0
7	49	32	8	0
9	81	40	8	0
11	121	48	8	0
13	169	56	8	0
15	225	64	8	
17	289	72		
19	361			

Similarly, it is clear that $\Delta^k(\propto_j \widehat{U}_j + \beta_j) = 0$, for all $k \geq 3$, $\propto_j + \beta = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$. The theorem is valid as observed from column 4 and 5.

The proof of (i), (ii) and (iii) are direct.

(i₁) $U_j = j \Rightarrow \propto U_j = \propto_j \Rightarrow \left(\propto U_j\right)^2 = \propto^2 j^2$

$\Rightarrow \quad (\propto^2 j^2) \quad = \propto^2 \Delta j^2 = \propto^2 [(j+1)^2 - j^2] = \propto^2 [j^2 + 2j + 1 - j^2]$

$= \propto^2 (2j+1) = \propto^2 (\text{odd}) = \text{even}.$

(i₂) $\widehat{U}_j = \hat{j} \Rightarrow \propto \widehat{U}_j + \beta = \propto \hat{j} + \beta$

$\left(\widehat{U}_j + \beta\right)^2 = \left(U_{\hat{j}} + \beta\right)^2 = \left(\propto_{\hat{j}}\right)^2 + 2\propto \beta_j + \beta^2$

$\Delta\left(\propto \widehat{U}_j + \beta_j\right)^2 = \Delta\left(\propto^2 j^2 + 2 \propto \beta_j + \beta^2\right)$

$= \propto^2 \Delta j^2 + 2 \propto \beta \Delta_j + \beta^2$

$= \propto^2 [(j+1)^2 - j^2] + 2\propto \beta[(j+1) - j] + 0$

$= \propto^2 (j^2 + 2j + 1 - j^2) + 2\propto \beta \quad (1)$

$= \propto^2 (2j+1) + 2\propto \beta$

$= 2\propto^2 j + \propto^2 + 2\propto \beta$

$= \propto^2 (2j+1) + 2\propto \beta$

$\Rightarrow \quad \Delta\left(\propto \widehat{U}_j + \beta_j\right)^2 = \propto^2 (2j+1) + 2\propto \beta = \text{even}$

For every $j \geq 0, \propto = 2, \beta = 1$.

(ii) $\Delta^2\left(\propto U_j\right)^2 \quad = \Delta\left(\Delta(\propto j)\right)^2 = \Delta[\propto^2 (2j+1)]$

$= \Delta(2 \propto^2 j + \propto^2) = 2 \propto^2 \Delta j + \Delta\propto^2$

$= 2 \propto^2 [(j+1) - j] + 0$

$= 2 \propto^2 (1) = 2 \propto^2$

$\Rightarrow \Delta^2\left(\propto U_j\right)^2 \quad = 2 \propto^2$

Similarly:

$\Delta^2\left(\propto \widehat{U}_j + \beta\right)^2 \quad = \Delta[\Delta(\propto j + \beta)^2]$

$= \Delta[\propto^2 (2j+1) + 2 \propto \beta]$

$= \Delta[2 \propto^2 j + \propto^2 + 2 \propto \beta]$

$$= 2 \propto^2 \Delta j + \Delta\propto^2 + 2\Delta\propto \beta$$

$$= 2 \propto^2 [(j+1) - j] + 0 + 0$$

$$= 2 \propto^2 (1) = 2 \propto^2$$

$$\Rightarrow \Delta^2\left(\propto \hat{U}_j + \beta\right) = 2 \propto^2$$

$$\Rightarrow \Delta^2\left(\propto U_j\right)^2 = \Delta^2\left(\propto \hat{U}_j + \beta\right) = 2 \propto^2. \text{ proving (ii)}$$

The principle of mathematical induction is needed in the proof of (iii).

For k = 3, $\Delta^3\left(\propto_j U_j\right)^2 = \Delta^3(\propto j)^2 = \Delta[\Delta^2(\propto j)^2] = \Delta(2 \propto^2) = 0$

Similarly, $\Delta^3\left(\propto \hat{U}_j + \beta\right)^2 = \Delta^3(\propto j + \beta)^2 = \Delta[\Delta^2(\propto j + \beta)^2] = \Delta(2 \propto^2) = 0$

Consequently, $\Delta^k\left(\propto \hat{U}_j\right)^2 = \Delta^{k-3}[\Delta^3(\propto j)^2] = \Delta^{k-3}(0) = 0$

and $\Delta^k\left(\propto \hat{U}_j + \beta\right) = \Delta^{k-3}[\Delta^3(\propto j + \beta)^2] = \Delta^{k-3}(0) = 0$

proving $\Delta^k\left(\propto U_j\right)^2 = \Delta^k\left(\propto \hat{U}_j + \beta\right)^2 = 0$, for all positive integer $k \geq 3$. This established the proof of (iii).

Thus, we have seen clearly that:

Theorem 3.1 $\Rightarrow$ $\Delta\left(\propto U_j\right)^2$ and $\Delta\left(\propto \hat{U}_j + \beta\right)^2$ are even .... (i)

$\Delta^2\left(\propto U_j\right)^2$ and $\Delta^2\left(\propto \hat{U}_j + \beta\right)^2 = \propto^2 (2!), \propto = 2$ .... (ii)

$\Delta^3\left(\propto U_j\right)^2$ and $\Delta^3\left(\propto \hat{U}_j + \beta\right)^2 = 0 \not\forall\, k \geq 3$ .... (iii)

In the sequel, we examine the computational disposition of $\Delta^k\left(\propto U_j\right)^p$ and $\Delta^k\left(\propto \hat{U}_j + \beta\right)^p$

for every integral j and positive integral k and p. The results are summarized as in the following theorem.

3.2 Main Theorem

Let $\propto U_j = \propto j$ and $\propto \hat{U}_j + \beta = \propto j + \beta$, where j is any integer. Then for arbitrary positive integer k and p, $\Delta^k\left(\propto U_j\right)^p$ and $\Delta^k\left(\propto \hat{U}_j + \beta\right)^p$ is given by:

$$\Delta^k(\propto U_j)^p = \Delta^k(\propto \widehat{U}_j + \beta)^p \begin{cases} 0, & \text{if } k > p \quad (a) \\ \sum_{i=1}^{p} \binom{p}{i} j^i, & \text{for } k = 1 \quad (b) \\ \propto^k k!, & \text{if } k = p \quad (c) \\ \text{even}, & \text{for } p \geq 1 \quad (d) \\ \text{even}, & \text{for } 2 \leq k < p \quad (e) \end{cases}$$

3.2.1 Proof of (a)

$$\Delta^2(\propto U_j) = \Delta(\Delta\propto j) = \Delta[\propto \Delta(j)] = \Delta[((j+1) - j)]$$
$$= \Delta[\propto (1)] = \Delta(\propto) = 0$$

Similarly,

$$\Delta^2(\propto U_j + \beta) = \Delta[\Delta(\propto j + \beta)] = \Delta[\propto \Delta j + \Delta\beta]$$
$$= \Delta[\propto ((j+1) - j)] = \Delta[\propto (1)]$$
$$= \Delta(\propto) = 0$$
$$\Rightarrow \quad \Delta^2(\propto U_j) = \Delta^2(\propto U_j + \beta) = 0.$$

So (a) is valid for $k = 2$ and $p = 1$, which are the least values for the respective exponents. Assume that (a) is valid for all pairs of integers $\hat{k}$, $\hat{p}$ for which $\hat{k} + \hat{p} \leq k + p$ for some positive integers $\hat{k}$ and $\hat{p}$ such that $p \geq 1$, $k \geq 2$, $p < k$.

Then $\Delta^k(\propto U_j)^p = \Delta[\Delta^k(\propto U_j + \beta)^p] = \Delta(0) = 0$

and $\Delta^{k+1}(\propto U_j + \beta) = \Delta[\Delta^k(\propto U_j + \beta)] = \Delta(0) = 0$

(by induction hypothesis hypothesis) = 0. Therefore, the validity of (a) is established.

3.2.2 Proof of (b)

$$(j+1)^p = \sum_{i=1}^{p} \binom{p}{i} j^i = \sum_{i=1}^{p-1} \binom{p}{i} j^i + \binom{p}{p} j^p$$

$$\Rightarrow \quad (j+1)^p - j^p = \sum_{i=1}^{p-1} \binom{p}{i} j^i \quad \text{proving (b)}$$

$$\Rightarrow \quad \Delta^1(\propto U_j)^p = \propto^p \left(\Delta U_j^p\right) = \propto^p \sum_{i=1}^{p-1} \binom{p}{i} j^i$$

$$\text{and} \quad \Delta^1(\propto U_j + \beta)^p = \sum_{i=1}^{p-1} \binom{p}{i} (\propto j)^{p-i} = \propto^p \sum_{i=1}^{p-1} \binom{p}{i} j^i$$

Observe that, since β = 1, implies that:

$$\propto \widehat{U}_j + \beta = \propto j + 1$$

$$\therefore \quad (\propto j + \beta)^p = \sum_{i=1}^{p-1} \binom{p}{i} (\propto j)^{p-i} \beta^i = \sum_{i=1}^{p-1} \binom{p}{i} (\propto j)^{p-i}$$

For simplicity of complexity of the odd form, we can logically express it as a single form with respect to even form, since they have same characteristic structures.

Claim that proof of even case is necessary and sufficient for the proof of the odd case.

Considering the even form for the remaining part of the proof ∵ proof even ⇔ proof of odd.

We have that:

$$\Delta(\propto U_j)^p = \Delta\left(\propto^p U_j^p\right) = \propto^p \left(\Delta U_j^p\right) = \propto^p \ (\Delta j^p)$$

$$= \propto^p [(j+1)^p - j^p] = \propto^p \sum_{i=1}^{p-1} \binom{p}{i} j^i$$

3.2.3 Proof of (c)

We examine $\Delta^k(\propto U_j)^p$: k =1

$\Rightarrow (\propto U_j) = \propto (\Delta j) = \propto (j + 1 - j) = \propto (1) = \propto = 2!$

$k = 2, \Delta^2(\propto U_j)^2 \quad = \propto^2 \Delta^2 U_j^2 = \Delta\propto^2 (\Delta j^2) = \propto^2 \Delta[(j+1)^2 - j^2]$

$= \propto^2 \Delta(j^2 + 2j + 1 - j^2) = \propto^2 \Delta(2j + 1)$

$= \propto^2 \Delta(2j + 1)$

$= 2 \propto^2 \Delta j + \Delta\propto^2$

$= 2 \propto^2 (j + 1 - j) = 0$

$= 2 \propto^2 = \propto^2 \ 2! = 2^2.2!$

$\Rightarrow \Delta^2(\propto U_j)^2 = \Delta^2(\propto \widehat{U}_j + \beta) = \propto^2 \ 2! = 2^2.2! = 8.$

by (ii) of theorem 3.1 ⇒ the theorem is valid for k ϵ {1, 2}. Assume the validity of the theorem for k ϵ {3,...,q} for some integer $q \geq 4$. Then $\Delta^q(\propto^q Uj^q) = \propto^q q! \Rightarrow$ $\Delta^q(\propto^q Uj^q) = \propto^q (\propto^q Uj^q) = \propto^q q!$

by induction hypothesis. Finally, we need to prove that:

$\Delta^{q+1}(\propto^{q+1} Uj^{q+1}) = \propto^{q+1} (q+1)!$

Claim Δ^r is reproductive. Reproductive is understood to mean the following:

$$\Delta^r \sum_{j=1}^{n} \widehat{\propto}_j\, g_j(x) = \sum_{j=1}^{n} \widehat{\propto}_j\, \Delta^r(g_j(x))$$

Where $\propto_j$ are arbitrary constants.

3.2.4 Proof of Claim

Consider $\Delta^r(\widehat{\propto}_1 U_{j_1}^{k_1} + \widehat{\propto}_2 U_{j_2}^{k_2}), r = \{1, 2 \ldots\ldots\}$.

for r = 1, $\Delta(\widehat{\propto}_1 U_{j_1}^{k_1} + \widehat{\propto}_2 U_{j_2}^{k_2}) = \Delta(\widehat{\propto}_1 U_{j_1}^{k_1} + \widehat{\propto}_2 U_{j_2}^{k_2})$

$$= \widehat{\propto}_1 (j+k)^{k_1} + \widehat{\propto}_2 (j+k)^{k_2} - \widehat{\propto}_1 U_{j_1}^{k_1} - \widehat{\propto}_2 U_{j_2}^{k_2}$$

$$= \widehat{\propto}_1 [(j_1+1)^{k_1} - j_i^{k_1}] + \widehat{\propto}_2 [(j_2+1)^{k_2} - j_2^{k_2}]$$

$$= \widehat{\propto}_1 \Delta(j_i^{k_1}) + \widehat{\propto}_2 \Delta(j_2^{k_2})$$

$$= \widehat{\propto}_1 \Delta\left(U_{j_i}^{k_1}\right) + \widehat{\propto}_2 \Delta\left(U_{j_2}^{k_2}\right)$$

⇒ Δ is a reproductive ⇒ the claim is valid for $2 \leq r \leq t$, for some integer $t \geq 3$.

Then $\Delta^t(\widehat{\propto}_1 U_{j_1}^{k_1} + \widehat{\propto}_2 U_{j_2}^{k_2}) = \widehat{\propto}_1 \Delta^t(U_{j_1}^{k_1}) + \widehat{\propto}_2 \Delta^t(U_{j_2}^{k_2})$

Finally,

$$\Delta^{t+1}(\widehat{\propto}_1 U_{j_1}^{k_1} + \widehat{\propto}_2 U_{j_2}^{k_2}) = \Delta^t[\Delta(\widehat{\propto}_1 U_{j_1}^{k_1} + \widehat{\propto}_2 U_{j_2}^{k_2})]$$

$$= \Delta^t(\widehat{\propto}_1 \Delta U_{j_1}^{k_1} + \widehat{\propto}_2 \Delta U_{j_2}^{k_2}) = \widehat{\propto}_1 \Delta^t(\Delta U_{j_1}^{k_1}) + \widehat{\propto}_2 \Delta^t(\Delta U_{j_2}^{k_2})$$

(by induction hypothesis)

$$= \widehat{\propto}_1 \Delta^{t+1}(U_{j_1}^{k_1}) + \widehat{\propto}_2 \Delta^{t+1}(U_{j_2}^{k_2})$$

So the theorem is valid for r = t+1 and hence valid for all positive integer r.

Now, $\Delta^{q+1}(\propto^{q+1} Uj^{q+1}) = \Delta^q[\propto^{q+1} \Delta(Uj^{q+1})]$

$$= \Delta^q[\propto^{q+1} ((j+1)^{q+1} - j^{q+1})]$$

$$= \Delta^q \left[\propto^{q+1} \sum_{i=1}^{q} \binom{q+1}{i} j^i \right]$$

$$= \Delta^q \left[\propto^{q+1} \sum_{i=1}^{q} \binom{q+1}{i} Uj^i \right]$$

$$= \propto^{q+1} \sum_{i=1}^{q} \binom{q+1}{i} \Delta^q \left(Uj^q\right) \quad \text{(by reproductive property)}$$

$$= \propto^{q+1} \binom{q+1}{i} \Delta^q (Uj^q) \quad \text{(by (a) of theorem 3.2)}$$

$$= \propto^{q+1} \binom{q+1}{i} q! \quad \text{(by the induction hypothesis)}$$

$$= \propto^{q+1} \binom{q+1}{i}!$$

Therefore, $\Delta^k\left(\propto^k {U_j}^k\right) = \propto^k k!$

For every positive integer k, proving (c).

3.2.5 Proof of (d)

$$\Delta\left(\propto U_j\right)^p = \propto^p \left[(j+1)^p - j\right]$$

$$\Rightarrow \quad \propto^p \left[(j+1)^p - j\right] = \begin{cases} \text{even} - \text{even} = \text{even for even case} \\ \text{odd} - \text{odd} = \text{odd for odd case} \end{cases}$$

So in all cases $\Delta\left(\propto U_j\right)^p$ and $\Delta\left(\propto \widehat{U}_j + \beta\right)^p$ are even, proving (d).

3.2.6 Proof of (e)

Consider $\Delta^k\left(\propto U_j\right)^p$ for k ≥ 2, k < p. If k = 2, then the condition k < p ⇒ p ≥ 3.

$$\Delta^2\left(\propto U_j\right)^p = \Delta\left[\Delta\left(\propto U_j\right)^p\right]$$

$$= \propto^p \Delta \left[\sum_{i=1}^{p-1} \binom{p}{i} j^i \right] = \Delta \text{ (an even)from (a)and (b)}$$

$$= \propto^p \Delta\left[(j+1)^p - j^p\right] = \left[\Delta\left[(j+1)^p\right] - \Delta\left[j^p\right]\right] \propto^p$$

(by the reproductive property.

$$= \propto^p \left[(j+2)^p - (j+1) - \left[(j+1)^p - j^p\right]\right]$$

$$= \propto^p \left[(j+2)^p - 2(j+1)^p - j^p\right] = \begin{cases} \text{even} - \text{even} + \text{even} = \text{even (for even case)} \\ \text{even} - \text{even} + \text{even} = \text{even (for odd case)} \end{cases}$$

So in all cases, $\Delta^2(\propto U_j)^p$ and $\Delta^2(\propto \widehat{U}_j + \beta)^p$ is even for p ≥ 2.

Assume the validity of (e) for all positive integer k and p such that k + p ≤ m, for some positive integer m.

Then

$$\Delta^{k+1}(\propto U_j)^p = \Delta^k(\Delta(\propto U_j)^p)$$

$$= \Delta^k \propto^p \left[[(j+2)^p - 2(j+1)^p + j^p]\right]$$

$$= \Delta^k \propto^p \left[[(j+2)^p] - 2\Delta^k[(j+1)^p] + \Delta^k[j^p]\right]$$

(by the reproductive property of Δ^k)

$$\left(\Delta^k\left(U^p_{j+2}\right) - 2\Delta^k\left(U^p_{j+1}\right) + \Delta^k\left(U^p_j\right)\right)\propto^p$$

= even – even + even = even.

$$= \propto^p \Delta^k\left[U^p_{j+2}\right] - 2\propto^p \Delta^k\left[U^p_{j+1}\right] + \propto^p \Delta^k\left[U^p_j\right]$$

Finally, we examine:

$\Delta^k(\propto U_j)^{p+1}$.

$$\Delta^k(\propto U_j)^{p+1} = \Delta^k(\propto j)^{p+1} = \Delta^{k-1}[\Delta(\propto j)^{p+1}]$$

$$= \Delta^{k-1}[\propto^{p+1}(j+1)^{p+1} - 2(j+1)^{p+1} + j^{p+1}]$$

$$= \Delta^{k-1}\left[\propto^{p+1}\left[U^p_{j+2}\right] - 2\propto^{p+1}\Delta^{k-1}\left[U^{p+1}_{j+1}\right] + \propto^{p+1}\Delta^{k-1}\left[U^{p+1}_j\right]\right]$$

= even – even + even = even.

(by the induction hypothesis). Since k – 1 + p + 1 = k + p ≤ m.

This completes the proof, that is the relation (e) hold for all +ve integers k and p for which k < p, k ≥ 2. Thus, the theorem is established.

3.3 Corollary

$$\Delta^k(\propto j)^k = \Delta^k\left(\propto^k j^k\right) = \propto^k \Delta^k j^k = \propto^k k!$$

For any integer j and for any positive integer k.

In other words, for any set of even and odd integers ($\propto U_j$ and $\propto \widehat{U}_j + \beta$), the k^{th} order difference of the k^{th} power of any integers is equal to $2^k k!$

The implication of (c) in theorem 3.2 is the following: "if any number of consecutive even or odd integers are raised to a positive integral power k, then the k^{th} order difference is equal to $2^k k!$

3.4 Remarks

The existence of the triddle between the difference operator Δ and the D operator (differential operator).

$$D^k(x^p) = 0 \text{ if } k > p$$

$$D^k(x^k) = k!$$

$$D^k(\propto^k x^k) = \propto^k k!$$

The coefficient of x^{p-k} in $D^k(x^p)$:

$$D^k(x^p) = \frac{p!}{(p-k)!} x^{p-k} \text{ for } k \geq 2, k < p \quad \text{is even.}$$

$$\Rightarrow \quad \propto^p D^k(x^p) = \frac{\propto^p p!}{(p-k)!} x^{p-k}, 2 \leq k < p \quad \text{is even}$$

IV. CONCLUSION

This article established the structures of finite orders with respect to powers of consecutive elements of even and odd sets. Specifically, the results reveal a similarity between the difference orders and the D operator powers of monomials with positive integral powers as reflected in (a) and (c) of theorem 3.2 and (I) and (II) of remark 3.4.

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The Discrete Ordinates and the Riccati Equation Methods in thee Estimation of Growth of Solutions of Systems of Two Linear RST Order Ordinary Dierential Equations

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Classification: MSC Code: 34C10

Language: English

Great Britain Journals Press

LJP Copyright ID: 925614
Print ISSN: 2631-8490
Online ISSN: 2631-8504

London Journal of Research in Science: Natural & Formal

Volume 24 | Issue 10 | Compilation 1.0

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Keywords: systems of equations with piecewise constant coecients, the Riccati equation, normal and extremal solutions, main, nonprincipal and ordinary solutions of the system, the theorem of Wazevski.

I. INTRODUCTION

Let $a_{jk}(t)\ \ (j,k=1,2)$ be complex-valued continuous functions on the interval $[t_0,+\infty)$. Consider the system

$$\begin{cases} \phi'(t) = a_{11}(t)\phi(t) + a_{12}(t)\psi(t); \\ \psi'(t) = a_{21}(t)\phi(t) + a_{22}(t)\psi(t), \end{cases} \tag{1.1}$$

$t \ge t_0$. Study of the question of stability of solutions of differential equations and systems of differential equations, in particular of system (1.1), is an important problem of qualitative theory of differential equations and many works are devoted to it (see [1], [2] and cited works therein [3 - 10]). The fundamental theorem of R. Bellman (see [2], pp. 168, 169) reduces the study of boundedness of solutions of wide class of nonlinear systems to the study of stability of linear systems of differential equations. Many problems of mechanics, physics and other natural sciences are connected with the study of stability of the linear

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systems of differential equations (in particular of the linear differential equations) too (see for example [6,7]). One of ways to study the mentioned above question is the use of different methods of estimations of solutions of systems of being studied equations (see [4]).

In this paper some estimates of solutions of the system (1.1) in terms of its coefficients are obtained. To obtain them it was used the method of approximation of solutions of (1.1) by solutions of system with piecewise constant coefficients (the discrete ordinates method).

Denote: $P(t) \equiv a_{12}(t)\exp\Big\{\int\limits_{t_0}^{t}[a_{22}(\tau) - a_{11}(\tau)]d\tau\Big\}$, $Q(t) \equiv a_{21}(t)\exp\Big\{\int\limits_{t_0}^{t}[a_{11}(\tau) - a_{22}(\tau)]d\tau\Big\}$. In this article we will study the following two principal cases:

$A)$ $P(t) > 0, \quad Q(t) < 0, \quad t \ge t_0;$

$B)$ $P(t) > 0, \quad Q(t) > 0, \quad t \ge t_0$

(the case $P(t) < 0,\ Q(t) > 0,\ t \ge t_0$, is similar to the case A), and the case $P(t) < 0$, $Q(t) < 0,\ t \ge t_0$, is reducible to the case B) by the simple substitution $\phi(t) \to -\phi(t)$). The case A) can be geometrically interpreted as a case, when the origin of coordinates of phase plane of variables u, v is a "center" or a "focus" and the case B) as a a "saddle" with respect to the curves $\{(u(t), v(t))\},\ t \ge t_0$, where $\{(u(t), v(t))\}$ are the solutions of the system

$$\begin{cases} u'(t) = \qquad P(t)v(t); \\ v'(t) = Q(t)u(t), \end{cases} \tag{1.2}$$

$t \ge t_0$. On examples the obtained results are compared with the results obtained by methods of Liapunov, Yu. S. Bogdanov, T. Wazevski, estimates of solutions by logarithmic norm of S. M. Lozinski and of freezing.

II. AUXILIARY PROPOSITIONS

Lemma 2.1. *For each solution $(\phi(t), \psi(t))$ of the system (1.1) and for each $\varepsilon > 0$ and $t_1 > t_0$ there exists piecewise constant functions $\widetilde{a}_{jk},\ t \ge t_0,\ j,k = 1,2,$ such, that the solutions $(\widetilde{\phi}(t), \widetilde{\ }(t))$ of the system*

$$\begin{cases} \phi'(t) = \widetilde{a}_{11}(t)\phi(t) + \widetilde{a}_{12}(t)\boldsymbol{\psi}(t); \\ \boldsymbol{\psi}'(t) = \widetilde{a}_{21}(t)\phi(t) + \widetilde{a}_{22}(t)\boldsymbol{\psi}(t), \end{cases} \tag{2.0}$$

$t \geq t_0$. *with* $\widetilde{\phi}(t_0) = \phi(t_0)$, $\widetilde{\psi}(t_0) = \psi(t_0)$ *satisfy the inequalities:* $|\widetilde{\phi}(t_1) - \phi(t_1)| \leq \varepsilon$, $|\widetilde{\psi}(t_1) - \psi(t_1)| \leq \varepsilon$.

The proof of this lemma is not difficult, and we omit it.

Remark 2.1. *By a solution of the system (2.0) we will mean a pair of absolutely continuous functions* $\phi(t)$ *and* $\psi(t)$, *satisfying (2.0) almost everywhere on* $[t_0, +\infty)$.

Let $t_0 < t_1 < ... < t_n < ...$ be a finite or infinite sequens, and let $\widetilde{p}(t) = p_j > 0$, $\widetilde{q}(t) = = q_j > 0$, $t \in [t_j; t_{j+1})$, $j = 0, 1, 2, ...$ Consider the Cauchy problem

$$\begin{cases} u'(t) = \qquad \widetilde{p}(t)v(t); \\ v'(t) = -\widetilde{q}(t)u(t); \\ u(t_0) = u_{(0)}, \;\; v(t_0) = v_{(0)}, \end{cases} \tag{2.1}$$

$t \geq t_0$. Any solution of this system we will seek in the form

$$u(t) = A_j \sin\left(\sqrt{l_j}t + \omega_j\right), \quad v(t) = A_j\sqrt{h_j}\cos\left(\sqrt{l_j}t + \omega_j\right), \quad t \in [t_j, t_{j+1}), \tag{2.2}$$

where $l_j \equiv p_j q_j$, $h_j \equiv \frac{q_j}{p_j}$ and A_j, ω_j are the sought constants, $j = 0, 1, 2, ...$ By virtue of initial conditions of problem (2.1) the unknowns A_0 and ω_0 we determine by solving the system

$$\begin{cases} A_0 \sin\left(\sqrt{l_0}t_0 + \omega_0\right) = u_{(0)}; \\ A_0\sqrt{h_0}\cos\left(\sqrt{l_0}t_0 + \omega_0\right) = v_{(0)}, \end{cases} \tag{2.3}$$

and the remaining unknowns by successive solving of the systems

$$\begin{cases} A_{j+1}\sin\left(\sqrt{l_{j+1}}t_{j+1} + \omega_{j+1}\right) = A_j \sin\left(\sqrt{l_j}t_{j+1} + \omega_j\right); \\ A_{j+1}\sqrt{h_{j+1}}\cos\left(\sqrt{l_{j+1}}t_{j+1} + \omega_{j+1}\right) = A_j\sqrt{h_j}\cos\left(\sqrt{l_j}t_{j+1} + \omega_j\right), \end{cases} \tag{2.4}$$

$j = 1, 2, ...$ From (2.3) it follows

$$A_0 = u^2_{(0)} + \frac{v^2_{(0)}}{h_0}. \tag{2.5}$$

Denote: $\alpha_j \equiv \sqrt{l_j}t_{j+1} + \omega_j$, $\beta_j \equiv \sqrt{l_{j+1}}t_{j+1} + \omega_{j+1}$, $j = 0, 1, 2, ...$ From (2.4) it is easy to derive the equalities:

$$A^2_{j+1} = A^2_j\left[\frac{h_j + h_{j+1}}{2h_{j+1}} + \frac{h_j - h_{j+1}}{2h_{j+1}}\cos 2\alpha_j\right], \quad A^2_j = A^2_{j+1}\left[\frac{h_j + h_{j+1}}{2h_j} + \frac{h_{j+1} - h_j}{2h_j}\cos 2\beta_j\right],$$

$j=0,1,2,....$ From here it follows

$$|A_{j+1}| \le |A_j| \le \sqrt{\frac{h_{j+1}}{h_j}}|A_{j+1}| \quad for \quad h_j \le h_{j+1}; \tag{2.6}$$

$$|A_j| \le |A_{j+1}| \le \sqrt{\frac{h_j}{h_{j+1}}}|A_j| \quad for \quad h_j \ge h_{j+1}; \tag{2.7}$$

$j=0,1,2,....$ From (2.2) it follows:

$$u^2(t)\frac{1}{h_j} + v^2(t) = A_j^2, \ \ t \in [t_j, t_{j+1}), \ \ j=0,1,2,.... \tag{2.8}$$

Let the initial values $u_{(0)}$ and $v_{(0)}$ be real. Then $u(t)$ and $v(t)$ will be real valued. Therefore, from (2.8) we will have:

$$\min\{1, h_j^1\}A_j^2 \le u^2(t) + v^2(t) \le \max\{1, h_j^1\}A_j^2, \ \ t \in [t_j, t_{j+1}), \tag{2.9}$$

where $h_j^1 \equiv \frac{p_j}{q_j}, \ \ j=0,1,2,....$

Definition 2.1. We shall say, that a continuous on the interval $[t_0;+\infty)$ function $f(t)$ belongs to the class $C_\sim = C_\sim[t_0,+\infty)$, if there exists an infinitely large sequence $\xi_0 = t_0 \le \xi_1 \le ... \le \xi_n \le ...$ such, that $f(t)$ is a nondecreasing on the interval $[\xi_{2n}, \xi_{2n+1}]$ and nonincreasing on the interval $[\xi_{2n+1}, \xi_{2n+2}]$ function, $n=0,1,2,....$ The numbers $\xi_n, \ \ n=0,1,2,...$ we shall call points of possible extremums of the function $f(t)$.

Let $S(t) \in C_\sim$, and let $\xi_n, \ \ n=0,1,2,...$ be the points of possible extremums of $S(t)$. Note, that if $\xi_{2n+1} = \xi_{2n+2}, \ \ n=0,1,2,...$, then $S(t)$ is a nondecreasing function and if $\xi_{2n} = \xi_{2n+1}, \ \ n=0,1,2,...$, then $S(t)$ is a nonincreasing function. Let $S(t) > 0, \ \ t \ge t_0$. Consider the functions

$$r_S^-(t) \equiv \begin{cases} 1, & t \in [\xi_0;\xi_1]; \\ \sqrt{\frac{S(\xi_1)}{S(t)}}, & t \in [\xi_1;\xi_2]; \\ \prod_{k=1}^{n} \sqrt{\frac{S(\xi_{2k-1})}{S(\xi_{2k})}}, & t \in [\xi_{2n};\xi_{2n+1}], \ \ n=1,2,...; \\ \left[\prod_{k=1}^{n-1} \sqrt{\frac{S(\xi_{2k-1})}{S(\xi_{2k})}}\right] \sqrt{\frac{S(\xi_{2n-1})}{S(t)}}, & t \in [\xi_{2n-1};\xi_{2n}], \ \ n=2,3,..., \end{cases}$$

$$r_S^+(t) \equiv \begin{cases} \sqrt{\frac{S(t)}{S(\xi_0)}}, & t \in [\xi_0; \xi_1]; \\ \prod_{k=0}^{n} \sqrt{\frac{S(\xi_{2k+1})}{S(\xi_{2k})}}, & t \in [\xi_{2n+1}; \xi_{2n+2}], \quad n = 0, 1, ...; \\ \left[\prod_{k=9}^{n-1} \sqrt{\frac{S(\xi_{2k-1})}{S(\xi_{2k})}}\right] \sqrt{\frac{S(t)}{S(\xi_{2n})}}, & t \in [\xi_{2n}; \xi_{2n+1}], \quad n = 1, 2, ..., \end{cases}$$

Let $S(t)$ be absolutely continuous. Then $\sqrt{\frac{S(\xi_{2n+1})}{S(t)}} = \exp\Big\{-\frac{1}{2} \int\limits_{\xi_{2n+1}}^{t} \frac{S'(\tau)}{S(\tau)} d\tau\Big\}, \quad n = 0, 1, 2, ...,$

$\sqrt{\frac{S(\xi_{2k-1})}{S(\xi_{2k})}} = \exp\Big\{-\frac{1}{2} \int\limits_{\xi_{2k-1}}^{\xi_{2k}} \frac{S'(\tau)}{S(\tau)} d\tau\Big\}, \quad k = 1, 2, ...$ Therefore

$$r_S^-(t) = \exp\Big\{\frac{1}{2} \int\limits_{t_0}^{t} \frac{S'_{(-)}(\tau)}{S(\tau)} d\tau\Big\}, \qquad t \geq t_0, \tag{2.10}$$

where $S_{(-)}(t) \equiv \begin{cases} -S(t), & \text{if exists } S'(t) > 0; \\ 0, & \text{otherwise}, \end{cases}$ $t \geq t_0$. By analogy it shows, that

$$r_S^+(t) = \exp\Big\{\frac{1}{2} \int\limits_{t_0}^{t} \frac{S'_{(+)}(\tau)}{S(\tau)} d\tau\Big\}, \qquad t \geq t_0, \tag{2.11}$$

where $S_{(+)}(t) \equiv \begin{cases} S(t), & \text{if exists } S'(t) > 0; \\ 0, & \text{otherwise}, \end{cases}$ $t \geq t_0$. It is clear that $S'(t) = S'_{(+)}(t) -$ $- S'_{(-)}(t)$, $|S'(t)| = S'_{(+)}(t) + S'_{(-)}(t)$ in all the points of existence of $S'(t)$. From here, from (2.10) and (2.11) it follows:

$$r_S^-(t) = \sqrt[4]{\frac{S(t_0)}{S(t)}} \exp\Big\{\frac{1}{4} \int\limits_{t_0}^{t} \frac{|S'(\tau)|}{S(\tau)} d\tau\Big\}, \qquad t \geq t_0, \tag{2.12}$$

$$r_S^+(t) = \sqrt[4]{\frac{S(t)}{S(t_0)}} \exp\Big\{\frac{1}{4} \int\limits_{t_0}^{t} \frac{|S'(\tau)|}{S(\tau)} d\tau\Big\}, \qquad t \geq t_0, \tag{2.13}$$

Let $p(t)$ and $q(t)$ be positive and continuous functions on the interval $[t_0; +\infty)$. Consider the system of equations

$$\begin{cases} u'(t) = \qquad p(t)v(t); \\ v'(t) = -q(t)u(t), \end{cases} \tag{2.14}$$

$t \ge t_0$. Let us introduce some notations, necessary in the sequel: $h(t) \equiv \frac{q(t)}{p(t)}$, $h_1(t) \equiv \frac{p(t)}{q(t)}$, $g(t) \equiv \min\{\sqrt[4]{h(t)}, \sqrt[4]{h_1(t)}\}$, $G(t) \equiv \max\{\sqrt[4]{h(t)}, \sqrt[4]{h_1(t)}\}$, $||(x(t), y(t))|| \equiv \equiv \sqrt{|x(t)|^2 + |y(t)|^2}$, $r_z(t) \equiv \exp\Big\{\frac{1}{4}\int\limits_{t_0}^{t} \frac{|z'(\tau)|}{z(\tau)} d\tau\Big\}$, $t \ge t_0$, where $x(t)$ and $y(t)$ are continuous functions on the interval $[t_0; +\infty)$, $z(t)$ is a absolutely continuous and positive function on the interval $[t_0; +\infty)$ with locally finite variation.

Lemma 2.2. *Let $h(t)$ be an absolutely continuous function with locally finite variation. Then for every solution $(u(t), v(t))$ of the system (2.14) the following inequalities hold:*

$$\frac{g(t_0)g(t)}{r_h(t)}||(u(t_0), v(t_0)|| \le ||(u(t), v(t))|| \le G(t_0)G(t)||(u(t_0), v(t_0))||r_h(t), \tag{2.15}$$

$t \ge t_0$.

Proof. Let us consider first the case, when $h(t)$ has the additional property: $h(t) \in C_{\sim}$. Let $(u_0(t), v_0(t))$ be a nontrivial real valued solution of the system (2.14), and let $\xi_0 = t_0 \le \xi_1 \le ... \le \xi_n \le ...$ be the possible extremums of the function $h(t)$. Let $t_1(> t_0)$ and $\varepsilon(> 0)$ be fixed. By virtue of Lemma 2.1 there exist piecewise constant functions $\widetilde{\widetilde{p}}(t)$ and $\widetilde{\widetilde{q}}(t)$ such, that the solution $(\widetilde{u}(t), \widetilde{v}(t))$ of the system

$$\begin{cases} u'(t) = \qquad \widetilde{\widetilde{p}}(t)v(t); \\ v'(t) = -\widetilde{\widetilde{q}}(t)u(t), \end{cases}$$

$t \ge t_0$, with $\widetilde{u}(t_0) = u_0(t_0)$, $\widetilde{v}(t_0) = v_0(t_0)$ satisfies the inequalities:

$$|\widetilde{u}(t_1) - u_0(t_1)| \le \varepsilon, \qquad |\widetilde{v}(t_1) - v_0(t_1)| \le \varepsilon. \tag{2.16}$$

Without loss of generality we will assume that $\widetilde{\widetilde{p}}(\xi_k) = p(\xi_k)$, $\widetilde{\widetilde{q}}(\xi_k) = q(\xi_k)$, $k = 0, 1, ...$, $\widetilde{\widetilde{p}}(t_1) = p(t_1)$, $\widetilde{\widetilde{q}}(t_1) = q(t_1)$; $\widetilde{\widetilde{p}}(t) > 0$, $\widetilde{\widetilde{q}}(t) > 0$, $t \ge t_0$; the function $\frac{\widetilde{\widetilde{q}}(t)}{\widetilde{\widetilde{p}}(t)}$ is nondecreasing on the intervals $[\xi_{2k}, \xi_{2k+1}]$ and nonincreasing on the intervals $[\xi_{2k+1}, \xi_{2k+2}]$, $k = 0, 1, 2, ...$ Then by (2.5) - (2.7), (2.9) the following inequalities hold

$$g_1(t_0)g_1(t_1)||(u_0(t_0), v_0(t_0))||\sqrt{\frac{h(\xi_0)}{h(t_1)}} \le ||(\widetilde{u}(t_1), \widetilde{v}(t_1)|| \le G_1(t_0)G_1(t_1)||(u_0(t_0), v_0(t_0))||,$$

$$\text{if}\ t_1 \in [\xi_0, \xi_1];$$

$$g_1(t_0)g_1(t_1)||(u_0(t_0), v_0(t_0))||\sqrt{\frac{h(\xi_0)}{h(t_1)}} \le ||(\widetilde{u}(t_1), \widetilde{v}(t_1))|| \le$$

$$\le G_1(t_0)G_1(t_1)||(u_0(t_0), v_0(t_0))||\sqrt{\frac{h(\xi_1)}{h(t_1)}}, \ \ if \ \ t_1 \in [\xi_1, \xi_2];$$

$$g_1(t_0)g_1(t_1)||(u_0(t_0), v_0(t_0))|| \left[\prod_{k=0}^{n}\sqrt{\frac{h(\xi_{2k})}{h(\xi_{2k+1})}}\right]\sqrt{\frac{h(\xi_{2n})}{h(t_1)}} \le ||(\widetilde{u}(t_1), \widetilde{v}(t_1))|| \le$$

$$\le G_1(t_0)G_1(t_1)||(u_0(t_0), v_0(t_0))|| \prod_{k=1}^{n}\sqrt{\frac{h(\xi_{2k-1})}{h(\xi_{2k})}}, \ \ if \ \ t_1 \in [\xi_{2n}, \xi_{2n+1}], \ \ n = 1, 2, ...;$$

$$g_1(t_0)g_1(t_1)||(u_0(t_0), v_0(t_0))|| \prod_{k=0}^{n}\sqrt{\frac{h(\xi_{2k})}{h(\xi_{2k+1})}} \le ||(\widetilde{u}(t_1), \widetilde{v}(t_1))|| \le$$

$$\le G_1(t_0)G_1(t_1)||(u_0(t_0), v_0(t_0))|| \left[\prod_{k=0}^{n-1},\sqrt{\frac{h(\xi_{2k+1})}{h(\xi_{2k+2})}}\right]\sqrt{\frac{h(\xi_{2n+1})}{h(t_1)}}, \ \ if \ \ t_1 \in [\xi_{2n+1}, \xi_{2n+2}],$$

$n = 1, 2, ...$, where $g_1(t) \equiv \min\{1, \sqrt{h_1(t)}\}$, $G_1(t) \equiv \max\{1, \sqrt{h_1(t)}\}$, $t \ge t_0$. It follows from here, that

$$\frac{g_1(t_0)g_1(t_1)||(u_0(t_0), v_0(t_0))||}{r_h^+(t_1)} \le ||(\widetilde{u}(t_1), \widetilde{v}(t_1)|| \le$$

$$\le G_1(t_0)G_1(t_1)||(u_0(t_0), v_0(t_0))||r_h^-(t_1). \tag{2.17}$$

By analogy (making the substitution $u(t) \to -u(t)$, interchanging $p(t)$ and $q(t)$, as well as interchanging $u(t)$ and $v(t)$) we come to the inequalities

$$\frac{g_2(t_0)g_2(t_1)||(u_0(t_0), v_0(t_0))||}{r_{h_1}^+(t_1)} \le ||(\widetilde{u}(t_1), \widetilde{v}(t_1))|| \le$$

$$\le G_2(t_0)G_2(t_1)||(u_0(t_0), v_0(t_0))||r_{h_1}^-(t_1),$$

where $g_2(t) \equiv \min\{1, \sqrt{h(t)}\}$, $G_2(t) \equiv \max\{1, \sqrt{h(t)}\}$, $t \ge t_0$. From here and from (2.17) we obtain:

$$\sqrt{\frac{g_1(t_0)g_1(t_1)g_2(t_0)g_2(t_1)}{r_h^+(t_1)r_{h_1}^+(t_1)}}||(u_0(t_0), v_0(t_0))|| \le ||(\widetilde{u}(t_1), \widetilde{v}(t_1)|| \le$$

$$\leq \sqrt{G_1(t_0)G_1(t_1)G_2(t_0)G_2(t_1)}||(u_0(t_0), v_0(t_0))||\sqrt{r_h^-(t_1)r_{h_1}^-(t_1)}. \qquad (2.18)$$

Obviously,

$$g_1(t)g_2(t) = g^2(t), \;\; G_1(t)G_2(t) = G^2(t), \qquad t \geq t_0. \qquad (2.19)$$

Note (due to (2.12) and (2.13)), that $r_{h_1}^{\pm}(t) = r_h^{\mp}(t), \;\; t \geq t_0$. Then $r_{h_1}^{\pm}(t_1)r_h^{\mp}(t_1) = r_h^2(t_1)$. From here, from (2.18) and (2.19) we obtain:

$$\frac{g(t_0)g(t_1)||(u_0(t_0), v_0(t_0))||}{r_h(t_1)} \leq ||(\widetilde{u}(t_1), \widetilde{v}(t_1)|| \leq$$

$$\leq G(t_0)G(t_1)||(u_0(t_0), v_0(t_0))||r_h(t_1).$$

From here and from (2.16) it follows:

$$\frac{g(t_0)g(t_1)||(u_0(t_0), v_0(t_0))||}{r_h(t_1)} - \sqrt{2}\varepsilon \leq ||(u_0(t_1), v_0(t_1)|| \leq$$

$$\leq G(t_0)G(t_1)||(u_0(t_0), v_0(t_0))||r_h(t_1) + \sqrt{2}\varepsilon.$$

By the arbitrariness of $\varepsilon(> 0)$ from here we will have:

$$\frac{g(t_0)g(t_1)||(u_0(t_0), v_0(t_0))||}{r_h(t_1)} \leq ||(u_0(t_1), v_0(t_1)|| \leq$$

$$\leq G(t_0)G(t_1)||(u_0(t_0), v_0(t_0))||r_h(t_1). \qquad (2.20)$$

Let $(u(t), v(t))$ be an arbitrary (complex) solution of the system (2.14). Since $(u(t), v(t)) =$

$= (u_1(t), v_1(t)) + i(u_2(t), v_2(t))$, where $(u_j(t), v_j(t)), \;\; (j = 1, 2)$ are some real solutions of the system (2.14), by (2.20) we will get:

$$\left[\frac{g(t_0)g(t_1)}{r_h(t_1)}\right]^2 \sum_{j=1}^{2} ||(u_j(t_0), v_j(t_0))||^2 \leq \sum_{j=1}^{2} ||(u_j(t_1), v_j(t_1))||^2 \leq$$

$$\leq [G(t_0)G(t_1)r_h(t_1)]^2 \sum_{j=1}^{2} ||(u_j(t_0), v_j(t_0))||^2.$$

Taking into account the equality $||(u(t), v(t))||^2 = \sum_{j=1}^{2} ||(u_j(t), v_j(t))||^2, \;\; t \geq t_0$, from here we will have:

$$\frac{g(t_0)g(t_1)||(u(t_0), v(t_0))||}{r_h(t_1)} \leq ||(u(t_1), v(t_1)|| \leq G(t_0)G(t_1)||(u(t_0), v(t_0))||r_h(t_1).$$

By virtue of arbitrariness of $t_1(> t_0)$ from here it follows (2.15). Thus, we have proved (2.15) under the additional assumption $h(t) \in C_\sim$. Let us prove it in the general case. Let $A_C[t_0;T]$ be the space of absolutely continuous functions $f(t)$ of finite variation on the interval $[t_0;T]$ with the norm $||f(t)|| \equiv \max\limits_{t\in[t_0;T]} |f(t)| + \int\limits_{t_0}^{T} |f'(\tau)|d\tau$. Obviously the set of rational functions is everywhere dense in $A_C[t_0,T]$. In view of this we choose polynomials $p_1(t)$ and $q_1(t)$ such, that $p_1(t) > 0$, $q_1(t) > 0$, $t \geq t_0$, and such, that for each fixed $t_1 \in (t_0,T]$ and $\varepsilon(> 0)$ the following inequalities hold

$$\left|\frac{g(t_0)g(t_1)}{r_h(t_1)} - \frac{\widetilde{g}(t_0)\widetilde{g}(t_1)}{r_{\widetilde{h}}(t_1)}\right| \leq \varepsilon, \quad \left|G(t_0)G(t_1)r_h(t_1) - \widetilde{G}(t_0)\widetilde{G}(t_1)r_{\widetilde{h}}(t_1)\right| \leq \varepsilon, \tag{2.21}$$

where $\widetilde{g}(t) \equiv \min\left\{\sqrt[4]{\widetilde{h}(t)}, \sqrt[4]{\widetilde{h}_1(t)}\right\}$, $\widetilde{G}(t) \equiv \max\left\{\sqrt[4]{\widetilde{h}(t)}, \sqrt[4]{\widetilde{h}_1(t)}\right\}$, $\widetilde{h}(t) \equiv \frac{q_1(t)}{p_1(t)}$, $\widetilde{h}_1(t) \equiv \frac{p_1(t)}{q_1(t)}$, $t \geq t_0$, as well as (by Lemma 2.1) the following inequalities hold

$$|u(t_1) - \widetilde{u}(t_1)| \leq \varepsilon, \qquad |v(t_1) - \widetilde{v}(t_1)| \leq \varepsilon, \tag{2.22}$$

where $(\widetilde{u}(t), \widetilde{v}(t))$ is the solution of the system

$$\begin{cases} u'(t) = \qquad\qquad p_1(t)v(t); \\ v'(t) = q_1(t)u(t), \end{cases}$$

$t \geq t_0$, with $\widetilde{u}(t_0) = u(t_0)$, $\widetilde{u}(t_0) = u(t_0)$. Since obviously $\widetilde{h}(t)|_{[t_0;+\infty)} \in C_\sim$, by already proven

$$\frac{\widetilde{g}(t_0)\widetilde{g}(t_1)||(u(t_0), v(t_0))||}{r_{\widetilde{h}}(t_1)} \leq ||(\widetilde{u}(t_1), \widetilde{v}(t_1)|| \leq \widetilde{G}(t_0)\widetilde{G}(t_1)||(u(t_0), v(t_0))||r_{\widetilde{h}}(t_1).$$

From here, from (2.21) and (2.22) it follows

$$\frac{g(t_0)g(t_1)||(u(t_0), v(t_0))||}{r_h(t_1)} - [\sqrt{2} + ||(u(t_0), v(t_0))||]\varepsilon \leq ||(u(t_1), v(t_1)|| \leq$$

$$\leq G(t_0)G(t_1)||(u(t_0), v(t_0))||r_h(t_1) + [\sqrt{2} + ||(u(t_0), v(t_0))||]\varepsilon.$$

By virtue of arbitrariness of $t_1 \in (t_0;T]$, $T(> t_0)$ and $\varepsilon(> 0)$ from here it follows (2.15). The lemma is proved.

Let us consider the Cauchy problem

$$\begin{cases} u'(t) = \qquad \widetilde{p}(t)v(t); \\ v'(t) = \widetilde{q}(t)u(t); \\ u(t_0) = u_{(0)}, \ \ v(t_0) = v_{(0)}, \end{cases} \tag{2.23}$$

$t \geq t_0$. Its solution we will seek in the form

$$\begin{cases} u(t) = A_j\sqrt{p_j}\exp\{\sqrt{l_j}t\} + B_j\sqrt{p_j}\exp\{-\sqrt{l_j}t\}; \\ v(t) = A_j\sqrt{q_j}\exp\{\sqrt{l_j}t\} - B_j\sqrt{q_j}\exp\{-\sqrt{l_j}t\}, \end{cases} \tag{2.24}$$

$t \in [t_j; t_{j+1}), \ \ j = 0, 1, 2, ...$ The unknowns A_0 and B_0 we can find by solving the system

$$\begin{cases} A_0\sqrt{p_0}\exp\{\sqrt{l_0}t)_0\} + B_0\sqrt{p_0}\exp\{-\sqrt{l_0}t_0\} = u_{(0)}; \\ A_0\sqrt{q_0}\exp\{\sqrt{l_0}t_0\} - B_0\sqrt{q_0}\exp\{-\sqrt{l_0}t_0\} = v_{(0)}, \end{cases} \tag{2.25}$$

and the remaining unknowns $A_j, B_j (j = 1, 2, ...)$ we can find by successive solving the systems

$$\begin{cases} A_j\sqrt{p_j}\exp\{\sqrt{l_j}t_j\} + B_j\sqrt{p_j}\exp\{-\sqrt{l_j}t_j\} = u(t_j); \\ A_j\sqrt{q_j}\exp\{\sqrt{l_j}t_j\} - B_j\sqrt{q_j}\exp\{-\sqrt{l_j}t_j\} = v(t_j), \end{cases}$$

$j = 1, 2, ...$ We have:

$$A_j = \frac{u(t_j)\sqrt{q_j} + v(t_j)\sqrt{p_j}}{2\sqrt{l_j}}\exp\{-\sqrt{l_j}t_j\}, \qquad B_j = \frac{u(t_j)\sqrt{q_j} - v(t_j)\sqrt{p_j}}{2\sqrt{l_j}}\exp\{-\sqrt{l_j}t_j\}.$$

From here, from (2.24) and (2.25) it follows

$$\begin{cases} u(t) = u(t_j)\,\mathrm{ch}\{\sqrt{l_j}(t - t_j)\} + v(t_j)\sqrt{\frac{p_j}{q_j}}\,\mathrm{sh}\{\sqrt{l_j}(t - t_j)\}; \\ v(t) = v(t_j)\,\mathrm{ch}\{\sqrt{l_j}(t - t_j)\} + u(t_j)\sqrt{\frac{q_j}{p_j}}\,\mathrm{sh}\{\sqrt{l_j}(t - t_j)\}, \end{cases} \tag{2.26}$$

$t \in [t_j; t_{j+1}), \ \ j = 0, 1, 2, ...$ From here it is easy to derive the equalities

$$\sqrt{q_j}u(t) + \sqrt{p_j}v(t) = [\sqrt{q_j}u(t_j) + \sqrt{p_j}v(t_j)]\exp\{\sqrt{l_j}(t - t_j)\}, \qquad t \in [t_j; t_{j+1}), \tag{2.27}$$

$j = 0, 1, \ldots$. From here it follows

$$\sqrt{q_{j+1}}u(t_{j+1})+\sqrt{p_{j+1}}v(t_{j+1}) = \sqrt{\frac{q_{j+1}}{q_j}}\left[\left(\sqrt{q_j}u(t_j) + \sqrt{\frac{h_j}{h_{j+1}}}\sqrt{p_j}v(t_j)\right) \times\right.$$

$$\left. \times \operatorname{ch}\{\sqrt{l_j}(t_{j+1} - t_j)\} + \left(\sqrt{\frac{h_j}{h_{j+1}}}\sqrt{q_j}u(t_j) + \sqrt{p_j}v(t_j)\right) \operatorname{sh}\{\sqrt{l_j}(t_{j+1} - t_j)\}\right], \quad (2.28)$$

$j = o, 1, 2, \ldots$. Let $u(t_0) \geq 0, \ \ v(t_0) \geq 0, \ \ u(t_0)^2 + v(t_0)^2 \neq 0$. Then from (2.26) it follows that

$$u(t) > 0, \qquad v(t) > 0, \qquad t > t_0. \quad (2.29)$$

From here, from (2.27) and (2.28) we get

$$\sqrt{\frac{q_{j+1}}{q_j}}\left[\sqrt{q_j}u(t_j) + \sqrt{p_j}v(t_j)\right] \exp\{\sqrt{l_j}(t_{j+1}-t_j)\} \leq \sqrt{q_{j+1}}u(t_{j+1})+\sqrt{p_{j+1}}v(t_{j+1}) \leq$$

$$\leq \sqrt{\frac{p_{j+1}}{p_j}}\left[\sqrt{q_j}u(t_j) + \sqrt{p_j}v(t_j)\right] \exp\{\sqrt{l_j}(t_{j+1} - t_j)\}, \quad (2.30)$$

for $h_j \geq h_{j+1}, \ \ j = 0, 1, \ldots$ and

$$\sqrt{\frac{p_{j+1}}{p_j}}\left[\sqrt{q_j}u(t_j) + \sqrt{p_j}v(t_j)\right] \exp\{\sqrt{l_j}(t_{j+1}-t_j)\} \leq \sqrt{q_{j+1}}u(t_{j+1})+\sqrt{p_{j+1}}v(t_{j+1}) \leq$$

$$\leq \sqrt{\frac{q_{j+1}}{q_j}}\left[\sqrt{q_j}u(t_j) + \sqrt{q_j}v(t_j)\right] \exp\{\sqrt{l_j}(t_{j+1} - t_j)\}, \quad (2.31)$$

for $h_j \leq h_{j+1}, \ \ j = 0, 1, \ldots$. Let us consider the system

$$\begin{cases} u'(t) = \qquad\qquad p(t)v(t); \\ v'(t) = q(t)u(t), \end{cases} \quad (2.32)$$

$t \geq t_0$. Denote: $e(t) \equiv \exp\left\{\int\limits_{t_0}^{t} \sqrt{p(\tau)q(\tau)}d\tau\right\}, \ \ t \geq t_0$.

Lemma 2.3. *Let $h(t)$ be absolutely continuous and has a local finite variation. Then for each solution $(u(t), v(t))$ of the system (2.32) with $u(t_0) \geq 0, \ \ v(t_0) \geq 0$ the following inequalities hold*

$$\frac{d(u,v)e(t)\sqrt{p(t)}}{\sqrt{p(t_0)}r_h^-(t)} \leq \sqrt{q(t)}u(t) + \sqrt{p(t)}v(t) \leq \frac{d(u,v)e(t)\sqrt{q(t)}}{\sqrt{q(t_0)}}r_h^-(t), \qquad t \geq t_0, \quad (2.33)$$

where $d(u,v) \equiv \sqrt{q(t_0)}u(t_0) + \sqrt{p(t_0)}v(t_0)$.

Proof. We prove the lemma only in the particular case when $h(t) \in C_\sim$. The proof in the general case by analogy of the last part of the proof of Lemma 2.2. Let $\varepsilon(>0)$ and $t_1(>t_0)$ be fixed, and let $\xi_0 = t_0 \le \xi_1 \le ... \le \xi_n \le ...$ be the possible extremums of the function $h(t)$. Let $(u(t), v(t))$ be a solution of the system (2.32) with $u(t_0) \ge 0,\ \ v(t_0) \ge 0$. By virtue of Lemma 2.1 there exist piecewise constant functions $\widetilde{\widetilde{p}}(t)$ and $\widetilde{\widetilde{q}}(t)$ such, that the solution $(\widetilde{u}(t), \widetilde{v}(t))$ of the system

$$\begin{cases} u'(t) = \qquad \widetilde{\widetilde{p}}(t)v(t); \\ v'(t) = \widetilde{\widetilde{q}}(t)u(t), \end{cases}$$

$t \ge t_0$, with $\widetilde{u}(t_0) = u_0(t_0),\ \ \widetilde{v}(t_0) = v_0(t_0)$ satisfies the inequalities

$$|u(t_1) - \widetilde{u}(t_1)| \le \frac{\varepsilon}{\sqrt{p(t_1)} + \sqrt{q(t_1)}},\quad |v(t_1) - \widetilde{v}(t_1)| \le \frac{\varepsilon}{\sqrt{p(t_1)} + \sqrt{q(t_1)}}. \tag{2.34}$$

Without loss of generality we will assume that $\widetilde{\widetilde{p}}(\xi_k) = p(\xi_k),\ \ \widetilde{\widetilde{q}}(\xi_k) = q(\xi_k),\ \ k = 0, 1, ...,\ \ \widetilde{\widetilde{p}}(t_1) = p(t_1),\ \ \widetilde{\widetilde{q}}(t_1) = q(t_1)$; the function $\frac{\widetilde{\widetilde{q}}(t)}{\widetilde{\widetilde{q}}(t)}$ is nondecreasing on the intervals $[\xi_{2k}; \xi_{2k+1}]$ and nonincreasing on the intervals $[\xi_{2k+1}; \xi_{2k+2}],\ \ k = 0, 1, ...$;

$$\left| \frac{d(u,v)\widetilde{e}(t_1)\sqrt{p(t_1)}}{\sqrt{p(t_0)}r_h^-(t_1)} - \frac{d(u,v)e(t_1)\sqrt{p(t_1)}}{\sqrt{p(t_0)}r_h^-(t_1)} \right| \le \varepsilon, \tag{2.35}$$

$$\left| \frac{d(u,v)\widetilde{e}(t_1)\sqrt{q(t_1)}r_h^-(t_1)}{\sqrt{q(t_0)}} - \frac{d(u,v)e(t_1)\sqrt{q(t_1)}r_h^-(t_1)}{\sqrt{q(t_0)}} \right| \le \varepsilon, \tag{2.36}$$

where $\widetilde{e}(t) \equiv \exp\{\int\limits_{t_0}^{t} \sqrt{\widetilde{\widetilde{p}}(\tau)\widetilde{\widetilde{q}}(\tau)}d\tau\},\ \ t \ge t_0$. Then by (2.30) and (2.31) we will have:

$$\frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{p(t_0)}}\sqrt{p(t_1)} \le \sqrt{q_1(t)}\widetilde{u}(t_1) + \sqrt{p_1(t)}\widetilde{v}(t_1) \le \frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{q(t_0)}}\sqrt{q(t_1)},\ \ if\ t_1 \in [\xi_0 l \xi_1];$$

$$\frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{p(t_0)}}\sqrt{\frac{h(t_1)}{h(\xi_1)}}\sqrt{p(t_1)} \le \sqrt{q_1(t)}\widetilde{u}(t_1) + \sqrt{p_1(t)}\widetilde{v}(t_1) \le \frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{q(t_0)}}\sqrt{\frac{h(\xi_1)}{h(t_1)}}\sqrt{q(t_1)},$$

if $t_1 \in [\xi_0; \xi_1]$;

$$\frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{p(t_0)}}\left(\prod_{k=0}^{n-1}\sqrt{\frac{h(\xi_{2k+1})}{h(\xi_{2k+2})}}\right)\sqrt{p(t_1)} \le \sqrt{q_1(t)}\widetilde{u}(t_1) + \sqrt{p_1(t)}\widetilde{v}(t_1) \le$$

$$\leq \frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{q(t_0)}}\left(\prod_{k=0}^{n-1}\sqrt{\frac{h(\xi_{2k+2})}{h(\xi_{2k+1})}}\right)\sqrt{q(t_1)}, \quad if \;\; t_1 \in [\xi_{2n};\xi_{2n+1}], \quad n=1,2,....;$$

$$\frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{p(t_0)}}\left(\prod_{k=0}^{n-1}\sqrt{\frac{h(\xi_{2k+1})}{h(\xi_{2k+2})}}\right)\sqrt{\frac{h(t_1)}{h(\xi_{2n+1})}}\sqrt{p(t_1)} \leq \sqrt{q_1(t)}\widetilde{u}(t_1)+\sqrt{p_1(t)}\widetilde{v}(t_1) \leq$$

$$\leq \frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{q(t_0)}}\left(\prod_{k=0}^{n-1}\sqrt{\frac{h(\xi_{2k+2})}{h(\xi_{2k+1})}}\right)\sqrt{\frac{h(\xi_{2n+1})}{h(t_1)}}\sqrt{q(t_1)}, \quad if \;\; t_1 \in [\xi_{2n+1};\xi_{2n+2}], \quad n=1,2,....;$$

Therefore

$$\frac{d(u,v)\widetilde{e}(t_1)\sqrt{p(t)}}{\sqrt{p(t_0)}r_h^-(t_1)} \leq \sqrt{q(t_1)}\widetilde{u}(t_1)+\sqrt{p(t_1)}\widetilde{v}(t_1) \leq \frac{d(u,v)\widetilde{e}(t_1)\sqrt{q(t_1)}}{q(t_0)}r_h^-(t_1).$$

From here and from (2.34) - (2.36) it follows that

$$\frac{d(u,v)e(t_1)\sqrt{p(t_1)}}{\sqrt{p(t_0)}r_h^-(t_1)} - 2\varepsilon \leq \sqrt{q(t_1)}u(t_1)+\sqrt{p(t_1)}v(t_1) \leq \frac{d(u,v)e(t_1)\sqrt{q(t_1)}}{q(t_0)}r_h^-(t_1)+2\varepsilon.$$

By virtue of arbitrariness of $\varepsilon(>0)$ and $t_1(>t_0)$ from here it follows (2.33). The lemma is proved.

Consider the Riccati equation

$$y'(t)+p(t)y^2(t)-q(t)=0, \qquad t \geq t_0. \tag{2.37}$$

The solutions $y(t)$ of this equation, existing on the interval $[t_1,t_2)(t_0 \leq t_1 < t_2 \leq +\infty)$ are connected with the solutions $(u(t),v(t))$ of the system (2.32) by equalities (see [11], pp. 153, 154):

$$u(t)=u(t_1)\exp\left\{\int_{t_2}^{t} p(\tau)y(\tau)d\tau\right\}, \qquad v(t)=y(t)u(t), \qquad t\in[t_1,t_2). \tag{2.38}$$

Let $y_0(t)$ be a solution of Eq. (2.37) with $y_0(t_0)>0$. It follows from Theorem 4.1 of work [12] (see [12],p. 26) that $y_0(t)$ exists on the interval $[t_0,+\infty)$ and

$$y_0(t)>0, \qquad t\geq t_0. \tag{2.39}$$

Since $p(t)>0, \;\; q(t)>0, \;\; t\geq t_0$, then from Theorem 3.1 of work [13] (see [13], p. 4) it follows that

$$y_0(s) > \frac{y_0(t)}{1+y_0(t)\int\limits_t^s p(\zeta)d\zeta}, \qquad s\geq t\geq t_0. \tag{2.40}$$

Consider the integral

$$\nu_{y_0}(t) \equiv \int\limits_t^{+\infty} p(\tau)\exp\Big\{-2\int\limits_t^{\tau} p(\xi)y_0(\xi)d\xi\Big\}d\tau, \qquad t \geq t_0.$$

From (2.39) and (2.40) it follows that

$$\nu_{y_0}(t) \leq \int\limits_t^{+\infty} p(\tau)\exp\Big\{-2\int\limits_t^{\tau} \frac{p(s)y_0(t)}{1+y_0(t)\int\limits_t^{s} p(\zeta)d\zeta}ds\Big\}d\tau = \frac{1}{y_0(t)}, \qquad t \geq t_0. \tag{2.41}$$

The function $y_*(t) \equiv y_0(t) - \frac{1}{\nu_{y_0}(t)}$, $t \geq t_0$, is an extremal solution of Eq. (2.37) (see [14], p. 194, Theorem 4). From (2.41) it follows that

$$y_*(t) \leq 0, \qquad t \geq t_0. \tag{2.42}$$

Let us show that

$$y_*(t) < 0, \qquad t \geq t_0. \tag{2.43}$$

Suppose that $y_*(t_1) = 0$ for some $t_1 \geq t_0$. Then by virtue of Theorem 4.1 of work [13] the following inequality holds $y_*(t) \geq 0$, $t \geq t_1$. From here and from (2.42) it follows that $y_*(t) \equiv 0$ on the interval $[t_1, +\infty)$, which is impossible. The obtained contradiction proves (2.43). Since $y_0(t_0) > 0$, then from (2.42) ((2.43)) and from Theorem 4 of work [14] it follows that $y_0(t)$ is a normal solution (a solution $y(t)$ of Eq. (2.37) is said to be normal if there exists a neighborhood of the point $y(t_0)$ such that every solution of Eq. (2.37) with initial value from this neighborhood exists on the interval $[t_0, +\infty)$). Then (see [14], p. 195)

$$\int\limits_{t_0}^{+\infty} p(\tau)[y_0(\tau) - y_*(\tau)]d\tau = +\infty. \tag{2.44}$$

Definition 2.1. *The solution $(u_*(t), v_*(t))$ of the system (2.32), satisfying the initial conditions $u_*(t_0) = 1$, $v_*(t_0) = y_*(t_0)$, will be called the canonical main solution of Eq. (2.32). The (a) solution $(u_0(t), v_0(t))$ of the system (2.32), satisfying the initial conditions $u_0(t_0) = v_0(t_0) = 1$ ($u_0(t_0) \geq 0$, $v_0(t_0) \geq 0$, $u_0^2(t_0) + v_0^2(t_0) \neq 0$), will be called the canonical nonprincipal (a real nonprincipal) solution of the system (2.32). The solutions $\lambda(u_*(t), v_*(t))$ and $\lambda(u_0(t), v_0(t))$, where λ is an arbitrary constant and $(u_0(t), v_0(t))$ is an real nonprincipal solution of the system (2.32), will be called a main and a principal* *solutions of the system (2.32) respectively. A solution of the system (2.32), which is not main solution will be called an ordinary solution of the system (2.32).*

From (2.43) it follows that the canonical main and nonprincipal solutions of the system (2.32) are linearly independent. Therefore for general solution $(u(t), v(t))$ of the system (2.32) the following representation holds

$$(u(t), v(t)) = \lambda_0(u_0(t), v_0(t)) + \lambda_*(u_*(t), v_*(t)), \quad \lambda_0 = const, \quad \lambda_* = const, \quad t \geq t_0. \quad (2.45)$$

On the strength of (2.38) we have

$$u_*(t) = \exp\Big\{\int_{t_0}^{t} p(\tau)y_*(\tau)d\tau\Big\}, \qquad v_*(t) = y_*(t)u_*(t), \qquad t \geq t_0; \qquad (2.46)$$

$$u_0(t) = \exp\Big\{\int_{t_0}^{t} p(\tau)y_0(\tau)d\tau\Big\}, \qquad t \geq t_0, \qquad (2.47)$$

where $y_0(t)$ is the solution of eq. (2.37) with $y_0(t_0) = 1$. From here and from (2.44) it follows that

$$\frac{u_*(t)}{u_0(t)} \to 0 \;\; for \;\; t \to +\infty. \qquad (2.48)$$

Let us show that

$$\frac{v_*(t)}{v_0(t)} \to 0 \;\; for \;\; t \to +\infty. \qquad (2.49)$$

In Eq. (2.37) we make the change $y(t) = \frac{1}{z(t)}$. We will come to the equation

$$z'(t) + q(t)z^2(t) - p(t) = 0, \qquad t \geq t_0. \qquad (2.50)$$

To prove (2.49) we need to the following

Lemma 2.4. *Suppose* $\int\limits_{t_0}^{+\infty} p(t)dt = +\infty$ *or* $\int\limits_{t_0}^{+\infty} q(t)dt = +\infty$. *Then* $z_*(t) \equiv \frac{1}{y_*(t)}, \;\; t \geq t_0$, *is the extremal solution of Eq. (2.50), where* $y_*(t)$ *is the extremal solution of Eq. (2.37).*

Proof. Above it was shown that $y_*(t) < 0, \;\; t \geq t_0$. Therefore $z_*(t)$ is defined correct. Obviously $z_*(t)$ is a solution to Eq. (2.50). Suppose $\int\limits_{t_0}^{+\infty} p(t)dt = +\infty$. Let $\widetilde{z}_*(t)$ be the extremal solution of Eq. (2.50). Suppose $z_*(t) \neq \widetilde{z}_*(t)$. Then the solution $z_N(t)$ of Eq. (2.50) with $z_N(t_0) = \frac{z_*(t_0)+\widetilde{z}_*(t_0)}{2}$ is a normal solution to Eq. (2.50). and $z_N(t) < 0, \;\; t \geq t_0$ (the graph of $z_N(t)$ is between the graphs of $z_*(t)$ and $\widetilde{z}_*(t)$). Therefore $y_N(t) \equiv \frac{1}{z_N(t)}, \;\; t \geq$

t_0 is a normal solution of Eq. (2.37). Hence $\nu_{y_N}(t_0) < +\infty$. On the other hand since $y_N(t) < 0,\ t \geq t_0$ we have $\nu_{y_N}(t_0) \geq \int\limits_{t_0}^{+\infty} p(t)dt = +\infty$. The obtained contradiction shows that $z_*(t)$ is extremal. Suppose now $\int\limits_{t_0}^{+\infty} q(t)dt = +\infty$. Let $z_*(t)$ not be extremal. the the integral $\int\limits_{t_0}^{+\infty} q(t)\exp\Big\{-\int\limits_{t_0}^{t} q(s)z_*(s)ds\Big\}dt$ is convergent. On the other hand since $z_*(t) < 0,\ t \geq t_0$, we have $\int\limits_{t_0}^{+\infty} q(t)\exp\Big\{-\int\limits_{t_0}^{t} q(s)z_*(s)ds\Big\}dt \geq \int\limits_{t_0}^{+\infty} q(t)dt = +\infty$. The obtained contradiction completes the proof of the lemma.

Obviously $z_0(t) \equiv \frac{1}{y_0(t)}$ is a normal solutions of Eq. (2.50). Then since $z_*(t)$ is extremal we have

$$\int\limits_{t_0}^{+\infty} q(\tau)[z_0(\tau) - z_*(\tau)]d\tau = +\infty. \tag{2.51}$$

Let $\widetilde{v}_0(t) \equiv \exp\Big\{\int\limits_{t_0}^{t} q(\tau)z_0(\tau)d\tau\Big\},\ \ \widetilde{u}_0(t) = z_0(t)\widetilde{v}_0(t),\ \ \widetilde{v}_*(t) \equiv \exp\Big\{\int\limits_{t_0}^{t} q(\tau)z_*(\tau)d\tau\Big\},$ $\widetilde{u}_*(t) = z_*(t)\widetilde{v}_*(t),\ \ t \geq t_0$. By virtue of (2.38) $(\widetilde{u}_0(t), \widetilde{v}_0(t))$ and $(\widetilde{u}_*(t), \widetilde{v}_*(t))$ are solutions of the system (2.32). From (2.51) it follows that

$$\frac{\widetilde{v}_*(t)}{\widetilde{v}_0(t)} \to 0 \ \ for \ \ t \to +\infty. \tag{2.52}$$

Since $\widetilde{u}_0(t_0) = \widetilde{v}_0(t_0) = u_0(t_0) = v_0(t_0) = 1,\ \widetilde{u}_*(t_0) = \frac{1}{y_*(t_0)},\ \widetilde{v}_*(t_0) = 1$, we have $(\widetilde{u}_0(t), \widetilde{v}_0(t)) = (u_0(t), v_0(t)),\ \ (\widetilde{u}_*(t), \widetilde{v}_*(t)) = \frac{1}{y_*(t_0)}(u_*(t), v_*(t)),\ \ t \geq t_0$. From here and from (2.52) it follows (2.49). From (2.44), (2.48) and (2.49) it follows

$$(u(t), v(t)) = \lambda_0(u_0(t), v_0(t))[1 + o(1)], \qquad t \to +\infty. \tag{2.53}$$

By (2.32) from (2.43) and (2.46) we will have

$$0 < u_*(t) \leq u_*(t_0), \qquad v_*(t_0) \leq v_*(t) < 0, \qquad t \geq t_0. \tag{2.54}$$

III. ESTIMATES OF THE SOLUTIONS OF THE SYSTEM (1.1)

In the system (1.1) we make the substitutions

$$\phi(t) = \exp\Big\{\int\limits_{t_0}^{t} a_{11}(\tau)d\tau\Big\}u(t), \qquad \psi(t) = \exp\Big\{\int\limits_{t_0}^{t} a_{22}(\tau)d\tau\Big\}v(t), \tag{3.1}$$

We will come to the system (1.2). In the sequel we will assume that the function $\frac{a_{12}(t)}{a_{21}(t)}$ is absolutely continuous and has a locally finite variation. Denote:

$$m(t) \equiv \min\Big\{\sqrt[4]{\Big|\frac{a_{12}(t)}{a_{21}(t)}\Big|}, \sqrt[4]{\Big|\frac{a_{21}(t)}{a_{12}(t)}\Big|}\Big\}, \qquad M(t) \equiv \max\Big\{\sqrt[4]{\Big|\frac{a_{12}(t)}{a_{21}(t)}\Big|}, \sqrt[4]{\Big|\frac{a_{21}(t)}{a_{12}(t)}\Big|}\Big\},$$

$$\mathcal{F}(t) \equiv \Big|\int\limits_{t_0}^{t}[Re\, a_{11}(\tau) - Re\, a_{22}(\tau)]d\tau\Big| + \int\limits_{t_0}^{t}\Big|\frac{1}{4}\Big(\frac{a_{12}(\tau)}{a_{21}(\tau)}\Big)'\frac{a_{21}(\tau)}{a_{12}(\tau)} + \frac{a_{22}(\tau) - a_{11}(\tau)}{2}\Big|d\tau, \ \ t \ge t_0.$$

Theorem 3.1. *Let the condition A) be satisfied. Then for each solution $(\phi(t), \psi(t))$ of the system (1.1) the following inequalities hold*

$$m(t_0)||(\phi(t_0), \psi(t_0))||m(t)\exp\Big\{\int\limits_{t_0}^{t}\Big[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)\Big]d\tau - \mathcal{F}(t)\Big\} \le ||(\phi(t), \psi(t))|| \le$$

$$\le M(t_0)||(\phi(t_0), \psi(t_0))||M(t)\exp\Big\{\int\limits_{t_0}^{t}\Big[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)\Big]d\tau + \mathcal{F}(t)\Big\}, \qquad t \ge t_0. \tag{3.2}$$

Proof. Let $(\phi(t), \psi(t))$ be a solution of the system (1.1), and let $(u(t), v(t))$ be the solution of the system (1.2), satisfying the initial conditions $u(t_0) = \phi(t_0), \ \ v(t_0) = \Psi(t_0)$. Then by virtue of (3.1) we have

$$|\phi(t)| = \exp\Big\{\int\limits_{t_0}^{t} Re\, a_{11}(\tau)d\tau\Big\}|u(t)|, \qquad |\Psi(t)| = \exp\Big\{\int\limits_{t_0}^{t} Re\, a_{22}(\tau)d\tau\Big\}|v(t)|, \qquad t \ge t_0.$$

From here it follows

$$||(\phi(t), \psi(t))|| = \sqrt{\exp\Big\{2\int\limits_{t_0}^{t} Re\, a_{11}(\tau)d\tau\Big\}|u(t)|^2 + \exp\Big\{2\int\limits_{t_0}^{t} Re\, a_{22}(\tau)d\tau\Big\}|v(t)|^2}, \qquad t \ge t_0.$$

Therefore

$$\exp\left\{\min\left\{\int_{t_0}^{t} Re\, a_{11}(\tau)d\tau, \int_{t_0}^{t} Re\, a_{22}(\tau)d\tau\right\}\right\}||(u(t),v(t))|| \le ||(\phi(t),\psi(t))|| \le, \qquad t \ge t_0.$$

$$\le \exp\left\{\max\left\{\int_{t_0}^{t} Re\, a_{11}(\tau)d\tau, \int_{t_0}^{t} Re\, a_{22}(\tau)d\tau\right\}\right\}||(u(t),v(t))||, \qquad t \ge t_0. \qquad (3.3)$$

Denote $H(t) \equiv \left|\frac{Q(t)}{P(t)}\right|, H_1(t) \equiv \left|\frac{P(t)}{Q(t)}\right|, w(t) \equiv \min\{\sqrt[4]{H(t)}, \sqrt[4]{H_1(t)}\}, \;\; W(t) \equiv$ $\equiv \max\{\sqrt[4]{H(t)}, \sqrt[4]{H_1(t)}\}, \;\; t \ge t_0$. By virtue of Lemma 2.2 from the condition of the theorem it follows

$$w(t_0)||(\phi(t_0),\psi(t_0))||\frac{w(t)}{r_H(t)} \le ||(u(t),v(t))|| \le W(t_0)W(t)||(\phi(t_0),\psi(t_0))||r_H(t), \qquad t \ge t_0.$$

From here and from (3.3) we will get

$$w(t_0)||(\phi(t_0),\psi(t_0))||\frac{w(t)}{r_H(t)}\exp\left\{\min\left\{\int_{t_0}^{t} Re\, a_{11}(\tau)d\tau, \int_{t_0}^{t} Re\, a_{22}(\tau)d\tau\right\}\right\} \le ||(\phi(t),\psi(t))|| \le$$

$$\le W(t_0)||(\phi(t_0),\psi(t_0))||W(t)\exp\left\{\max\left\{\int_{t_0}^{t} Re\, a_{11}(\tau)d\tau, \int_{t_0}^{t} Re\, a_{22}(\tau)d\tau\right\}\right\}r_H(t), \qquad t \ge t_0.$$

Since $w(t_0) = m(t_0), \;\; W(t_0) = M(t_0)$,

$$m(t)\exp\left\{\min\left\{\int_{t_0}^{t}\frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau, \int_{t_0}^{t}\frac{Re\, a_{22}(\tau) - Re\, a_{11}(\tau)}{2}d\tau\right\}\right\} \le w(t),$$

$$W(t) \le M(t)\exp\left\{\max\left\{\int_{t_0}^{t}\frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau, \int_{t_0}^{t}\frac{Re\, a_{22}(\tau) - Re\, a_{11}(\tau)}{2}d\tau\right\}\right\}, \qquad t \ge t_0,$$

taking into account the equalities

$$\min\left\{\int_{t_0}^{t} Re\, a_{11}(\tau)d\tau, \int_{t_0}^{t} Re\, a_{22}(\tau)d\tau\right\} =$$

$$= \int_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)\right]d\tau - \left|\int_{t_0}^{t}\frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau\right|,$$

$$\max\left\{\int_{t_0}^{t} Re\, a_{11}(\tau)d\tau, \int_{t_0}^{t} Re\, a_{22}(\tau)d\tau\right\} =$$

$$= \int_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)\right]d\tau + \left|\int_{t_0}^{t}\frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau\right|,$$

$$\min\left\{\int_{t_0}^{t}\frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau, \int_{t_0}^{t}\frac{Re\, a_{22}(\tau) - Re\, a_{11}(\tau)}{2}d\tau\right\} =$$

$$= -\left|\int_{t_0}^{t}\frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau\right|,$$

$$\max\left\{\int_{t_0}^{t}\frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau, \int_{t_0}^{t}\frac{Re\, a_{22}(\tau) - Re\, a_{11}(\tau)}{2}d\tau\right\} =$$

$$= \left|\int_{t_0}^{t}\frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau\right|,$$

$t \geq t_0$. from (3.4) we will get (3.2). The theorem is proved.

Remark 3.1. *Let $a(t)$ and $b(t)$ be some continuous functions on $[t_0; +\infty)$ and let $b(t) > 0,\ \ t \geq t_0$. Consider the system*

$$\begin{cases} \phi'(t) = a(t)\phi(t) + b(t)\psi(t); \\ \psi'(t) = -b(t)\phi(t) + a(t)\psi(t), \quad t \geq t_0. \end{cases}$$

For this system we have $\mathcal{F}(t) \equiv 0,\ \ m(t) = M(t) \equiv 1$. Therefore by Theorem 3.1 for its general solution $(\phi(t), \psi(t))$ the inequalities

$$||(\phi(t_0), \psi(t_0))|| \exp\left\{\int_{t_0}^{t} Re\, a(\tau)d\tau\right\} \leq ||(\phi(t), \psi(t))|| \leq ||(\phi(t_0), \psi(t_0))|| \exp\left\{\int_{t_0}^{t} Re\, a(\tau)d\tau\right\},$$

$t \geq t_0$, are fulfilled. Hence

$$||(\phi(t), \psi(t))|| = ||(\phi(t_0), \psi(t_0))|| \exp\left\{\int_{t_0}^{t} Re\, a(\tau)d\tau\right\}, \qquad t \geq t_0,$$

and in this sense the estimates (3.2) are sharp.

Example 3.1. Consider the system

$$\begin{cases} \phi'(t) = (-\lambda + \sin t)\phi(t) + t^{\alpha}\psi(t); \\ \psi'(t) = -t^{\beta}\phi(t) + (-\mu + \cos t)\psi(t), \end{cases} \tag{3.5}$$

$t \geq \frac{\pi}{4}$, where $\lambda,\ \mu,\ \alpha$ and β are some real numbers. For this system $m(t) = t^{-\frac{|\alpha-\beta|}{2}}$, $M(t) = t^{\frac{|\alpha-\beta|}{2}}$, $\mathcal{F}(t) = |\sqrt{2} + \lambda - \mu + (\lambda - \mu)t| + \int\limits_{\pi/4}^{t} \left| \frac{\alpha-\beta}{4\tau} + \frac{\lambda-\mu}{2} + \frac{\sqrt{2}}{2}\cos(\tau + \frac{\pi}{4})\right| d\tau$, $t \geq \frac{\pi}{4}$. Using Theorem 3.1 it is easy to find the following regions of values of the parameters $\lambda,\ \mu,\ \alpha,\ \beta$ for which the system (3.5) is asymptotically stable:

$O_1 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda + \mu > 3|\lambda - \mu| + \sqrt{2},\ \ \lambda > 0,\ \ \mu > 0\}$;

$O_2 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda = \mu > \frac{\sqrt{2}}{2\pi}\}$;

$O_3 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda + \mu > 3|\lambda - \mu| > 3\sqrt{2},\ \ \lambda > 0,\ \ \mu > 0\}$;

$O_3 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda + \mu > 3|\lambda - \mu| = 3\sqrt{2},\ \ \lambda > 0,\ \ \mu > 0,\ \ \alpha = \beta\}$; and the following regions of values of parameters $\lambda,\ \mu,\ \alpha,\ \beta$ for which system (3.5) is unstable:

$O_5 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda + \mu + 3|\lambda - \mu| + \sqrt{2} < 0,\ \ \lambda < 0,\ \ \mu < 0\}$;

$O_6 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda = \mu < -\frac{\sqrt{2}}{2\pi}\}$;

$O_7 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda + \mu + 3|\lambda - \mu| < 0,\ \ |\lambda - \mu| > \sqrt{2},\ \ \lambda < 0,\ \ \mu < 0\}$;

$O_7 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda + \mu + 3|\lambda - \mu| < 0,\ \ |\lambda - \mu| = \sqrt{2},\ \ \lambda < 0,\ \ \mu < 0,\ \ \alpha = \beta\}$;

It is not difficult to verify that the application of the estimates of Liapunov (see [4], p. 432), Yu. S. Bogdanov (see [4], p. 433) and estimate by freezing method (see [4], p 441) to the system (3.5) give no result. The estimates by logarithmic norms γ_I and γ_{II} of S. M. Lozinski (see [4], pp. 435, 436) give result only for $\lambda > 0,\ \ \mu > 0,\ \ \alpha < -1,\ \ \beta < -1$. For comparison now we use the theorem of Wazevski to the system (3.5) (see [4], p. 434). By virtue of this theorem for each solution $(\phi(t), \psi(t))$ of the system (3.5) the following inequalities hold

$$||(\phi(t),\psi(t))|| \exp\left\{\int\limits_{\pi/4}^{t} \omega_{-}(\tau)d\tau\right\} \leq ||(\phi(t),\psi(t))|| \leq$$

$$\leq ||(\phi(t),\psi(t))|| \exp\left\{\int\limits_{\pi/4}^{t} \omega_{+}(\tau)d\tau\right\}, \qquad t \geq \frac{\pi}{4}, \tag{3.6}$$

where $\omega_{\pm} \equiv \frac{-\lambda-\mu+\sin t+\cos t\pm\sqrt{(\lambda-\mu+cost-\sin t)^2+(t^{\alpha}-t^{\beta})^2}}{2}$. If $\alpha \neq \beta > 0$ or $\beta \neq \alpha > 0$, then from (3.6) does not follow neither asymptotic stability nor instability of the system (3.5) for every values of λ and μ.

Definition 3.1. *A solution $(\phi(t), \psi(t))$ of the system (1.1), satisfying the condition B), is said to be a main (a nonprincipal, an ordinary) solution of the system (1.1), if $\phi(t_0) = u(t_0)$, $\psi(t_0) = v(t_0)$, where $(u(t), v(t))$ is a main (a nonprincipal, an ordinary) solution of the system (1.2).*

Theorem 3.2. *Let the condition B) be satisfied and let*

$$C)\ \int\limits_{t_0}^{+\infty} a_{12}(t)\exp\Big\{\int\limits_{t_0}^{t}\Big[a_{22}(s)-a_{11}(s)\Big]ds\Big\}dt = +\infty \ \ or$$

$$\int\limits_{t_0}^{+\infty} a_{21}(t)\exp\Big\{\int\limits_{t_0}^{t}\Big[a_{11}(s)-a_{22}(s)\Big]ds\Big\}dt = +\infty.$$

Then if:
i) $(\phi(t), \psi(t))$ is a nonprincipal solution of the system (1.1), then

$$D(\phi,\psi)m(t)\exp\left\{\int\limits_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau) + \sqrt{a_{12}(\tau)a_{21}(\tau)}\right]d\tau - \mathcal{F}(t)\right\} \leq |\phi(t)| + |\psi(t)| \leq$$

$$\leq D(\phi,\psi)M(t)\exp\left\{\int\limits_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau) + \sqrt{a_{12}(\tau)a_{21}(\tau)}\right]d\tau + \mathcal{F}(t)\right\}, \quad \psi t \geq t_0, \quad (3.7)$$

where $D(\phi,\psi) \equiv \sqrt[4]{\left|\frac{a_{21}(t_0)}{a_{12}(t_0)}\right|}|\phi(t_0)| + \sqrt[4]{\left|\frac{a_{12}(t_0)}{a_{21}(t_0)}\right|}|\psi(t_0)|;$

ii) $(\phi(t), \psi(t))$ is a main solution of the system (1.1), then

$|\phi(t)|+|\ \ (t)| \leq$

$$\leq (|\phi(t_0)| + |\psi(t_0)|)\exp\left\{\int\limits_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)\right]d\tau + \left|\int\limits_{t_0}^{t}\frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau\right|\right\}, \quad (3.8)$$

$t \geq t_0$;
iii) $(\phi(t), \psi(t))$ is an ordinary solution of the system (1.1), then

$$c_1 m(t)\exp\left\{\int\limits_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)+\sqrt{a_{12}(\tau)a_{21}(\tau)}\right]d\tau-\mathcal{F}(t)\right\} \leq |\phi(t)|+|\psi(t)| \leq$$

$$\leq c_2 M(t)\exp\left\{\int\limits_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)+\sqrt{a_{12}(\tau)a_{21}(\tau)}\right]d\tau+\mathcal{F}(t)\right\},\qquad t\geq t_0, \qquad (3.9)$$

where $c_j = const > 0, \ \ j = 1, 2.$

Proof. Let $(\phi(t), \psi(t))$ be a solution of the system (1.1), and $(u(t), v(t))$ be the solution of the system (1.2) with $u(t_0) = \phi(t_0), \ \ v(t_0) = \psi(t_0)$. Then by (3.1) we have

$$|\phi(t)|+|\psi(t)|=\exp\left\{\int\limits_{t_0}^{t} Re\, a_{11}(\tau)d\tau\right\}|u(t)|+\exp\left\{\int\limits_{t_0}^{t} Re\, a_{22}(\tau)d\tau\right\}|v(t)|,\qquad t\geq t_0.$$

From here it follows

$$\exp\left\{\min\left\{\int\limits_{t_0}^{t} Re\, a_{11}(\tau)d\tau,\int\limits_{t_0}^{t} Re\, a_{22}(\tau)d\tau\right\}\right\}(|u(t)|+|v(t)|)\leq|\phi(t)|+|\psi(t)|\leq$$

$$\leq\exp\left\{\max\left\{\int\limits_{t_0}^{t} Re\, a_{11}(\tau)d\tau,\int\limits_{t_0}^{t} Re\, a_{22}(\tau)d\tau\right\}\right\}(|u(t)|+|v(t)|),\qquad t\geq t_0,$$

or, which is the same,

$$\exp\left\{\int\limits_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)\right]d\tau-\left|\int\limits_{t_0}^{t}\frac{Re\, a_{11}(\tau)-Re\, a_{22}(\tau)}{2}d\tau\right|\right\}(|u(t)|+|v(t)|)\leq$$

$$\leq|\phi(t)|+|\psi(t)|\leq$$

$$\leq\exp\left\{\int\limits_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)\right]d\tau+\left|\int\limits_{t_0}^{t}\frac{Re\, a_{11}(\tau)-Re\, a_{22}(\tau)}{2}d\tau\right|\right\}(|u(t)|+|v(t)|),\qquad (3.10)$$

$t \geq t_0$. Let $(u_0(t), v_0(t))$ be a real nonprincipal solution of the system (1.2). Then by virtue of Lemma 2.3 and (2.29) we have

$$\frac{\widetilde{D}E(t)\sqrt{P(t)}}{\sqrt{P(t_0)}r_H^{-}(t)}\leq\sqrt{Q(t)}u_0(t)+\sqrt{P(t)}v_0(t)\leq\frac{\widetilde{D}E(t)\sqrt{Q(t)}}{\sqrt{Q(t_0)}}r_H^{-}(t),\qquad t\geq t_0,\qquad (3.11)$$

where $\widetilde{D} \equiv \sqrt{Q(t_0)}u_0(t) + \sqrt{P(t_0)}v_0(t) = \sqrt{a_{21}(t_0)}\phi(t_0) + \sqrt{a_{12}(t_0)}\psi(t_0), \quad E(t) \equiv$

$$\equiv \exp\left\{\int\limits_{t_0}^{t} \sqrt{P(\tau)Q(\tau)}d\tau\right\} = \exp\left\{\int\limits_{t_0}^{t} \sqrt{a_{12}(\tau)a_{21}(\tau)}d\tau\right\}, \qquad t \geq t_0.$$ Obviously

$$\min\{\sqrt{P(t)}, \sqrt{Q(t)}\}[u_0(t)+v_0(t)] \leq \sqrt{Q(t)}u_0(t)+\sqrt{P(t)}v_0(t) \leq$$
$$\leq \max\{\sqrt{P(t)}, \sqrt{Q(t)}\}[u_0(t) + v_0(t)], \qquad t \geq t_0.$$

Therefore,

$$\min\left\{\frac{1}{\sqrt{P(t)}}, \frac{1}{\sqrt{Q(t)}}\right\}\left[\sqrt{Q(t)}u_0(t)+\sqrt{P(t)}v_0(t)\right] \leq u_0(t)+v_0(t) \leq$$
$$\leq \max\left\{\frac{1}{\sqrt{P(t)}}, \frac{1}{\sqrt{Q(t)}}\right\}\left[\sqrt{Q(t)}u_0(t) + \sqrt{P(t)}v_0(t)\right], \qquad t \geq t_0.$$

From here and from (3.11) we will get

$$\widetilde{D}\min\left\{\sqrt[4]{\frac{P(t)}{Q(t)}}, \sqrt[4]{\frac{Q(t)}{P(t)}}\right\}\frac{E(t)}{\sqrt[4]{P(t_0)Q(t_0)}r_H(t)} \leq u_0(t)+v_0(t) \leq$$
$$\leq \widetilde{D}\max\left\{\sqrt[4]{\frac{P(t)}{Q(t)}}, \sqrt[4]{\frac{Q(t)}{P(t)}}\right\}\frac{E(t)}{\sqrt[4]{P(t_0)Q(t_0)}}r_H(t), \qquad t \geq t_0.$$

Therefore,

$$\frac{\widetilde{D}m(t)}{\sqrt[4]{a_{12}(t_0)a_{21}(t_0)}}\exp\left\{-\left|\int\limits_{t_0}^{t} \frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau\right|\right\}\frac{E(t)}{r_H(t)} \leq u_0(t)+v_0(t) \leq$$
$$\leq \frac{\widetilde{D}M(t)}{\sqrt[4]{a_{12}(t_0)a_{21}(t_0)}}\exp\left\{\left|\int\limits_{t_0}^{t} \frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau\right|\right\}E(t)r_H(t), \qquad t \geq t_0.$$

Taking into account the equalities $(u(t), v(t)) = \lambda_0(u_0(t), v_0(t)), \quad \lambda_0 = const \neq 0, \quad |\phi(t_0)| =$

$= |\lambda_0|u_0(t_0), \quad |\psi(t_0)| = |\lambda_0|v_0(t_0)$ from here we will get

$$D(\phi,\psi)\exp\left\{-\left|\int\limits_{t_0}^{t} \frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau\right|\right\}\frac{m(t)E(t)}{r_H(t)} \leq |u(t)| + |v(t)$$

$$\leq D(\phi,\psi)\exp\left\{\left|\int_{t_0}^{t}\frac{Re\,a_{11}(\tau)-Re\,a_{22}(\tau)}{2}d\tau\right|\right\}M(t)E(t)r_H(t),\qquad t\geq t_0. \quad (3.12)$$

From here and from (3.10) it follows (3.7). The assertion i) is proved. Let us prove ii). Let $(\phi(t),\psi(t))$ be a main solution of the system (1.1). Then by (3.1) we have

$$\phi(t)=\phi(t_0)\exp\left\{\int_{t_0}^{t}a_{11}(\tau)d\tau\right\}u_0(t),\qquad \psi(t)=\boldsymbol{\psi}(t_0)\exp\left\{\int_{t_0}^{t}a_{22}(\tau)d\tau\right\}v_0(t), \quad (3.13)$$

$t\geq t_0$, where $(u_0(t),v_0(t))$ is the canonical main solution of the system (1.2). By (2.54) from C) it follows $|\phi_0(t_0)|u_*(t)+|\phi_0(t_0)||v_*(t)|\leq|\phi_0(t_0)|u_*(t_0)+|\phi_0(t_0)||v_*(t_0)|,\ \ t\geq t_0$. By (3.10) from here and from (3.13) it follows (3.8). The assertion ii) is proved. Let us prove iii). Let $(\phi(t),\psi(t))$ be an ordinary solution of the system (1.1). By (3.1) we have

$$\phi(t)=\exp\left\{\int_{t_0}^{t}a_{11}(\tau)d\tau\right\}\Big[\lambda_0u_0(t)+\lambda_*u_*(t)\Big],\qquad t\geq t_0, \quad (3.14)$$

$$(t)=\exp\left\{\int_{t_0}^{t}a_{11}(\tau)d\tau\right\}\Big[\lambda_0v_0(t)+\lambda_*v_*(t)\Big],\qquad t\geq t_0, \quad (3.15)$$

where $(u_*(t),v_*(t))$ and $(u_0(t),v_0(t))$ are the canonical main and canonical nonprincipal solutions of the system (1.2) respectively, and $\lambda_0\neq 0$. Then by (2.29) and (2.52) we can deduce from C) that $\widetilde{c}_1[u_0(t)+v_0(t)]\leq|\lambda_0u_0(t)+\lambda_*u_0(t)|+|\lambda_0v_0(t)+\lambda_*u_*(t)|\leq$ $\leq\widetilde{c}_2[u_0(t)+v_0(t)],\ \ t\geq t_0,\ \ \widetilde{c}_j=const,\ \ j=1,2$. By virtue of (3.10) from here, from (3.14) and (3.15) we obtain

$$\widetilde{c}_1\exp\left\{\int_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2}Re\,a_{jj}(\tau)\right]d\tau-\left|\int_{t_0}^{t}\frac{Re\,a_{11}(\tau)-Re\,a_{22}(\tau)}{2}d\tau\right|\right\}[u_0(t)+v_0(t)]\leq$$

$$\leq|\phi(t)|+|\boldsymbol{\psi}(t)|\leq$$

$$\leq\widetilde{c}_2\exp\left\{\int_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2}Re\,a_{jj}(\tau)\right]d\tau+\left|\int_{t_0}^{t}\frac{Re\,a_{11}(\tau)-Re\,a_{22}(\tau)}{2}d\tau\right|\right\}[u_0(t)+v_0(t)],$$

$\boldsymbol{\psi}t\geq t_0$. By (3.12) from here it follows (3.9). The assertion iii), and therefore, the theorem are proved.

Remark 3.2. *Let $a(t)$ and $b(t)$ be the same as in Remark 3.1. Consider the system*

$$\begin{cases} \phi'(t) = a(t)\phi(t) + b(t)\psi(t); \\ \psi'(t) = b(t)\phi(t) + a(t)\psi(t), \quad t \geq t_0. \end{cases}$$

For this system we have $\mathcal{F}(t) \equiv 0$, $m(t) = M(t) \equiv 1$. Therefore by Theorem 3.2 for its each nonprincipal solution $(\phi(t), \psi(t))$ the inequalities

$$|\phi(t)| + |\psi(t)| = (|\phi(t_0)| + |\psi(t_0)|) \exp\left\{\int_{t_0}^{t} \Big[Re\, a(\tau) + b(\tau)\Big] d\tau\right\} \leq |\phi(t)| + |\psi(t)| \leq$$

$$\leq (|\phi(t_0)| + |\quad(t_0)|) \exp\left\{\int_{t_0}^{t} \Big[Re\, a(\tau) + b(\tau)\Big] d\tau\right\}, \qquad t \geq t_0,$$

are fulfilled. Hence

$$|\phi(t)| + |\psi(t)| = (|\phi(t_0)| + |\psi(t_0)|) \exp\left\{\int_{t_0}^{t} \Big[Re\, a(\tau) + b(\tau)\Big] d\tau\right\}, \qquad t \geq t_0,$$

and in this sense the estimates (3.7) are sharp.

Example 3.2. Let us consider the system

$$\begin{cases} \phi'(t) = (-\lambda + \sin t)\phi(t) + t^{\alpha}\psi(t); \\ \psi'(t) = t^{\beta}\phi(t) + (-\mu + \cos t)\psi(t), \end{cases} \tag{3.16}$$

$t \geq \frac{\pi}{4}$, where λ, μ, α and β are some real constants. For this system the functions $m(t)$, $M(t)$ and $\mathcal{F}(t)$ are the same, which are in the example 3.1. Applying Theorem 3.2 to (3.16) it is easy to find the following regions of parameters λ, μ, α, β for which Eq. (3.16) is asymptotically stable:

$O_1^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| + \sqrt{2},\ \ \lambda > 0,\ \ \mu > 0,\ \ \lambda \neq \mu,\ \ \alpha + \beta < 0\}$;
$O_2^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| + 2 + \sqrt{2},\ \ \lambda > 0,\ \ \mu > 0,\ \ \lambda \neq \mu,\ \ \alpha + \beta = 0\}$;
$O_3^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| > 3\sqrt{2},\ \ \lambda > 0,\ \ \mu > 0,\ \ \lambda \neq \mu,\ \ \alpha + \beta < 0\}$;
$O_4^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| + 2 > 3\sqrt{2} + 2,\ \ \lambda > 0,\ \ \mu > 0,\ \ \lambda \neq \mu,\ \ \alpha + \beta = 0\}$;
$O_5^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| = 3\sqrt{2},\ \ \lambda > 0,\ \ \mu > 0,\ \ \lambda \neq \mu,\ \ \alpha = \beta < 0\}$;
$O_6^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| + 1 = 3\sqrt{2} + 2,\ \ \lambda > 0,\ \ \mu > 0,\ \ \lambda \neq \mu,\ \ \alpha = \beta = 0\}$;

and the following regions of parameters λ, μ, α, β for which eq. (3.16) is instable:

$O_7^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda \neq \mu, \ \ \alpha + \beta > 0\};$
$O_8^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu + 3|\lambda - \mu| + \sqrt{2} < 2, \ \ \lambda < 0, \ \ \mu < 0, \ \ \alpha + \beta = 0\};$
$O_9^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda = \mu < -\frac{\sqrt{2}}{2\pi}\};$
$O_{10}^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu + 3|\lambda - \mu| < 2, \ \ |\lambda - \mu| \geq \sqrt{2}, \ \ \alpha + \beta = 0\}.$

As in the case of the system (3.5) the application of the estimates of Liapunov, Yu. S. Bogdanov and estimate by freezing method to the system (3.16) give no result and the estimates by logarithmic norms γ_I and γ_{II} of S. M. Lozinski give result only for $\lambda > 0, \ \ \mu > 0, \ \ \alpha < -1, \ \ \beta < -1$. For the case $\alpha > 0$ or $\beta > 0$ it is impossible by use of the theorem of Wazevski to verify neither asymptotic stability nor instability of system (3,16).

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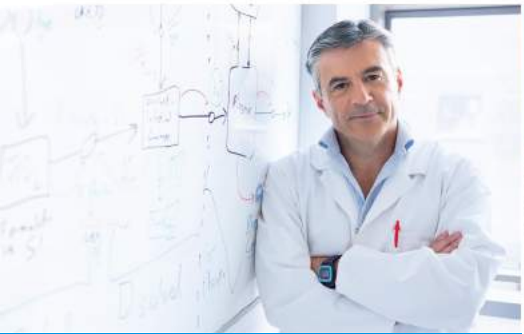

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