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LONDON JOURNAL OF RESEARCH IN SCIENCE: NATURAL AND FORMAL

Volume 23 | Issue 4 | Compilation 1.0

Print ISSN: 2631-8490
Online ISSN: 2631-8504
DOI: 10.17472/LJRS

COMPILED IN UNITED KINGDOM

ENGLISH

Great Britain Journals Press

London Journal of Research in Science: Natural and Formal

Volume 23 | Issue 4 | Compilation 1.0

PUBLISHER

Great Britain Journals Press
1210th, Waterside Dr,Opposite Arlington Building, Theale, Reading
Phone:+444 0118 965 4033 Pin: RG7-4TY United Kingdom

SUBSCRIPTION

Frequency: Quarterly

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$500USD for 2 year
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Guy Armand Onambélé, Alexandre Biaou, Dominique Dedegbe, Gildas Nangbe & Evariste Gounou

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Food availability is influenced by the means of production. A decrease in production affects market prices. In Benin, food production is changing. This paper seeks to analyze the influence of the professionalization of work, the use of new technologies, and individualism on agricultural production. Two data sources are used: the National Agricultural Census core module and the Harmonized Framework based on the World Food Programme food security survey, the results of the 2021-2022 agricultural season, the caloric proxy, shocks, historical variations in agricultural production, agricultural commodity prices, access to drinking water and improved toilets, and the seasonal calendar. Contributing factors related to food availability and access influence food consumption and livelihoods. On farms, 81.6% of farms are part-time, with 2/12 months for non-professionals, and 7-8/12 months for professionals. Respectively 2.0%, 6.9% and 11.9% keep accounting, have access to credit, and are affiliated in agricultural producers' organizations.

Keywords: professionalization, farm work, farming, tools, productivity.

Classification: FOR CODE: 070199

Language: English

London Journals Press

LJP Copyright ID: 925631
Print ISSN: 2631-8490
Online ISSN: 2631-8504

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ABSTRACT

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Keywords: professionalization, farm work, farming, tools, productivity.

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I. INTRODUCTION

Agricultural policies primarily affect farmers' income (Jacques R., 1987). To implement them and hope for a positive outcome, it is necessary to identify who is a farmer. A clear answer to this question remains a challenge in many African countries. Although the majority of African managers have emerged from the agricultural and peasant environment where they spent their early childhood, they do not spontaneously accept to practice as professionals in agriculture. Even when they have a degree in agronomy, becoming an agricultural professional is not always a priority option. Is this a deliberate choice or a situational constraint? The explanations must be sought in their daily lives. The shortcomings of the agricultural sector to date are rooted mainly in the lure of profit.

Population growth translates into a growing need for food and agricultural products, amplified by changes in diets and the types of products consumed, induced by economic development and its rapid urbanization (Pauline Marty, 2015). African countries are doing enough to establish this causality. Governments have become aware of the backlog in rural and agricultural development. For several decades, oil rents, advances in communications and tourism made sourcing from world markets easier than increasing local production and improving transport and distribution channels between hinterlands and capitals (Chantal Le Mouël, 2015). Multinational companies are supporting massive imports to southern countries; pushing some agricultural sectors are substituted by cash crops without taking into account the career profile of the farmer. Benin is not on the fringe of this dynamic.

In Benin, agricultural production is in deep mutation. It is increasingly documented and receives special attention from public authorities. This contribution aims to analyze the influence of the professionalization of work, the use of new technologies, and individualism on agricultural production. The direct effects on food security, in general, are not omitted. The characteristics of

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professional agrarian work, and its configuration in Benin should be investigated. Identifying the factors that influence it will also lead to the perverse effects of a lack of professionalism in the agricultural sector. Two data sources are used: the National Agricultural Census and the Harmonized Framework based on the WFP food security survey.

II. DATA AND METHODS

An orthodox analysis leads to a search for the determinants of the professionalization of agricultural work among different factors. These factors are

a. The profile of the agricultural family (sex of the farmer, age, level of education, marital status, parity, number of members in the family, type of land owned by the household (gift/inheritance/rental/lease/metayage/other);
b. Bookkeeping;
c. Access to credit;
d. Membership in an Agricultural Producers' Organization;
e. Average farm size with the possibility of plot rotation;
f. The type of farm (crop/animal/fish production);
g. Control of the value chain (production, processing, marketing);
h. Work tools and means of operation.

national census of agriculture in Benin was made public in the first quarter of 2022. It exhaustively collected all the variables of interest to analyze the Agro-Sylvio-Pastoral sectors in the 77 Communes of the country. The variables collected made it possible to draw up a profile of agrarian households and to identify the sectors of activity, the means of operation, the types of activities, the level of professionalization, the use of mechanization, the areas sown, etc.

As the analysis of the basic module of the national agricultural census is still in progress, the option has been taken to present in this paper the effects associated with the low professionalization of agricultural work in Benin. It will be done through the Harmonized Framework (HF), which maps vulnerability to food insecurity. The analysis of the Harmonized Framework of March 2022 was conducted with the contribution of government technicians, and civil society actors such as NGOs. The March 2022 HF session analyzed all 77 communes in Benin. The analysis consisted of an inventory of available evidence. It consisted of outcome indicators from the WFP's Global Analysis of Vulnerability, Food Security and Nutrition (GIVSAN), contributing factors related to hazard and vulnerability, and the four (4) dimensions of food security. The food security indicators are food consumption score, household dietary diversity score, source of food consumed, coping strategies, livelihoods and income sources, household expenditure structure and access to credit, shocks and vulnerability. Subsequently, the evidence was analyzed and reliability scores were assigned to the various pieces of evidence, communes and populations were classified in their current and projected situations, food-insecure populations were estimated and maps and results were produced.

III. RESULTS

Benin has 926,539 agricultural households, according to the first national agricultural census. 80.6% of non-professional workers in the agro-silvicultural sector work part-time, i.e., two months out of twelve. Professionals work annually for seven months out of twelve. The characteristics that limit the professionalization of agricultural work in Benin have an impact on household food security.

3.1 Limits to the professionalization of agricultural work

The limits to the professionalization of agricultural work have their sources in the structural, social, cultural, environmental, technological and economic spheres.

3.2 The structural sphere

Control of the means of production is the primary factor in professionalization. 95.6% of agricultural households are involved in crop production, and 65.4%, 5.4%, 0.4% and 6.2%, respectively are engaged in animal production, fishing, aquaculture or forestry. Access to inputs and small agricultural equipment is a constraint for small producers. The government of Benin, through its new guidelines, wants to take up this challenge by encouraging civil society to give itself the means to access them. The rate of mechanization of soil work is 12.4%. Of the total exploitable land, 43.9% are sown, with an average size of 3.3 ha per farming household. Of the total exploitable land, 6.2% is fallowed, and 1.76% is irrigated (MAEP, RNA, 2019). The prices of agricultural products on the market are not very remunerative when the farm is small, is not structured, and is because of the current economic situation.

3.3 Social Aspects

The average age of the heads of agricultural households is 43.5 years (MAEP, RNA, 2019). At this age, the home society must have proof of contribution from its non-disabled members. Usually those who have not completed primary or secondary school return to work the land. The school orientation towards agricultural fields is not widespread in Benin. The opening of a national university of agriculture in Kétou will undoubtedly change this situation.

Benin has been subdivided into seven (7) agricultural development poles (PDA). The size and scope of the farms vary according to the poles. PDAs 7, 4 and 5 have more farms (PDA 7 Ouémé, Atlantique, Mono, Littoral : 222,078; PDA 4 Borgou Sud, Donga, Collines : 221,201; PDA 5 Zou, Couffo: 177,639) than the other four (4) (MAEP, RNA, 2019). These are essentially areas that offer more outlets for agricultural products. They are close to areas where cross-border trade is very dynamic. Crossed from North to South, and from East to West by international roads, these areas benefit from significant investments in trade and communication infrastructure.

3.4 The Cultural

15.7% of agricultural households are headed by women (MAEP, RNA, 2019). This statistic calls into question the cross-cutting themes of "Gender", "Cross-cutting protection", "Positive discrimination" and "Inclusion". One of the characteristics of farms in Benin is that they are the result of individual initiatives at the family level, with a heritage/land capital transferred from generation to generation. There is little room for the association of energies in the form of collective entrepreneurship or consortium.

3.5 The Environment

The agricultural calendar, with its rainy seasons, is associated with professionalization. The agroecological ecosystem and soil mapping, which are poorly documented, also have an impact. There are four seasons in the South and two seasons in the North. There is a long and short rainy season and a long and short dry season in the south. In the northern zone, there is only one rainy season and one dry season. The lack of water control is a result of climatic variations and rainfall breaks.

3.6 Technology

The use of information and communication technologies in Benin is increasing and covers very little of the agro-sylvo-pastoral sphere. A platform bringing together research centers in this area has been set up under the leadership of the National Institute of Agricultural Research of Benin (INRAB), which acts as its secretariat. One of the goals of this platform is to disseminate innovations in the agricultural field to producers. Despite these efforts, only 2% of farms in Benin keep accounts. The processing of agricultural products is the work of 25.0% of farm households (MAEP, RNA, 2019). Few farms use a full-time skilled labor force. Instead, they use seasonal workers and sharecroppers. This personnel has no formalized contractual relationship with their employers. This leaves room for violations of several rights. Graduates of agricultural schools often prefer salaried jobs far from the farms. In addition, they often need additional training in technology, project management, and the use of agricultural machinery.

3.7 The Economy

For various reasons, 22.8% of farm households manage to market their products. In terms of the rate of banking in the West African Monetary Union zone, Benin leads with 31.2%, followed by Togo (27.0%), Burkina Faso (20.6%), Côte d'Ivoire (20.4%) and Senegal (19.6%) (Wadagni Romuald, Economie, 2020). Despite the proliferation of microfinance institutions, 6.9% of farmers have access to credit. 11.9% of agricultural households are affiliated with an OPA (MAEP, RNA 2019). Although they are plural, the limits of agrarian professionalization are not insurmountable. However, until effective solutions are found, different levels of life in society suffer.

3.8 Effects of non-professionalization

The analogies of the non-professionalization of agricultural work affect the basic social unit, the micro-economy, and the macroeconomy.

3.9 In the Basic Social Unit

37.7% of Beninese have a relatively acceptable index of accessibility and quality of services. They are more concentrated in Cotonou (UNDP, 2022). The lack of professionalization of agricultural work limits the access of agricultural households to a substantial remuneration and therefore to basic sanitary/social infrastructures: drinking water, improved toilets, health care, education for children, decent housing, appropriate means of communication, etc. According to the latest Benin Sustainable Human Development Report (UNDP, 2022), access to basic sanitation and hygiene was 32.2% for ECOWAS in 2020, while in Benin, the same indicator was 17% (WFP/INSTAD, AGVSAN, 2017). The main reason is low purchasing power. 47% of Benin's population experiences extreme poverty and material deprivation (UNDP, 2022). This translates into poor access to food. As a result, 9.6% of households have a moderate to severe food security index. Of these households, 15.2% spend more than 65% of their income on food and 27.8% use crisis or emergency coping strategies (WFP/INStAD, AGVSAN, 2017). Access to technology and innovations is a luxury for them. 2/10 Beninese have a broadband internet subscription (UNDP, 2022). This places Benin among the last four (4) countries in the sub-region in this area.

3.10 At the Microeconomic Level

The lack of professionalization is declining production and productivity in the agro-sylvo-pastoral sectors. Low production naturally leads to a limited supply of markets and their dysfunction. Limited income leads households to renew their means of existence without, however, guaranteeing significant purchasing power.

3.11 At the Macroeconomic level

The low level of professionalization of agricultural work affects GDP and tends to increase imports while limiting formal exports. Observation of agricultural statistics over the last three decades shows historical variations in agricultural production.

3.12 For the meta-analysis

According to the harmonized framework exercise, the mapping of food insecurity makes it possible to classify the analysis zones into phases: minimal, pressure, crisis, emergency, famine. In the current situation (March to May 2022), eleven communes (Aplahoué, Klouékanmè, Lalo, Toviklin, Dogbo, Djakotomey, Athiémè, Toffo, Allada, Bassila, and Glazoué) are in phase 3 "Crisis". These communes have been exposed to shocks (drought, floods) that have weakened their resilience. The populations in the Crisis to Worst phase at the national level are estimated at 1,225,957 people (or 9.49%). 41 communes are in phase 2 with 2,754,478 people in borderline food security. 25 communes are classified as minimal phase with 8,934,565 people in food security. In the projected situation (June to August 2022), one (01) Commune is in the Crisis phase (Bassila), 32 Communes are in Phase 2 (Under Pressure) and 44 are in the Minimum phase. At the national level, the food-insecure population is estimated at 830,150 people. The number of borderline food-insecure people is 2,318,231. The estimated food-insecure population is 9,766,619 people (MAEP, CH, March 2022).

Overall, the contributing factors of availability (overall increase in production compared to the average of the past five years) and access (increase in the price level of staples compared to the average of the past five years) have affected the majority of municipality food consumption and livelihoods, therefore household food security (APRM, CH, March 2022).

IV. DISCUSSION & CONCLUSION

This contribution provides evidence of the profitability of agricultural work linked to professionalism in this sector in Benin. The temporal limits observed in the sector can be explained either structurally or cyclically, individually or collectively, or by the family sphere or the physical environment. The consequences of the low professionalization of agricultural work are visible in the daily life of the basic social unit, the family. They are reflected in the dynamics of the local economy and in the economic aggregates. The meta-analysis carried out through the Harmonized Framework provides an up-to-date and factual mapping of the severity of food insecurity due in part to the limited availability of food, essentially linked to the professionalism of the work that should produce it.

This study is trying to combining empirical analysis with meta analysis. It does not pretend to present a cause-and-effect relationship between the professionalization of agriculture and food security. Still, it attempts to identify the explanatory elements of the level of profitability of the agro-sylvo-pastoral domains. This is part of the roots of current food security situation in Benin. The present contribution provides updated statistics that open up avenues of research in fields as varied as rural sociology, agronomy, technology, human resource management, rural development, etc. The results presented are comparable to previous work.

In Madagascar, the average age of farmers is higher than in Benin (48 years versus 43.5 years) (Razafimahatratra Mamy Hanitriniaina et al., 2017). In France, this age is even higher. Over half of farmers are ranged 50 years old or older (Olivier Chardon et al., 2020). In 2019, three-quarters of French farmer-operators were men, a proportion that has been increasing over the past forty years. In this country, on average, farmers work longer hours per week than all employed people, and they usually work in the wek-end :Saturdays or Sundays. Four times fewer farmers work than forty years ago. While in Benin, the working time in the agricultural sector is decreasing. On an annual basis, professionals work only 7/12 months.

If in Benin individualism seems to be more important, in Madagascar, farms associate several members of the siblings. This guarantees labor and ensures a good redistribution of the farm's income as well as self-support (Razafimahatratra Mamy Hanitriniaina et al., 2017).

The encouragement to invest in agriculture is underway in Benin to the benefit of economic liberalization. Indeed, it is an asset for boosting the agricultural sector, but it should not be misunderstood because, at times, it fuels more non-agricultural growth, exerts land pressure, aggravated locally by unequal access to land, water and value added. Such a structural transformation calls for adapted institutional and technical innovations (Bruno Dorin and Claire Aubron, 2016). In India, for example, we are at 0.65 ha/agricultural asset, whereas in Benin, we are already at 3.3 ha per farm. This also seems high in contrast to the average size of farms in Madagascar which is 215.6 ares (Razafimahatratra Mamy Hanitriniaina et al., 2017).

The low professionalization of agricultural work is not only the case in Benin. Dumont Antoinette has, in her PhD thesis, shown that the contracts and statuses of agricultural workers are very variable in Wallonia Belgium: 37% seasonal, 20% CDI, 12% undeclared, 10% local aid employment contracts, 9% CDD, 5% students, 7% independent. (Dumont, Antoinette, 2017). Such a quantification of the agricultural labor market in Benin is not yet a reality. It would be utopian to address the issue of contracts in the agricultural sector. However, it is the document that formalizes the links between the employer and the employee. In the absence of such evidence, it is difficult to distinguish career prospects, nor to guarantee full enjoyment of the rights of agricultural workers.

In essence, the agricultural sector requires several types of profiles depending on the requirements of the positions: mechanical positions with repetitive acts, coordination positions, thinking positions that require initiative and creativity, surveillance and security positions, positions that require gentleness and sensitivity. In Benin, it is common to find the same person performing several of these functions on farms. This person is sometimes paid in kind. The problem of continuous training in the agricultural sector is thus raised.

Given the average age of Beninese farmers (43.5 years), there are opportunities. If a farmer has ambition, he must cultivate himself to produce differently. Tommy Collin-Vallée et al. have posed fundamental questions for the professionalization of the farm worker: how to produce differently? how to learn to do so? what to learn? who should learn? under what conditions? What consequences can be drawn from new learning? (Tommy Collin-Vallée, Maryvonne Merri, 2020). It is by identifying the appropriate answers to these questions that the professionalization of agricultural work in Benin can become an undeniable reality.

Conflict of Interest Statement

The authors state that there is no conflict of interest.

ACKNOWLEDGEMENT

We wish to acknowledge all the contributors such as field staff, health and administrative authorities without whom this survey could have not been carried out. And thanks to all reviewers.

Consent for Publication

The authors grant his consent for publication of this paper.

Funding

The writing of this paper has not been funded or sponsored. It was done at the author's expense.

Data Availability Statement

The data used in this paper is fully available and can be accessed upon request.

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Facts Accounting for an Impediment to Financial Sustenance in Favour of Marine Artisanal Fishery-Related Activities at Sassandra, Southwestern Côte d'Ivoire

*Laurent BAHOU**

ABSTRACT

The study was carried out in February, March and August 2021, using a questionnaire intended for the professionals working for the benefit of the marine artisanal fishery located at Sassandra. The interviewees (220 persons) were approached at the landing site and principal marketplace of Sassandra. After a brief inquiry about the different types of activities in which they engaged and the number of people within their respective corporations, each interviewee was to give personal view on health conditions and safety at work, financial support displacement for his/her activity, as well as on childhood education. The results indicated that people who engaged in fishery-related activities comprised experienced men and women having knowledge and skill acquired over many years. They served their apprenticeship with members of their family. They would also work with apprentices of the same ethnic group or with fellow countrymen and countrywomen. A large part of the fishery´s workforce rests in the family circle.

Keywords: ethnic patterns, financial support displacement, fishery´s workforce, health conditions, landing site, members of family, safety-work conditions, sassandra.

Classification: FOR CODE: 040305

Language: English

London Journals Press

LJP Copyright ID: 925632
Print ISSN: 2631-8490
Online ISSN: 2631-8504

London Journal of Research in Science: Natural and Formal

Volume 23 | Issue 4 | Compilation 1.0

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Keywords: ethnic patterns, financial support displacement, fishery´s workforce, health conditions, landing site, members of family, safety-work conditions, sassandra.

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I. INTRODUCTION

As component parts of activities of vital importance to food security and employment generation, fishing and post-harvesting tasks plainly contribute to people´s welfare and the economy of the African countries. Yet, they do not always benefit from financial support, since they would lack consistent financial displacement in any form. In Côte d'Ivoire, fishing and aquaculture are believed to generate up to 70000 direct jobs, while about 400000 people and more would make a livelihood, essentially with artisanal fishing and processing of fishery products **(Document COMHAFAT, 2014).**

Unfortunately, in the coastal areas of Côte d'Ivoire, particularly those in the southwestern part such as Sassandra and San-Pédro, marine artisanal fishing has long been considered as unpopular profession by the natives. It can currently be regarded as an area of activity over which migrant fishers of Ghanaian-origin have influence. Today, a progressive change in people´s thinking and behaviour is quite noticeable about the previous commonly-held opinion on fishing at sea. For the substantial revenue derived from fishing and trading for fish aroused the natives´ consciousness in such a way that they would resort to fishing as a promising way out for their increasing needs for animal proteins of fishery source. **Delaunay (1991)** recalled a certain number of historical events (in the eastern, central and western parts of Côte d'Ivoire) that contributed to the setting of influence of Ghanaian migrant fishers on the marine artisanal fishing domain, comparing such a situation to a « colonization of fishery-kind » (**Delaunay, 1989**). Additional facts are presented by **Bahou et al. (2022)**, enhancing readers´ comprehension of the particular set of circumstances that existed at a period of time starting from 1893, especially when European countries established a former colony in Côte d'Ivoire. That situation led to the Ghanaian migrant fisher flocks becoming dominant in the field of marine artisanal fishing.

Sassandra, southwestern Côte d'Ivoire, is known for its virtually thriving fishery-related facilities region-wide. However, the lack of financial support, which can be viewed as an impediment to fishery-related activities´ takeoff, created persistent setbacks that need be dealt with. The overall objective of the current study was to show that problems facing the fisher flock and the professionals working for the benefit of the marine artisanal fishing sector of Sassandra are real. A specific goal was to enumerate the facts, which generally account for reasons for financial Institutions´ cautiousness.

II. MATERIALS AND METHODS

Study design was not quite different from the previous ones by **Bahou (2022)** and **Bahou et al (2022)**. The study was carried out, arranging for individual interviews, using a questionnaire to which the professionals of the marine artisanal fishing sector of Sassandra submitted themselves. In particular, people were to answer the following questions: 1. How long have you been carrying out this type of activity? 2. Do you work on your own? 3. With whom did you serve your apprenticeship? 4. How many apprentices do you have? 5. What is your nationality? People were asked extra questions about problems facing them in their efforts to carry out their fishery-related activities. Those series of questions, which were wishes-like, were recorded in a Table as shown below:

Type of wishes intended for the professionals of the marine artisanal fishing sector of Sassandra	Would rather benefit from.... / Get constructed nearby work places....
Wish Number 1...	An Hospital / A free health centre
Wish Number 2...	A primary school constructed for your kids´ benefit
Wish Number 3...	Benefit from Insurance services
Wish Number 4...	Benefit from Bank services / Get financial support
Wish Number 5...	Get life jackets for fishers´ safety at sea

Respondents´ answers were taken in note form on duly-designed sheets of paper. In total, 220 people submitted themselves to the questionnaire. Yet, 206 respondents did participate in the interview throughout, answering all questions and giving helpful additional details to enhance the interviewers´ comprehension. These people talked about their fishery-related activities and year of experience, indicating their age and what they needed most to further their activities. The estimated age at which the professionals commenced their jobs was determined making a subtraction between their present age and year of experience. Age classes of 5 intervals were determined. The data collected were registered in an Excel file to facilitate processing the data and making calculations, while figures and tables were used as illustrations.

III. RESULTS

3.1 Work Environment

Table 1 indicates that 60% of smoke-curing agents, 30.49% of retailers and 27.03% of wholesalers served their apprenticeship with their mothers. Likewise, 59.52% of the marine artisanal fishers did learn fishing working at their fathers´ sides, while 19.05% of them served their apprenticeship with their uncles. However, some other professionals got involved in fishery-related activities on their own initiative, without the guidance of anyone for their early beginnings. That was the case of male and female wholesalers as well as retailers, of whom 51.35% and 52.43% among the interviewees respectively confirmed the fact (Table 1). Overall, fathers and mothers undoubtedly played key roles in teaching and showing guidance to 13.11% and 30.10% of early beginners respectively, when these ones started their activities, based on the four types of activities listed in Table 1. Some other 37.38% of the professionals, however, admitted that they just started working on their own to reach the current stage of their careers.

Table 1: Professionals´ opinions about their early beginning and the persons with whom they served their apprenticeship in the marine artisanal fishing sector of Sassandra, southwestern Côte d'Ivoire

Professionals	Admitted that they served their apprenticeship with	Responses obtained from the interviewees	Percentages relating to the corporations	Percentages obtained, considering a total of 206 respondents
Smoke-curing agents (N = 45)	Aunt	1	2.22	–
	« Fanti » people	1	2.22	–
	Sister	3	6.67	–
	O I	13	28.89	–
	Mother	27	60.00	–
Retailers (N = 82)	Aunt	2	2.44	–
	« Fanti » people	2	2.44	1.46
	Friend	2	2.44	0.97
	Sister	8	9.76	–
	Mother	25	30.49	–
	O I	43	52.43	–
Male and Female Wholesalers (N = 37)	"Father's marriage-mate"	1	2.70	0.49
	Cousin	1	2.70	0.48
	Sister	1	2.70	5.83
	Father	2	5.41	–
	Aunt	3	8.11	2.91
	Mother	10	27.03	30.10
	O I	19	51.35	–
Artisanal fishers (N = 42)	Brother-in-law	1	2.38	0.48
	O I	2	4.76	37.38
	Friend	2	4.75	0.97
	Brother	4	0.52	1.94
	Uncle	8	19.05	3.88
	Father	25	59.52	13.11

Note: O I = the interviewees admitted that they started their activities (or their jobs) by their own initiative.

Figure 1 shows in each corporation the approximate age the professionals reached when they engaged in their fishery-related activities. For instance, smoke-curing agents ranged in age from 7 to 48 years, while retailers were 5 to 54 years old, as beginners. In addition, some wholesalers engaged in their activities when they turned ten, while others completed their 58 years old. Likewise, fishers got involved in fishing activity at early age (sometimes when they were 7 years old) to learn how to make their first appearance in a job that is premised on the knowledge that older fishers pass on to younger ones, so that they may likely become fully competent before they reach 56 years old. Overall, age range varied according to type of activity, but it was not necessarily linked up to the age of the people. For instance, 13 smoke-curing agents admitted that they started that job when they were 20 to 25 years old, in stark contrast with what other 2 smoke-curing agents said; for they made their first appearance in that job when they were 7 and 10 years old, respectively. A large number of retailers (16+16) admitted that they began their activities when in the age brackets of 15-20 and 20-25, respectively. Only one person of that corporation said that he started that job when he was 49. A female wholesaler revealed that she started the business of buying the fish in large quantities and selling them especially to retailers for resale, when she was 10; whereas another wholesaler told that she began showing interest in that business when she turned her 58. Four persons revealed that they began learning fishing at an early age (when 7 to 10 years old), while more than half of fishers (9+8+9) started fishing when in the age bracket of 20-35 (Figure 1).

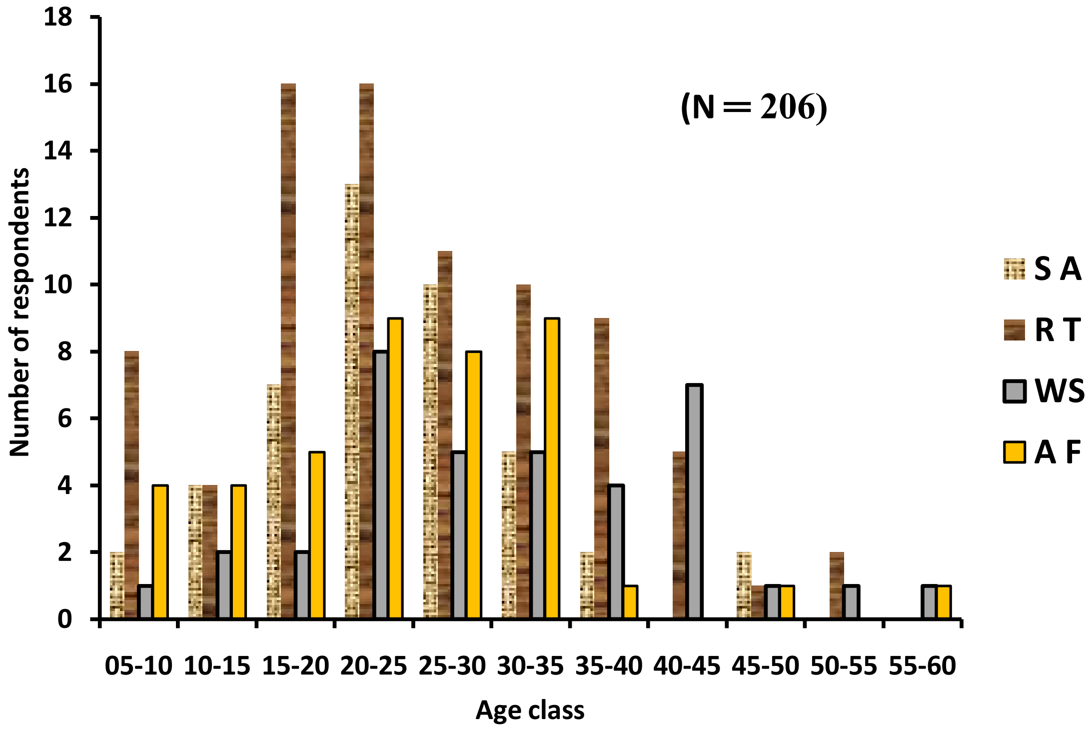

Figure 1: Histograms showing the approximate age in each category of professionals when they started their fishery-related activities in the marine artisanal fishing sector of Sassandra (southwestern Côte d'Ivoire). Note: **SA** = smoke-curing agents; **RT** = retailers; **WS** = wholesalers; **AF** = artisanal fishers.

Figure 2 shows years of experience of the people working for the benefit of the marine artisanal fishing sector of Sassandra, as revealed by the interviewees. Overall, year of experience varied from 1 to 50. Within each corporation, people with greater years of experience were fewer than those who had less experience. For example, smoke-curing agents ranged in experience from 2 to 45 years, with many people (11 + 10 + 9) who had 5 to 20 years of experience. A large number of retailers (26 + 30) had 2 to 10 years of experience while years of experience within that corporation ranged from 1 to 35. A large majority of wholesalers (7 + 10 + 8) had between 4 and 15 years of experience, though years of experience within that corporation varied from 2 to 45. Younger fishers (and consequently the less experienced) were more numerous than older ones who were more experienced. Based on the four fishery-related activities, fishers represented the only corporation that had the highest year of experience (50 years), whereas wholesalers and smoke-curing agents, in addition to retailers, had the lowest years of experience (1 to 2 years, Figure 2).

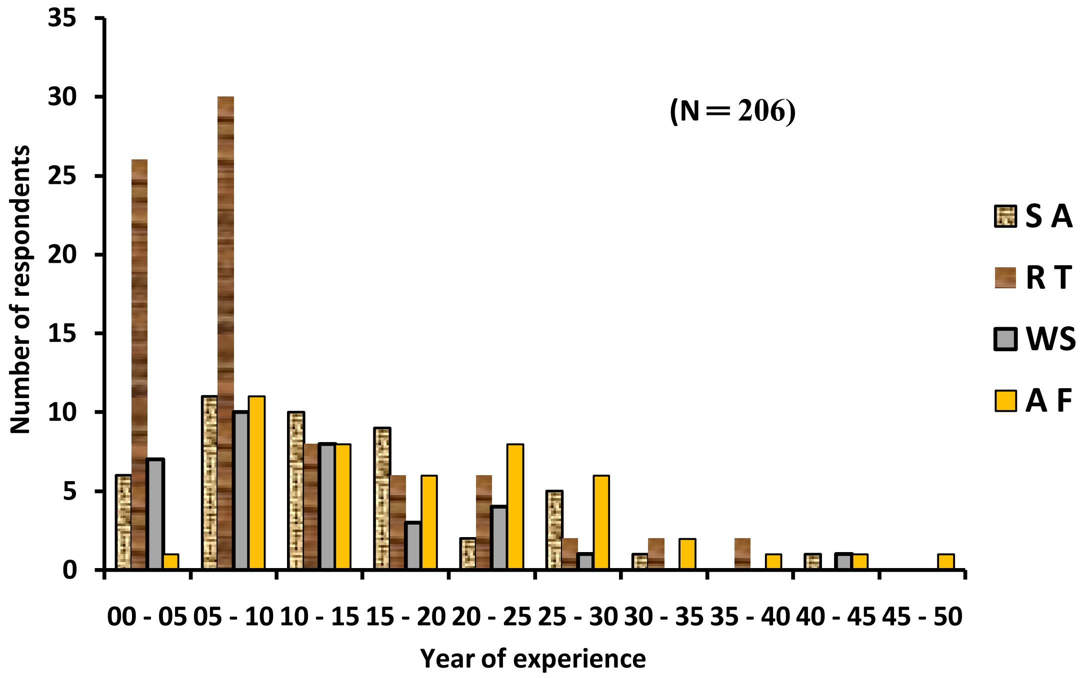

Figure 2: Histograms showing the year of experience in each category of professionals who engaged in fishery-related activities in the marine artisanal fishing sector of Sassandra (southwestern Côte d'Ivoire). Note: **SA** = smoke-curing agents; **RT** = retailers; **WS** = wholesalers; **AF** = artisanal fishers.

3.2 Behaviour and social characteristics

Table 2 indicates the responses of the professionals while they were telling about their work conditions. Those responses can be likened to scores. Surprisingly, we expected that all respondents rush the opportunity to express wishes, without someone to show no interest in the wishes listed as a proposal. Overall, the interviewees responded favourably, enabling us to rank their wishes this way: W1 > W2 > W4 > W3 > W5, taking into account the number of people who showed a remarkable interest in the wishes. In fact, the scores attributed to wishes were the more so high that the wish included the ambitions and desires of professionals of all kinds. Therefore, wishes W1 and W2 had the highest scores and percentages (Table 2). In addition, wish W4, which deals with the commonest desire all the professionals have in focus (i.e. get financial support from the Bank or from any other financial Institution), gained collective interest because it seems more inclusive. However, the specific nature of some other wishes, which seem exclusive, like wish W5 dealing with an equipment intended for fishers and canoe-owners, allows for a distinction to be made as regards safety-work conditions. All respondents told us about the Bank and Insurances´ cautiousness, the former refraining from placing financial support at their disposal, and the later generally seeking more trust.

Table 2: Professionals´ opinions about issues concerning their health and security at work, schooling of their children, Insurance services, financial support displacement and fishers´ safety at sea

Opinions of people engaged in the different types of activities observed at Sassandra	Number of responses	Percentages (P_i) relating to type of responses	Percentages obtained, considering a total of 220 respondents
People who showed no interest in the wishes	4	1.82	0.79
Positive responses to the 5 wishes at a time	8	3.64	1.57
Positive responses to the first 4 wishes at a time	21	9.55	4.13
Positive responses to the latest 4 wishes at a time	4	1.82	0.79
People who showed interest in wish Number 1	149	67.73	29.27
People who showed interest in wish Number 2	121	55.00	23.77
People who showed interest in wish Number 3	74	33.64	14.54
People who showed interest in wish Number 4	103	46.82	20.23
People who showed interest in wish Number 5	25	11.36	4.91

Note: Percentages in the fourth column were obtained respectively, dividing each percentage (P_i) by the sum of all percentages (ΣP_i). Wish (W1): ***An Hospital / A free health centre*** *; Wish (W2):* ***A primary school constructed for your kids´ benefit*** *; Wish (W3):* ***Benefit from Insurance services*** *; Wish (W4):* ***Benefit from Bank services / Get financial support*** *; Wish (W5):* ***Get life jackets for fishers´ safety at sea****.*

Figure 3 shows the number of collaborators in each corporation. Overall, number of apprentices or collaborators is dependent on type of activity. For instance, smoke-curing agents, retailers and wholesalers would work with 1 to 4 persons. In stark contrast, fishers would work with as much as 2 to 18 collaborators, though number of collaborators would vary according to factors such as the type of fishing gear, which may need that fishers work as a team or not (e.g. line, hooks, gillnet, or seine) and according to title of the fisher (e.g. chief of fishers, canoe-owner, a mere member of the fishing-crew). Nearly all smoke-curing agents (80% of them) worked with 1 or 2 persons, as 92.68% of retailers and 81.08% of wholesalers did (Figure 3). However, cases in which the professionals worked alone were reported among smoke-curing agents (35.56% of them), retailers (53.66% of them), and wholesalers (21.62% of them). Overall, 68 persons (33.01% of the interviewees) admitted that they usually work alone.

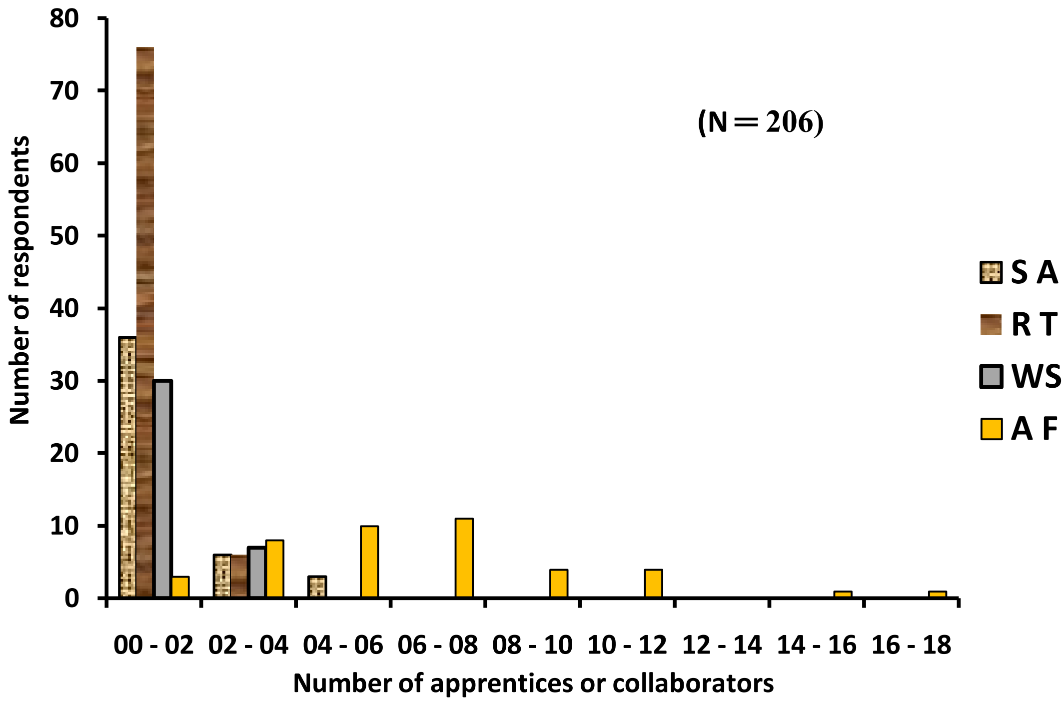

Figure 3: Histograms showing the number of apprentices or collaborators in each category of professionals who engaged in activities relating to the marine artisanal fishing sector of Sassandra (southwestern Côte d'Ivoire). Note: **SA** = smoke-curing agents; **RT** = retailers; **WS** = wholesalers; **AF** = artisanal fishers.

IV. DISCUSSION

Some specific circumstances people go through generally served for a driving force behind their choosing a particular type of fishery-related activity. As for the youths, their early beginnings taking on the fishery-related activities were a manner for them to serve as a helping hand for their mothers and fathers, especially when those youths who were scholars spent a holiday. In such a case, their involvement in fishery-related activities is on a part-time job basis. However, some other youths´ taking part in those activities was dictated by their parents´ will to early initiate their offspring into them. It is obvious that at an early age, the choice for a particular activity may not be definitive. Adult men and women would however willingly decide for themselves and choose the type of activity they prefer. Talking of artisanal fishing in Guinea (a West African country), **Koita (2017)** said that it is carried out by men and women, even by teenagers engaged in such an occupation as a job, describing it as a type of activity where knowledge is passed on from a father to a son. In fact, for the jobs that do not require any particular time for training (e.g. smoke-curing, retailing and wholesaling), the hardships people went through sometimes forced them to make a choice. For some women we interviewed told us that their choice was dictated by circumstances they faced after they became widows or divorced, in order to make both ends meet as single parents, being alone to care for their children´s needs and schooling. In such cases, whether they turned their forties or fifties, people could choose a type of activity, without prior requirement for training.

To reach the actual stage of their careers, the professionals of the marine artisanal fishing sector of Sassandra did work hard, seeking guidance and accepting to learn, generally at their parents´ side. Yet some of them did progress by their own efforts. Others acknowledged that some of their first efforts at retailing or wholesaling were pretty awful. In each corporation, people customary work with members of family or persons of the same ethnic group. By so doing, the knowledge of fishing for instance rests concealed in family circle, being passed on to the new generations with the passing of the years. This undoubtedly contributed to the success migrant fishers generally had in the fishing activities. Moreover, as the Ghanaian fisher flock is quite dominant, they do have influence over almost all fishery-related activities **(Delaunay, 1991**; **Bignoumba, 1998)**, which confirm the view of « colonization of fishery-kind » **Delaunay (1989**) held. It seems that for some types of activity, there is no need to rely on guidance from former trained-persons. That was certainly what justified the cases in which the professionals worked alone. Additionally, as in the case of the marine artisanal fishery of Pointe Noire, Congo **(Gobert, 1985)**, where the fisher flock originated from Benin has influence over the activities, a large part of the fishery´s workforce at Sassandra rests in the family circle. Consequently, when the period for fishing for tunas and Sardinellas or that for festivals usually held at regular intervals in Ghana are drawing closer, many fishers would return to Ghana (their home-country) with their families. This situation reinforces the unsteadiness of the fishery-related activities at Sassandra, southwestern Côte d'Ivoire.

Within the fisher flock, the wishes seem to be tied in priorities set by the professionals of the fishing sector, mainly according to their professional needs and daily life conditions. In fact, most of the professionals have to care for their families and at the same time pay for their children´s schooling and face current expenditures; which need be put above all other things. Yet, the professionals are aware that they can successfully face dependents if only their activities flourish. That is why they look to the Banks or any financial Institution, expecting them to provide financial support for the furtherer of their activities. In this regard, are there any sound reasons for financial Institutions´ cautiousness? In fact, financial problems facing people working in the artisanal fishing sector are widespread in West Africa, as discussed by **Bignoumba (1998)**, referring to the marine artisanal fishery of Gabon. The lack of financial support displacement for activities was the main problem the professionals of the fishing sector of Sassandra have in share. Apparently, people carrying out the four types of fishery-related activities the current study deals with are unaware that features pertaining to their behaviour and work conditions are the main reasons for financial Institutions´ and Insurances´ cautiousness. For instance, work condition was shaped by three outstanding facts: (i) unsteadiness of the fishery-related activities, (ii) a strong tendency towards informal work, and (iii) equipments for fishing are very expensive and fishers prefer to buy cheaper ones in Ghana. As regards behaviour and social characteristics, three remarkable facts generally occur: (i) the high proportion of migrants, (ii) the lack of truthfulness and reliability, and (iii) most professionals regard themselves as temporary residents, preferring making investments in their home-country (i.e. Ghana). Overall, these are the main reasons why the Banks and Insurances are cautious, refraining from engaging their responsibility and money in a risky adventure (i.e. fishery-related activities) that is not constant in purpose or actions, lacking reliability as a result, and showing no steady and maximum profits ahead.

Finally, features we discussed in the current study are not specific to the marine artisanal fishery of Sassandra, southwestern Côte d'Ivoire. They are common elsewhere, namely in Gabon (a West African country), where the artisanal fishery is mainly characterized by the weakness of its production tool, the supremacy of migrant fishermen and a strong tendency towards informal work **(Bignoumba, 2011)**. Additionally, involvement of members of family in the fishing business was observed elsewhere. In Senegal (another West African country), **Cormier (1981)** noticed that within the "Lébou" community, fishers at times would trade for the fish, which was an activity ordinarily carried out by their sisters or by fishers´ wives.

V. CONCLUSION

The professionals of the marine artisanal fishing sector of Sassandra, southwestern Côte d'Ivoire, actually range in year of experience, according to type of fishery-related activity. They generally worked alone or with few people as collaborators, except the fishers. Most of them have served their apprenticeship with members of their family, or tried to work on their own initiative to achieve goals. As the professionals prominently worked with members of family, the fishery´s workforce would rest in the family circle, making the work environment be shaped by persistency of ethnic patterns. Consequently, this situation resulted in an impediment to financial sustenance in favour of the fishery-related activities as regards financial help from the Banks and other financial Institutions whose cautiousness was plainly supported by actual facts.

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Risk Hedging in Financial Markets

N.S. Gonchar

ABSTRACT

A recursive method of martingale measures construction for a wide class of evolutions of risky asset is proposed. An integral representation for each equivalent martingale measure is obtained. A complete description of all martingale measures is established. The formulas for both infimum and supremum for the average values of payment functions of call and put options with respect to all equivalent martingale measures are established. The invariance of the set of all martingales with respect to a certain class of evolutions of risky assets is proved. A parametric class of evolutions of risky asset is introduced, which includes ARCH and GARCH models and their generalizations. A parameter estimation method for the introduced parametric models is proposed.

Necessary and sufficient conditions are obtained under which the martingale measure is unique. A significant number of examples of the discounted evolution of risky assets are presented for which the existence of a single martingale measure is established.

Keywords: random process; spot set of measures; parametric model of evolution; unique martingale measure; martingale; assessment of derivatives.

Classification: FOR CODE: 150299

Language: English

London
Journals Press

LJP Copyright ID: 925632
Print ISSN: 2631-8490
Online ISSN: 2631-8504

London Journal of Research in Science: Natural and Formal

Volume 23 | Issue 4 | Compilation 1.0

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Necessary and sufficient conditions are obtained under which the martingale measure is unique. A significant number of examples of the discounted evolution of risky assets are presented for which the existence of a single martingale measure is established. An explicit construction of a single martingale measure in these cases is given. Formulas for fair price of options contracts and investor hedging strategies are provided. A parametric model of evolution of risky asset is introduced so that the single martingale measure does not depend on the entered parameters. A complete description of the family of martingale measures is given for multinomial models of the evolution of risky asset. Each martingale measure is a finite sum of the introduced spot measures. The attractive side of such models is that the lower and upper price of the interval non arbitrage prices are, respectively, the minimum and maximum of the average values of the payment functions on a set of spot measures.

A class of parametric models is introduced that describe the multinomial evolution of risky asset such that the family of martingale measures does not depend on the entered parameters.

Keywords: random process; spot set of measures; parametric model of evolution; unique martingale measure; martingale; assessment of derivatives.

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I. INTRODUCTION

This paper continues the papers [1] - [5] and generalizes them to the case of different evolutions of risky assets. These examples of evolutions are quite realistic because they contain the memory of the past and describe the phenomenon of clustering and other effects. Our results concerning construction of risk neutral measures are quite general relative to volatility evolution and therefore they contain a wide class of evolutions of risky asset. The construction of the set of martingale measures for the above class of evolution of risky asset is based on the result of the work [4] (see Lemma 5) where, for a given random variable and a measure on an abstract

[1]This work was partially supported by the Program of Fundamental Research of the Department of Physics and Astronomy of the National Academy of Sciences of Ukraine "Construction and research of financial market models using the methods of non-equilibrium statistical physics and the physics of nonlinear phenomena" N 0123U100362.

probability space, the set of all measures equivalent to the original one and such that the average value over such measures of the considered random variable is equal to zero is described. The notion of consistency of a family of measures with filtration introduced in this paper and the proven Lemma 5 [4] made it possible to propose a new method for constructing a family of martingale measures equivalent to a given measure, which is different from the Escher transformation and generalizations of Girsanov's theorem. The ideas proposed in this paper [4] made it possible to propose a recursive method for constructing a set of risk neutral measures and to give a complete description of them for a certain class of evolutions of risky asset. It turned out that it is possible to introduce a set of spot martingale measures in a recurrent way and prove that any equivalent martingale measure to the original measure is an integral over the set of spot martingale measures. The latter made it possible to establish formulas for the boundaries of non-arbitrage prices for nonnegative contingent claims, as well as a formula for the fair price of a complete hedging of systematic risk. In the paper [3], formulas for the interval of non-arbitrage prices for put and call options are found for the evolution of a risky asset occurring in accordance with the geometric Brownian motion. The work [2] contains a general construction of building risk-neutral measures by the recursive method.

In the present paper, a significant generalization of the class of evolutions of risky assets is made, which contains ARCH and GARCH processes and their generalizations.

The study of non-arbitrage markets was begun for the first time in Bachelier's work [6]. Then, in the famous works of Black F. and Scholes M. [7] and Merton R. S. [8] the formula was found for the fair price of the standard call option of European type. The absence of arbitrage in the financial market has a very transparent economic sense, since it can be considered reasonably arranged. The concept of non arbitrage in financial market is associated with the fact that one cannot earn money without risking, that is, to make money you need to invest in risky or risk-free assets. The exact mathematical substantiation of the concept of non arbitrage was first made in the papers [9], [10] [11] for the finite probability space and in the general case in the paper [12]. In the continuous time evolution of risky asset, the proof of absent of arbitrage possibility see in [13]. The value of the established Theorems is that they make it possible to value assets. They got a special name "The First and The Second Fundamental Asset Pricing Theorems." Generalizations of these Theorems are contained in papers [14], [15], [16].

If the martingale measure is not the only one for a given evolution of a risky asset, then a rather difficult problem of describing all martingale measures arises in order to evaluate, for example, derivatives.

Assessment of risk in various systems was begun in papers [17], [18], [19], [20].

Statistical studies of the time series of the logarithm of the price ratio of risky assets contain heavy tails in distributions with strong elongation in the central region. The temporal behavior of these quantities exhibits the property of clustering and a strong dependence on the past. All this should be taken into account when building models for the evolution of risky assets.

In this paper, we generalize the results of the papers [1] - [5] and construct the evolution of risky assets for which we completely describe the set of equivalent martingale measures.

The aim of this study is to describe the family of martingale measures for a wide class of risky asset evolutions. The paper proposes the general concept for constructing the family of martingale measures equivalent to a given measure for a wide class of evolutions of risky assets. In particular, it also contains the description of the family of martingale measures for the evolution of risky assets given by the ARCH [21] and GARCH [22], [23] models. In section 2, we formulate the conditions relative to the evolution of risky assets and give the examples of risky asset evolution satisfying these conditions. Section 3 contains the construction of measures by recurrent relations. It is shown that under the conditions relative to the evolution of risky asset such construction is meaningful. It is proved that the constructed set of measures is equivalent to an initial measure. In theorem 1, we are proved that under certain integrability conditions of risky asset evolution the set of constructed measures is a set of martingale measures relative to this evolution of risky asset. In section 4, a family of spot martingale measures is introduced and a set of measures is constructed from it and a family of random variables, and it is shown in Theorem 2 that the constructed family of measures is absolutely continuous with respect to the original measure. And in Theorem 3, it is proved that the family of measures constructed in this way is a family of martingale measures which are equivalent to the original measure. A complete description of all martingale measures is found in Theorem 4. Theorem 7 establishes that the infimum and supremum of the mean value of payment functions all over martingale measures equals, correspondingly, infimum and supremum of the mean value of payment functions all over spot martingale measures. Theorem 8 establishes that the constructed class of martingale measures is invariant with respect to a certain class of evolutions of risky assets. This statement is important and makes it possible to build parametric models of financial markets. In Section 5, estimates for both the lower and upper limits of the interval of non-arbitrage prices are found for the constructed parametric model. The proposed parametric model based on the canonical model of the evolution of risky asset (9), which takes into account both memory and clustering, takes into account the fact that the price of a risky asset cannot fall to zero. As a consequence of these estimates, explicit formulas for the fair prices of a superhedge in the case of the payment functions of a standard call and put options are found in Theorems 11, 12. Analogous results are found in Theorems 13 and 14 for the payment functions of Asian-type call and put options.

Theorem 15 provides estimates for the parameters through realizations of the random parametric evolution of the risky asset. In Theorems 16 - 19 the formulas for interval of non arbitrage prices and the fair prices of superhedge are given through the obtained parameter estimates.

Another parametric model of the evolution of risky assets is considered in Section 6. It differs from the previous one in that it considers the discounted evolution of risky asset. Theorems 20 - 21 are proved, in which estimates are obtained both from above and from below and established. Theorems 22 - 23 derive formulas for the fair price of a superhedge for the payment functions of call and put options, respectively. A similar result is obtained in Theorems 24 - 25 for the payment functions of Asian-type put and call options. In Theorems 26 - 29, based on the sample for the evolution of the risky asset, the formulas for the fair price of the superhedge through parameter estimation are presented. Section 7 establishes Theorem 30, which gives the necessary and sufficient conditions for the unity of an equivalent martingale measure.

In Section 8, Proposition 2 proposes a model of the financial market with a single martingale measure that is invariant with respect to the evolution of each of the assets. In Theorems 32 and 33, various examples of discounted evolutions of risky assets are presented, conditions for the existence of a single martingale measure are found, and its explicit construction is given. Formulas for fair pricing options contracts and investor hedging strategies are provided. In proposition 3, a parametric model of the evolution of risky asset is proposed; the single martingale measure constructed for this evolution does not depend on these parameters. Estimates of the model parameters were built based on the realizations of the random evolution of asset.

Section 9 contains a description of all martingale measures for the multinomial evolution of risk assets. This result is obtained in Theorem 35.

In section 10, models of incomplete financial markets are proposed for which inequalities are established for the fair price of a superhedge for various models of the evolution of risky asset. Theorem 37 establishes that for a certain class of payment functions and for a wide class of evolutions of risky assets, the fair price of the superhedge is strictly less than the price of the underlying asset. Among such payment functions is the payment function of the standard call option of the European type. Theorems 39, 40 give various examples of discounted evolutions of risky assets that satisfy the conditions of the proved theorems 35 - 37, and find the conditions under which the family of martingale measures is nonempty. Formulas for a fair superhedge price have been found. Proposition 5 contains the construction of a parametric model of an incomplete financial market, a family of martingale measures of which does not depend on the considered parameters. Proposition 6 provides an estimates of the parameters of the constructed models of incomplete markets through realizations of the considered evolutions of risky asset.

II. GENERAL ASSUMPTIONS RELATIVE TO EVOLUTIONS OF RISKY ASSETS

Let $\{\Omega_N, \mathcal{F}_N, P_N\}$ be a direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, $i = \overline{1,N}$, $\Omega_N = \prod_{i=1}^{N} \Omega_i^0$, $P_N = \prod_{i=1}^{N} P_i^0$, $\mathcal{F}_N = \prod_{i=1}^{N} \mathcal{F}_i^0$, where the σ-algebra $\mathcal{F}_N$ is a minimal σ-algebra, generated by the sets $\prod_{i=1}^{N} G_i$, $G_i \in \mathcal{F}_i^0$. On the measurable space $\{\Omega_N, \mathcal{F}_N\}$, under the filtration $\mathcal{F}_n$, $n = \overline{1,N}$, we understand the minimal σ-algebra generated by the sets $\prod_{i=1}^{N} G_i$, $G_i \in \mathcal{F}_i^0$, where $G_i = \Omega_i^0$ for $i > n$. We also introduce the probability spaces $\{\Omega_n, \mathcal{F}_n, P_n\}, n = \overline{1,N}$, where $\Omega_n = \prod_{i=1}^{n} \Omega_i^0$, $\mathcal{F}_n = \prod_{i=1}^{n} \mathcal{F}_i^0$, $P_n = \prod_{i=1}^{n} P_i^0$. There is a one-to-one correspondence between the sets of the σ-algebra $\mathcal{F}_n$, belonging to the introduced filtration, and the sets of the σ-algebra $\mathcal{F}_n = \prod_{i=1}^{n} \mathcal{F}_i^0$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}, n = \overline{1,N}$. Therefore, we don't introduce new denotation for the σ-algebra $\mathcal{F}_n$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}$, since it always will be clear the difference between the above introduced σ-algebra $\mathcal{F}_n$ of filtration on the measurable space $\{\Omega_N, \mathcal{F}_N\}$ and the σ-algebra $\mathcal{F}_n$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}, n = \overline{1,N}$.

We assume that the evolution of risky asset $\{S_n\}_{n=0}^N$, given on the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, is consistent with the filtration $\mathcal{F}_n$, that is, S_n is a $\mathcal{F}_n$-measurable. Due to the above one-to-one correspondence between the sets of the σ-algebra $\mathcal{F}_n$, belonging to the introduced filtration, and the sets of the σ-algebra $\mathcal{F}_n$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}, n = \overline{1,N}$, we give the evolution of risky assets in the form

$$\{S_n(\omega_1, \ldots, \omega_n)\}_{n=0}^N, \tag{1}$$

where $S_n(\omega_1, \dots, \omega_n)$ is an $\mathcal{F}_n$-measurable random variable, given on the measurable space $\{\Omega_n, \mathcal{F}_n\}$. It is evident that such evolution is consistent with the filtration $\mathcal{F}_n$ on the measurable space $\{\Omega_N, \mathcal{F}_N, P_N\}$.

Further, we assume that

$$P_n((\omega_1, \dots, \omega_n) \in \Omega_n,\ \Delta S_n > 0) > 0,$$

$$P_n((\omega_1, \dots, \omega_n) \in \Omega_n,\ \Delta S_n < 0) > 0, \quad n = \overline{1, N}, \tag{2}$$

where $\Delta S_n = S_n(\omega_1, \dots, \omega_n) - S_{n-1}(\omega_1, \dots, \omega_{n-1}),\ n = \overline{1, N}$.

Let us introduce the denotations

$$\Omega_n^- = \{(\omega_1, \dots, \omega_n) \in \Omega_n,\ \Delta S_n \leq 0\}, \quad \Omega_n^+ = \{(\omega_1, \dots, \omega_n) \in \Omega_n,\ \Delta S_n > 0\}, \tag{3}$$

$$\Delta S_n^- = -\Delta S_n \chi_{\Omega_n^-}(\omega_1, \dots, \omega_n), \quad \Delta S_n^+ = \Delta S_n \chi_{\Omega_n^+}(\omega_1, \dots, \omega_n), \tag{4}$$

$$V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2) = \Delta S_n^-(\omega_1, \dots, \omega_{n-1}, \omega_n^1) + \Delta S_n^+(\omega_1, \dots, \omega_{n-1}, \omega_n^2),$$

$$(\omega_1, \dots, \omega_{n-1}, \omega_n^1) \in \Omega_n^-, \quad (\omega_1, \dots, \omega_{n-1}, \omega_n^2) \in \Omega_n^+. \tag{5}$$

Our assumptions relative to Ω_n^- and Ω_n^+ are the following

$$\Omega_n^- = \Omega_{n-1} \times \Omega_n^{0-}, \quad \Omega_n^+ = \Omega_{n-1} \times \Omega_n^{0+}, \quad \Omega_n^{0-},\ \Omega_n^{0+} \in \Omega_n^0, \quad n = \overline{1, N}, \tag{6}$$

where

$$\Omega_n^{0-} \cup \Omega_n^{0+} = \Omega_n^0, \quad n = \overline{1, N}, \tag{7}$$

$$P_n^0(\Omega_n^{0+}) > 0, \quad P_n^0(\Omega_n^{0-}) > 0, \quad n = \overline{1, N}. \tag{8}$$

Below, we give the examples of evolutions $\{S_n(\omega_1, \dots, \omega_n)\}_{n=1}^N$, for which the conditions (6) - (8) are true. Let us consider the evolution of risky asset given by the law

$$S_n(\omega_1, \dots, \omega_{n-1}, \omega_n) = S_0 \prod_{i=1}^n e^{\sigma_i(\omega_1, \dots, \omega_{i-1})\varepsilon_i(\omega_i)}, \quad n = \overline{1, N}, \quad S_0 > 0, \tag{9}$$

relative to which we assume that the conditions

$$\sigma_i(\omega_1, \dots, \omega_{i-1}) \geq \sigma_i^0 > 0, \quad P_i^0(\varepsilon_i(\omega_i) > 0) > 0, \quad P_i^0(\varepsilon_i(\omega_i) < 0) > 0, \quad i = \overline{1, N},$$

are true. For the evolution of risky asset (9), we have

$$\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$S_{n-1}(\omega_1,\dots,\omega_{n-1})(e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n)} - 1) = \tag{10}$$

$$d_n(\omega_1,\dots,\omega_{n-1},\omega_n)(e^{\sigma_n^0\varepsilon_n(\omega_n)} - 1),$$

where

$$d_n(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$S_{n-1}(\omega_1,\dots,\omega_{n-1})\frac{(e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n)} - 1)}{(e^{\sigma_n^0\varepsilon_n(\omega_n)} - 1)}. \tag{11}$$

It is evident that $d_n(\omega_1,\dots,\omega_{n-1},\omega_n) > 0$ and for Ω_n^-, Ω_n^+ the representations (6) are true with

$$\Omega_n^{0-} = \{\omega_n \in \Omega_n^0, \varepsilon_n(\omega_n) \le 0\}, \quad \Omega_n^{0+} = \{\omega_n \in \Omega_n^0, \varepsilon_n(\omega_n) > 0\}.$$

The more general example of risky asset evolution, satisfying the conditions (6) - (8), is given by the formula

$$S_n(\omega_1,\dots,\omega_n) =$$

$$S_0\prod_{i=1}^{n}(1 + a_i(\omega_1,\dots,\omega_i)\eta_i(\omega_i)), \ \{\omega_1,\dots,\omega_{n-1},\omega_n\} \in \Omega_n, \ n = \overline{1,N}, \quad S_0 > 0, \tag{12}$$

where the random values $a_n(\omega_1,\dots,\omega_{n-1},\omega_n)$, $\eta_n(\omega_n)$, $n = \overline{1,N}$, given on the probability space $\{\Omega_n, \mathcal{F}_n, P_n\}$, satisfy the conditions

$$a_n(\omega_1,\dots,\omega_{n-1},\omega_n) > 0, \quad \sup_{\omega_n\in\Omega_n^0,\eta_n^-(\omega_n)>0} \eta_n^-(\omega_n) < \infty,$$

$$\sup_{\{\omega_1,\dots,\omega_n\}\in\Omega_n} a_n(\omega_1,\dots,\omega_{n-1},\omega_n) < \frac{1}{\sup_{\omega_n\in\Omega_n^0,\eta_n^-(\omega_n)>0} \eta_n^-(\omega_n)}. \tag{13}$$

So, for $\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)$, $n = \overline{1,N}$, the representation

$$\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$S_{n-1}(\omega_1,\dots,\omega_{n-1})a_n(\omega_1,\dots,\omega_{n-1},\omega_n)\eta_n(\omega_n) =$$

$$d_n(\omega_1,\dots,\omega_{n-1},\omega_n)\eta_n(\omega_n), \quad n = \overline{1,N}, \tag{14}$$

is true, where $d_n(\omega_1,\dots,\omega_{n-1},\omega_n) > 0$. From the representation (14) we obtain $\Omega_n^- = \Omega_{n-1} \times \Omega_n^{0-}$, $\Omega_n^+ = \Omega_{n-1} \times \Omega_n^{0+}$, where $\Omega_n^{0-} = \{\omega_n \in \Omega_n^0, \eta_n(\omega_n) \le 0\}$, $\Omega_n^{0+} = \{\omega_n \in \Omega_n^0, \eta_n(\omega_n) > 0\}$.

Further, we assume that $P_n^0(\Omega_n^{0-}) > 0$, $P_n^0(\Omega_n^{0+}) > 0$. The measure P_n^{0-} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0-} = \Omega_n^{0-} \cap \mathcal{F}_n^0$, P_n^{0+} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0+} = \Omega_n^{0+} \cap \mathcal{F}_n^0$.

Below we give an example of discount evolution having the representation (12). Suppose that risky asset evolution is given by the formula (9) and an evolution of non risky asset is given by the law

$$B_n = \prod_{i=1}^{n} e^{r_i}, \quad 0 < r_n < \infty, \quad n = \overline{1,N}. \tag{15}$$

Let us assume that

$$P_i^0(\{\omega_i \in \Omega_i^0, \ \sigma_i^0 \varepsilon_i(\omega_i) - r_i < 0\}) > 0,$$

$$P_i^0(\{\omega_i \in \Omega_i^0, \quad \sigma_i^0 \varepsilon_i(\omega_i) - r_i > 0\}) > 0, \quad i = \overline{1,N}. \tag{16}$$

Then for the discount evolution

$$S_n^1(\omega_1, \ldots, \omega_{n-1}, \omega_n) = \frac{S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)}{B_n}, \quad n = \overline{1,N}, \tag{17}$$

the representation (12) is true, where

$$a_i(\omega_1, \ldots, \omega_{i-1}, \omega_i) = \frac{e^{\sigma_i(\omega_1, \ldots, \omega_{i-1})\varepsilon_i(\omega_i) - r_i} - 1}{e^{\sigma_i^0 \varepsilon_i(\omega_i) - r_i} - 1} \geq 1, \quad \eta_i(\omega_i) = e^{\sigma_i^0 \varepsilon_i(\omega_i) - r_i} - 1.$$

In this case,

$$\Omega_i^{0-} = \{\omega_i \in \Omega_i^0, \ \varepsilon_i(\omega_i) \leq \frac{r_i}{\sigma_i^0}\}, \quad \Omega_i^{0+} = \{\omega_i \in \Omega_i^0, \varepsilon_i(\omega_i) > \frac{r_i}{\sigma_i^0}\}, \tag{18}$$

$$\Omega_i^- = \Omega_{i-1} \times \Omega_i^{0-}, \quad \Omega_i^+ = \Omega_{i-1} \times \Omega_i^{0+}. \tag{19}$$

The evolution of risky asset, given by the formula (9), includes a wide class of evolutions of risky assets, used in economics. For example, under an appropriate choice of probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$ and a choice of sequence of independent random values $\varepsilon_i(\omega_i)$ with the normal distribution $N(0,1)$, we obtain ARCH model (Autoregressive Conditional Heteroskedastic Model) introduced by Engle in [21] and GARCH model (Generalized Autoregressive Conditional Heteroskedastic Model) introduced later by Bollerslev in [22]. In these models, the random variables $\sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $i = \overline{1,N}$, are called the volatilities which satisfy the nonlinear set of equations.

Further, we do not restrict ourselves only the above considered case of evolutions of risky assets. We assume that the random variables $\sigma_i(\omega_1, \ldots, \omega_{i-1})$ entering in the formulas (9) satisfy only the inequalities $\sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $i = \overline{1,N}$, and the random values $\varepsilon_i(\omega_i)$, $i = \overline{1,N}$, are non correlated between themselves. For example, they may be independent random values having the normal distribution with zero mean value and not only.

III. RECURSIVE CONSTRUCTION OF THE SET OF MARTINGALE MEASURES

In this section, we present the construction of the set of measures on the basis of evolution of risky asset, given by the formula (1), satisfying the conditions (6) - (8). For this purpose, we use the set of nonnegative random values $\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})$, given on the probability space $\{\Omega_n^- \times \Omega_n^+, \mathcal{F}_n^- \times \mathcal{F}_n^+, P_n^- \times P_n^+\}$, $n=\overline{1,N}$, where $\mathcal{F}_n^- = \mathcal{F}_n \cap \Omega_n^-$, $\mathcal{F}_n^+ = \mathcal{F}_n \cap \Omega_n^+$. The measure P_n^- is a contraction of the measure P_n on the σ-algebra $\mathcal{F}_n^-$ and the measure P_n^+ is a contraction of the measure P_n on the σ-algebra $\mathcal{F}_n^+$. After that, we prove that this set of measures is equivalent to the measure P_N. At last, Theorem 1 gives the sufficient conditions under which the constructed set of measures is a set of martingale measures for the considered evolution of risky asset. Sometimes, we use the abbreviated denotations $\{\omega_1^1,\ldots,\omega_n^1\}=\{\omega\}_n^1$, $\{\omega_1^2,\ldots,\omega_n^2\}=\{\omega\}_n^2$.

We assume that the set of random values $\alpha_n(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\}) = \alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)$, $(\{\omega\}_n^1;\{\omega\}_n^2) \in \Omega_n^- \times \Omega_n^+$, $n=\overline{1,N}$, satisfies the following conditions:

$$P_n^- \times P_n^+((\{\omega\}_n^1;\{\omega\}_n^2) \in \Omega_n^- \times \Omega_n^+, \alpha_n(\{\omega\}_n^1;\{\omega\}_n^2) > 0) =$$

$$P_n(\Omega_n^-) \times P_n(\Omega_n^+), \quad n=\overline{1,N}; \tag{20}$$

$$\int\limits_{\Omega_n^0 \times \Omega_n^0} \chi_{\Omega_n^-}(\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1)\chi_{\Omega_n^+}(\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2)\times$$

$$\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)} dP_n^0(\omega_n^1)dP_n^0(\omega_n^2) < \infty,$$

$$(\{\omega_1^1,\ldots,\omega_{n-1}^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2\}) \in \Omega_{n-1}\times\Omega_{n-1},$$

$$(\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}, \quad n=\overline{1,N}; \tag{21}$$

$$\int\limits_{\Omega_n^0 \times \Omega_n^0} \chi_{\Omega_n^-}(\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1)\chi_{\Omega_n^+}(\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2)\times$$

$$\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})dP_n^0(\omega_n^1)dP_n^0(\omega_n^2) = 1,$$

$$(\{\omega_1^1,\ldots,\omega_{n-1}^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2\}) \in \Omega_{n-1}\times\Omega_{n-1}, \quad n=\overline{1,N}. \tag{22}$$

In the next Lemma 1, we give the sufficient conditions under which the conditions (20) - (22) are valid.

Lemma 1. *Suppose that the evolution of risky asset, given by the formula (1), satisfies the conditions (6) - (8). If the inequalities*

$$\int\limits_{\Omega_N} \Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n)dP_N < \infty, \qquad n=\overline{1,N}, \tag{23}$$

are true, then the set of bounded random values $\alpha_n(\{\omega\}_n^1; \{\omega\}_n^2)$, satisfying the conditions (20) - (22), is a nonempty set.

Proof. If the random values

$$0 < c_1 \leq \alpha_n^1(\{\omega_1^1, \ldots, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_n^2\}) \leq c_2 < \infty, \ n = \overline{1, N}, \tag{24}$$

are bounded as from below and above, then the random values

$$\alpha_n(\{\omega_1^1, \ldots, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_n^2\}) =$$

$$\frac{\alpha_n^1(\{\omega_1^1, \ldots, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_n^2\})}{T(\{\omega_1^1, \ldots, \omega_{n-1}^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2\})}, \ n = \overline{1, N}, \tag{25}$$

where

$$T(\{\omega_1^1, \ldots, \omega_{n-1}^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2\}) =$$

$$\int\limits_{\Omega_n^0} \int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1^1, \ldots, \omega_n^1) \chi_{\Omega_n^+}(\omega_1^2, \ldots, \omega_n^2) \alpha_n^1(\{\omega_1^1, \ldots, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_n^2\}) \times$$

$$dP_n^0(\omega_n^1) dP_n^0(\omega_n^2),$$

is also bounded as from below and above. Really,

$$\frac{c_1}{c_2 P_n^0(\Omega_n^-) P_n^0(\Omega_n^+)} \leq \alpha_n(\{\omega_1^1, \ldots, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_n^2\}) \leq$$

$$\frac{c_2}{c_1 P_n^0(\Omega_n^-) P_n^0(\Omega_n^+)} = C_n < \infty, \quad (\{\omega\}_n^1; \{\omega\}_n^2) \in \Omega_n^- \times \Omega_n^+, \quad n = \overline{1, N}. \tag{26}$$

It is evident that the random values (25) satisfy the condition (20) - (21). Really, due to the inequalities (26), the random values $\alpha_n(\{\omega\}_n^1; \{\omega\}_n^2\})$, $n = \overline{1, N}$, are strictly positive. Therefore, the conditions (20) are true.

Owing to the boundedness of $\alpha_n(\{\omega\}_n^1; \{\omega\}_n^2\}) \leq C_n$, $n = \overline{1, N}$, the inequalities

$$\int\limits_{\Omega_n^0 \times \Omega_n^0} \chi_{\Omega_n^-}(\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1) \chi_{\Omega_n^+}(\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2) \times$$

$$\alpha_n(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\}) \times$$

$$\frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} dP_n^0(\omega_n^1) dP_n^0(\omega_n^2) \leq$$

$$C_n \int\limits_{\Omega_n^0} \Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) dP_n^0(\omega_n^1) < \infty, \quad n = \overline{1, N}, \tag{27}$$

are true for almost everywhere $(\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}$, $n=\overline{1,N}$, relative to the measure P_{n-1}, owing to the inequalities (23) and Foubini Theorem. This proves the inequality (21). The equality (22) is also satisfied, due to the construction of $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)$. Lemma 1 is proved.

On the basis of the set of random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)$, $n=\overline{1,N}$, constructed in Lemma 1, let us introduce into consideration the family of measure $\mu_0(A)$ on the measurable space $\{\Omega_N,\mathcal{F}_N\}$ by the recurrent relations

$$\mu_N^{(\omega_1,\ldots,\omega_{N-1})}(A)=\int\limits_{\Omega_N^0\times\Omega_N^0}\chi_{\Omega_N^-}(\omega_1,\ldots,\omega_{N-1},\omega_N^1)\chi_{\Omega_N^+}(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\times$$

$$\alpha_N(\{\omega_1,\ldots,\omega_{N-1},\omega_N^1\};\{\omega_1,\ldots,\omega_{N-1},\omega_N^2\})\times$$

$$\left[\frac{\Delta S_N^+(\omega_1,\ldots,\omega_{N-1},\omega_N^2)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N^1)}(A)+\right.$$

$$\left.\frac{\Delta S_N^-(\omega_1,\ldots,\omega_{N-1},\omega_N^1)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N^2)}(A)\right]dP_N^0(\omega_N^1)dP_N^0(\omega_N^2), \tag{28}$$

$$\mu_{n-1}^{(\omega_1,\ldots,\omega_{n-1})}(A)=\int\limits_{\Omega_n^0\times\Omega_n^0}\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\alpha_n(\{\omega_1,\ldots,\omega_{n-1},\omega_n^1\};\{\omega_1,\ldots,\omega_{n-1},\omega_n^2\})\times$$

$$\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}(A)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}(A)\right]dP_n^0(\omega_n^1)dP_n^0(\omega_n^2),\quad n=\overline{2,N}, \tag{29}$$

$$\mu_0(A)=\int\limits_{\Omega_1^0\times\Omega_1^0}\chi_{\Omega_1^-}(\omega_1^1)\chi_{\Omega_1^+}(\omega_1^2)\alpha_1(\omega_1^1;\omega_1^2)\times$$

$$\left[\frac{\Delta S_1^+(\omega_1^2)}{V_1(\omega_1^1,\omega_1^2)}\mu_1^{(\omega_1^1)}(A)+\frac{\Delta S_1^-(\omega_1^1)}{V_1(\omega_1^1,\omega_1^2)}\mu_1^{(\omega_1^2)}(A)\right]dP_1^0(\omega_1^1)dP_1^0(\omega_1^2), \tag{30}$$

where we put

$$\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N)}(A)=\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N),\quad A\in\mathcal{F}_N. \tag{31}$$

Lemma 2. *Suppose that the conditions of Lemma 1 are true. For the measure $\mu_0(A)$, $A\in\mathcal{F}_N$, constructed by the recurrent relations (28) - (30), the representation*

$$\mu_0(A)=\int\limits_{\Omega_N}\prod_{n=1}^N\psi_n(\omega_1,\ldots,\omega_n)\chi_A(\omega_1,\ldots,\omega_N)\prod_{i=1}^N dP_i^0(\omega_i) \tag{32}$$

is true and $\mu_0(\Omega_N) = 1$, that is, the measure $\mu_0(A)$ is a probability measure, being equivalent to the measure P_N, where we put

$$\psi_n(\omega_1,\ldots,\omega_n) = \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n)\psi_n^1(\omega_1,\ldots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n)\psi_n^2(\omega_1,\ldots,\omega_n), \quad (33)$$

$$\psi_n^1(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\int\limits_{\Omega_n^0} \chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\alpha_n(\{\omega_1,\ldots,\omega_{n-1},\omega_n\};\{\omega_1,\ldots,\omega_{n-1},\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^2), \quad (\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}, \quad (34)$$

$$\psi_n^2(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\alpha_n(\{\omega_1,\ldots,\omega_{n-1},\omega_n^1\};\{\omega_1,\ldots,\omega_{n-1},\omega_n\})\times$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1), \quad (\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}. \quad (35)$$

Proof. Due to Lemma 1 conditions, the set of strictly positive bounded random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)$, $n = \overline{1,N}$, satisfying the conditions (20) - (22), is a non empty set. The proof of formula (32) see in [2]. To prove Lemma 2, we need to prove that $\psi_n(\omega_1,\ldots,\omega_n) > 0$, $n = \overline{1,N}$. Really,

$$\psi_n^1(\omega_1,\ldots,\omega_{n-1},\omega_n) \geq$$

$$\frac{c_1}{c_2}\int\limits_{\Omega_n^{0+}} \frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^2) > 0, \quad (\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}, \quad (36)$$

$$\psi_n^2(\omega_1,\ldots,\omega_{n-1},\omega_n) \geq$$

$$\frac{c_1}{c_2}\int\limits_{\Omega_n^{0-}} \frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1) > 0, \quad (\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}. \quad (37)$$

From the inequalities (36), (37) we have what we need. To prove that $\mu_0(\Omega_N) = 1$, let us prove the equality

$$\int\limits_{\Omega_n^0} \psi_n(\omega_1, \ldots, \omega_n) dP_n^0(\omega_n) = 1, \quad (\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1}, \qquad n = \overline{1, N}. \tag{38}$$

We have

$$\int\limits_{\Omega_n^0} \psi_n(\omega_1, \ldots, \omega_n) dP_n^0(\omega_n) =$$

$$\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) \chi_{\Omega_n^+}(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \times$$

$$\alpha_n(\{\omega_1, \ldots, \omega_{n-1}, \omega_n^1\}; \{\omega_1, \ldots, \omega_{n-1}, \omega_n^2\}) \times$$

$$\left[\frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} + \right.$$

$$\left. \frac{\Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} \right] dP_n^0(\omega_n^1) dP_n^0(\omega_n^2) =$$

$$\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) \chi_{\Omega_n^+}(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \times$$

$$\alpha_n(\{\omega_1, \ldots, \omega_{n-1}, \omega_n^1\}; \{\omega_1, \ldots, \omega_{n-1}, \omega_n^2\}) dP_n^0(\omega_n^1) dP_n^0(\omega_n^2) = 1. \tag{39}$$

The last equality follows from the fact that the set of random values $\alpha_n(\{\omega_1\}_n^1; \{\omega_1\}_n^2)$, $n = \overline{1, N}$, satisfies the condition (22). The equalities (38) proves that every measure (32), defined by the set of random values $\alpha_n(\{\omega_1^1, \ldots, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_n^2\})$, $n = \overline{1, N}$, satisfying the conditions (20) - (22), is a probability measure, being equivalent to the measure P_N.

This proves Lemma 2.

Note 1. *Assume that for* $\alpha_n(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\})$, *constructed in Lemma 1, the inequalities*

$$0 < c_n \leq \alpha_n(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\}) \leq C_n < \infty,$$

are true. Suppose that the conditions

$$\Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n) \leq B_n < \infty, \quad n = \overline{1, N}, \tag{40}$$

are valid, where c_n, C_n, B_n *are constant, then the set of equivalent measures to the measure* P_N*, described in Lemma 2, is nonempty one.*

Proof. Due to Lemma 2 conditions, the equality (20) is true. Further,

$$\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1)\chi_{\Omega_n^+}(\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2)\times$$

$$\alpha_n(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)\leq B_n,$$

$$(\{\omega_1^1,\dots,\omega_{n-1}^1\};\{\omega_1^2,\dots,\omega_{n-1}^2\})\in\Omega_{n-1}\times\Omega_{n-1},\quad (\omega_1,\dots,\omega_{n-1})\in\Omega_{n-1},$$

$$\int\limits_{\Omega_n^0\times\Omega_n^0} \chi_{\Omega_n^-}(\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1)\chi_{\Omega_n^+}(\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2)\times$$

$$\alpha_n(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)=1,$$

$$(\{\omega_1^1,\dots,\omega_{n-1}^1\};\{\omega_1^2,\dots,\omega_{n-1}^2\})\in\Omega_{n-1}\times\Omega_{n-1}. \tag{41}$$

The last inequality and the equality (41) means that the conditions (20) - (22) are satisfied. Note 1 is proved.

For a nonnegative random value $f_N(\omega_1,\dots,\omega_N)$ let us define the integral relative to the measure $\mu_0(A)$, given by the formula

$$E^{\mu_0}f_N=\int\limits_{\Omega_N}\prod_{n=1}^N\psi_n(\omega_1,\dots,\omega_n)f_N(\omega_1,\dots,\omega_{N-1},\omega_N)\prod_{i=1}^N dP_i^0(\omega_i). \tag{42}$$

Theorem 1. *Suppose that the conditions of Lemma 1 are true. Then, the set of nonnegative random values* $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2), n=\overline{1,N}$, *satisfying the conditions*

$$E^{\mu_0}|\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)|=$$

$$\int\limits_{\Omega_N}\prod_{i=1}^N\psi_i(\omega_1,\dots,\omega_i)|\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)|\prod_{i=1}^N dP_i^0(\omega_i)<\infty,\quad n=\overline{1,N}, \tag{43}$$

is a nonempty one and the convex linear span of the set of measures (32), defined by the random values $\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})$, $n=\overline{1,N}$, *and satisfying the conditions (43), is a set of martingale measures, being equivalent to the measure* P_N.

Proof. Taking into account the equality (38), the right hand side of equality (43) can be written in the form

$$\int\limits_{\Omega_N} \prod_{i=1}^{N} \psi_i(\omega_1, \ldots, \omega_i) |\Delta S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)| \prod_{i=1}^{N} dP_i^0(\omega_i) =$$

$$\int\limits_{\Omega_n} \prod_{i=1}^{n} \psi_i(\omega_1, \ldots, \omega_i) |\Delta S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)| \prod_{i=1}^{n} dP_i^0(\omega_i) =$$

$$2 \int\limits_{\Omega_{n-1}} \prod_{i=1}^{n-1} \psi_i(\omega_1, \ldots, \omega_i) \int\limits_{\Omega_n^0} \int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) \chi_{\Omega_n^+}(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \times$$

$$\alpha_n(\{\omega_1, \ldots, \omega_{n-1}, \omega_n^1\}; \{\omega_1, \ldots, \omega_{n-1}, \omega_n^2\}) \times$$

$$\frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} \times$$

$$dP_n^0(\omega_n^1) dP_n^0(\omega_n^2) \prod_{i=1}^{n-1} dP_i^0(\omega_i), \quad n = \overline{1, N}. \tag{44}$$

Since the conditions of Lemma 1 are true, then the set of bounded random values $\alpha_n(\{\omega_1^1, \ldots, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_n^2\})$, $n = \overline{1, N}$, satisfying the conditions (20) - (22), is nonempty one. From the equality (44) for the set of bounded random values $\alpha_n(\{\omega\}_n^1; \{\omega\}_n^2)$, $n = \overline{1, N}$, figuring in Lemma 1, we obtain the inequality

$$\int\limits_{\Omega_N} \prod_{i=1}^{N} \psi_i(\omega_1, \ldots, \omega_i) |\Delta S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)| \prod_{i=1}^{N} dP_i^0(\omega_i) \leq$$

$$\prod_{n=1}^{N} \frac{2c_2}{c_1 P_n^0(\Omega_n^-) P_n^0(\Omega_n^+)} \int\limits_{\Omega_N} \Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) dP_N < \infty, \quad n = \overline{1, N}. \tag{45}$$

This proves that the set of nonnegative random values $\alpha_n(\{\omega_1^1, \ldots, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_n^2\})$, $n = \overline{1, N}$, satisfying the conditions (43), is a non empty set.

Let us prove that

$$\int\limits_{\Omega_n^0} \psi_n(\omega_1, \ldots, \omega_n) \Delta S_n(\omega_1, \ldots, \omega_n) dP_n^0(\omega_n) = 0,$$

$$(\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1}, \quad n = \overline{1, N}. \tag{46}$$

Really,

$$\int\limits_{\Omega_n^0} \psi_n(\omega_1, \ldots, \omega_n) \Delta S_n(\omega_1, \ldots, \omega_n) dP_n^0(\omega_n) =$$

$$\int_{\Omega_n^0}\int_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\alpha_n(\{\omega_1,\ldots,\omega_{n-1},\omega_n^1\};\{\omega_1,\ldots,\omega_{n-1},\omega_n^2\})\times$$

$$\Big[-\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)+$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\Big]\, dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)=0, \qquad (47)$$

due to the condition (21).

To complete the proof of Theorem 1, let A belong to the filtration $\mathcal{F}_{n-1}$, then $A = B\times\prod_{i=n}^{N}\Omega_i^0$, where B belongs to the σ-algebra $\mathcal{F}_{n-1}$ of the measurable space $\{\Omega_{n-1},\mathcal{F}_{n-1}\}$. Taking into account the equality (39), (47), we have, due to Foubini theorem,

$$\int_{\Omega_N}\prod_{i=1}^{N}\psi_i(\omega_1,\ldots,\omega_i)\chi_A(\omega_1,\ldots,\omega_N)\Delta S_n(\omega_1,\ldots,\omega_n)\prod_{i=1}^{N}dP_i^0(\omega_i)=$$

$$\int_{\Omega_n}\prod_{i=1}^{n}\psi_i(\omega_1,\ldots,\omega_i)\chi_B(\omega_1,\ldots,\omega_{n-1})\Delta S_n(\omega_1,\ldots,\omega_n)\prod_{i=1}^{n}dP_i^0(\omega_i)=$$

$$\int_{\Omega_{n-1}}\prod_{i=1}^{n-1}\psi_i(\omega_1,\ldots,\omega_i)\chi_B(\omega_1,\ldots,\omega_{n-1})\prod_{i=1}^{n-1}dP_i^0(\omega_i)\times$$

$$\int_0\psi_n(\omega_1,\ldots,\omega_n)\Delta S_n(\omega_1,\ldots,\omega_n)dP_n^0(\omega_n)=0. \qquad (48)$$

The last means that $E^{\mu_0}\{S_n(\omega_1,\ldots,\omega_n)|\mathcal{F}_{n-1}\} = S_{n-1}(\omega_1,\ldots,\omega_{n-1})$. Since every measure, belonging to the convex linear span of the measures considered above, is a finite sum of such measures, then it is a martingale measure, being equivalent to the measure P_N. Theorem 1 is proved.

Our aim is to describe this convex span of martingale measures.

IV. INTEGRAL REPRESENTATION FOR MARTINGALE MEASURES

In this section we consider the spot measures $\mu_{\{\omega_n^1,\omega_n^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$, introduced in [2]. Let us consider the random values

$$\psi_n(\omega_1,\ldots,\omega_n)=\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n)\psi_n^1(\omega_1,\ldots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n)\psi_n^2(\omega_1,\ldots,\omega_n), \qquad (49)$$

where

$$\psi_n^1(\omega_1,\dots,\omega_{n-1},\omega_n)=\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\times$$

$$\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)},\quad (\omega_1,\dots,\omega_{n-1})\in\Omega_{n-1}, \tag{50}$$

$$\psi_n^2(\omega_1,\dots,\omega_{n-1},\omega_n)=\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\times$$

$$\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)},\quad (\omega_1,\dots,\omega_{n-1})\in\Omega_{n-1}. \tag{51}$$

Definition 1. *Let the evolution of risky asset, given by the formula (1), satisfies the conditions (6) - (8). On the measurable space* $\{\prod_{i=1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}],\prod_{i=1}^{N}[\mathcal{F}_i^{0-}\times\mathcal{F}_i^{0+}]\}$, *being the direct product of the measurable spaces* $\{\Omega_i^{0-}\times\Omega_i^{0+},\mathcal{F}_i^{0-}\times\mathcal{F}_i^{0+}\}$, *for every point* $\{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}\}\in\prod_{i=1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]$ *let us introduce the set of spot measures (see also [2])*

$$\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)=$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\chi_A(\omega_1^{i_1},\dots,\omega_N^{i_N}),\quad A\in\mathcal{F}_N, \tag{52}$$

where $\psi_n(\omega_1,\dots,\omega_n)$ *is determined by the formulas (49) - (51).*

Let us define the integral for the random value $f_N\ (\omega_1,\dots,\omega_{N-1},\omega_N)$ relative to the measure $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ by the formula

$$\int_{\Omega_N} f_N(\omega_1,\dots,\omega_{N-1},\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}=$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})f_N(\omega_1^{i_1},\dots,\omega_N^{i_N}). \tag{53}$$

To describe the convex set of equivalent martingale measures, we introduce the family of α-spot measures, depending on the point $(\{\omega_1^1,\{\omega_1^2\},\dots,\{\omega_N^1,\{\omega_N^2\})$ belonging to $\prod_{i=1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]$ and the set of strictly positive random values

$$\alpha_n(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\}),\quad n=\overline{1,N}, \tag{54}$$

at points $W_n=(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})$, being constructed by the point $(\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\})$.

Let us determine the random values

$$\psi_n^\alpha(\omega_1,\dots,\omega_n)=\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n)\psi_n^{1,\alpha}(\omega_1,\dots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n)\psi_n^{2,\alpha}(\omega_1,\ldots,\omega_n), \tag{55}$$

$$\psi_n^{1,\alpha}(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\int\limits_{\Omega_n^0} \chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^2), \quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}, \tag{56}$$

$$\psi_n^{2,\alpha}(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})\times$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1), \quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}. \tag{57}$$

Let us define the set of the measures on the σ-algebra $\mathcal{F}_N$ by the formula

$$\mu_0(A) = \int\limits_{\prod_{i=1}^N[\Omega_i^{0-}\times\Omega_i^{0+}]} \prod_{i=1}^N \alpha_i(\{\omega_1^1,\ldots,\omega_i^1\};\{\omega_1^2,\ldots,\omega_i^2\})\times$$

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)\prod_{i=1}^N d[P_i^0\times P_i^0], \ A\in\mathcal{F}_N. \tag{58}$$

Theorem 2. *Suppose that the strictly positive random value*

$$\alpha_n(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\}), \ n=\overline{1,N}, \tag{59}$$

given on the measurable space $\{\prod_{i=1}^N[\Omega_i^{0-}\times\Omega_i^{0+}],\prod_{i=1}^N[\mathcal{F}_i^{0-}\times\mathcal{F}_i^{0+}]\}$, *satisfies the conditions of Lemma 1, then for the measure* $\mu_0(A)$, *given by the formula (58), the representation*

$$\mu_0(A) =$$

$$\int\limits_{\Omega_N}\prod_{i=1}^N\psi_i^\alpha(\omega_1,\ldots,\omega_i)\chi_A(\omega_1,\ldots,\omega_N)dP_N \tag{60}$$

is true.

Proof. Due to Lemma 1, the set of random values $\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})$, $n=\overline{1,N}$, satisfying the conditions (20) - (22), is a non empty set. Introduce into consideration the sequence of measures

$$\mu_{n-1}^{\omega_1,\dots\omega_{n-1}}(A)=\int\limits_{\prod_{i=n}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]}\prod_{i=n}^{N}\alpha_i(\{\omega_1^1,\dots,\omega_i^1\};\{\omega_1^2,\dots,\omega_i^2\})\times$$

$$\sum_{i_n=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{j=n}^{N}\psi_j(\omega_1,\dots\omega_{n-1},\omega_n^{i_n},\dots,\omega_j^{i_j})\chi_A(\omega_1,\dots\omega_{n-1},\omega_n^{i_n},\dots,\omega_N^{i_N})\times$$

$$dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)\dots dP_N^0(\omega_N^1)dP_N^0(\omega_N^2),\quad n=\overline{1,N},\tag{61}$$

and find the recurrent relations between them. Using Fubini Theorem, we have

$$\mu_{n-1}^{\omega_1,\dots\omega_{n-1}}(A)=$$

$$\int\limits_{\Omega_n^{0-}}\int\limits_{\Omega_n^{0+}}dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})\sum_{i_n=1}^{2}\psi_n(\omega_1,\dots\omega_{n-1},\omega_n^{i_n})$$

$$\int\limits_{\prod_{i=n+1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]}\prod_{i=n+1}^{N}\alpha_i(\{\omega_1^1,\dots,\omega_i^1\};\{\omega_1^2,\dots,\omega_i^2\})\times$$

$$\sum_{i_{n+1}=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{j=n+1}^{N}\psi_j(\omega_1,\dots\omega_{n-1},\omega_n^{i_n},\dots,\omega_j^{i_j})\chi_A(\omega_1,\dots\omega_{n-1},\omega_n^{i_n},\dots,\omega_N^{i_N})\times$$

$$dP_{n+1}^0(\omega_{n+1}^1)dP_{n+1}^0(\omega_{n+1}^2)\dots dP_N^0(\omega_N^1)dP_N^0(\omega_N^2)=$$

$$\int\limits_{\Omega_n^{0-}}\int\limits_{\Omega_n^{0+}}\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})\sum_{i_n=1}^{2}\psi_n(\omega_1,\dots\omega_{n-1},\omega_n^{i_n})\times$$

$$\mu_n^{\omega_1,\dots\omega_{n-1},\omega_n^{i_n}}(A)dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)=$$

$$\int\limits_{\Omega_n^{0-}}\int\limits_{\Omega_n^{0+}}\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})\psi_n(\omega_1,\dots\omega_{n-1},\omega_n^1)\times$$

$$\mu_n^{\omega_1,\dots\omega_{n-1},\omega_n^1}(A)dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)+$$

$$\int\limits_{\Omega_n^{0-}}\int\limits_{\Omega_n^{0+}}\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})\psi_n(\omega_1,\dots\omega_{n-1},\omega_n^2)\times$$

$$\mu_n^{\omega_1,\dots\omega_{n-1},\omega_n^2}(A)dP_n^0(\omega_n^1)dP_n^0(\omega_n^2).\tag{62}$$

In accordance with the formulas (49) - (51), for $\psi_n(\omega_1,\ldots,\omega_n)$ we have

$$\psi_n(\omega_1,\ldots,\omega_n^1)=\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\psi_n^1(\omega_1,\ldots,\omega_n^1)+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\psi_n^2(\omega_1,\ldots,\omega_n^1)=$$

$$\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\times$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}=$$

$$\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}. \qquad (63)$$

Further,

$$\psi_n(\omega_1,\ldots,\omega_n^2)=\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\psi_n^1(\omega_1,\ldots,\omega_n^2)+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\psi_n^2(\omega_1,\ldots,\omega_n^2)=$$

$$\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\times$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}=$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\times$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}. \qquad (64)$$

Substituting (63), (64) into (62), we obtain the recurrent relations

$$\mu_{n-1}^{\omega_1,\ldots\omega_{n-1}}(A)=$$

$$\int\limits_{\Omega_n^{0-}}\int\limits_{\Omega_n^{0+}}\alpha_n(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\})\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{\omega_1,\ldots\omega_{n-1},\omega_n^1}(A)+\right.$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{\omega_1,\ldots\omega_{n-1},\omega_n^2}(A)\Bigg] dP_n^0(\omega_n^1)dP_n^0(\omega_n^2). \tag{65}$$

To prove Theorem 2, we need to prove that the recurrent relations for

$$\mu_n^{\omega_1\ldots\omega_n}(A) =$$

$$\int\limits_{\prod\limits_{k=n+1}^{N}\Omega_k^0} \prod_{i=n+1}^{N} \psi_i^\alpha(\omega_1,\ldots,\omega_i)\chi_A(\omega_1,\ldots,\omega_N) \prod_{i=n+1}^{N} dP_i^0(\omega_i^0)$$

are the same as (65). Really,

$$\mu_n^{\omega_1\ldots\omega_n}(A) =$$

$$\int\limits_{\Omega_{n+1}^0} dP_{n+1}^0(\omega_{n+1})\psi_{n+1}^\alpha(\omega_1,\ldots,\omega_n,\omega_{n+1})\times$$

$$\int\limits_{\prod\limits_{k=n+2}^{N}\Omega_k^0} \prod_{i=n+2}^{N} \psi_i^\alpha(\omega_1,\ldots,\omega_i)\chi_A(\omega_1,\ldots,\omega_N) \prod_{i=n+2}^{N} dP_i^0(\omega_i^0)dP_{n+1}^0(\omega_n^0) =$$

$$\int\limits_{\Omega_{n+1}^0} \psi_{n+1}^\alpha(\omega_1,\ldots,\omega_n,\omega_{n+1})\mu_{n+1}^{\omega_1\ldots\omega_n,\omega_{n+1}}(A)dP_{n+1}^0(\omega_{n+1}). \tag{66}$$

Substituting (55) - (57) into (66), we obtain

$$\mu_n^{\omega_1\ldots\omega_n}(A) =$$

$$\int\limits_{\Omega_{n+1}^{0-}} \chi_{\Omega_{n+1}^-}(\omega_1,\ldots,\omega_n,\omega_{n+1}^1)\psi_{n+1}^{1,\alpha}(\omega_1,\ldots,\omega_n,\omega_{n+1}^1)\mu_{n+1}^{\omega_1\ldots\omega_n,\omega_{n+1}^1}(A)dP_{n+1}^0(\omega_{n+1}^1)+$$

$$\int\limits_{\Omega_{n+1}^{0+}} \chi_{\Omega_{n+1}^+}(\omega_1,\ldots,\omega_n,\omega_{n+1}^2)\psi_{n+1}^{2,\alpha}(\omega_1,\ldots,\omega_n,\omega_{n+1}^2)\mu_{n+1}^{\omega_1\ldots\omega_n,\omega_{n+1}^2}(A)dP_{n+1}^0(\omega_{n+1}^2) =$$

$$\int\limits_{\Omega_{n+1}^{0-}} \chi_{\Omega_{n+1}^-}(\omega_1,\ldots,\omega_n,\omega_{n+1}^1) \int\limits_{\Omega_{n+1}^{0+}} \chi_{\Omega_{n+1}^+}(\omega_1,\ldots,\omega_n,\omega_{n+1}^2)\times$$

$$\alpha_{n+1}(\{\omega_1^1,\ldots,\omega_{n+1}^1\};\{\omega_1^2,\ldots,\omega_{n+1}^2\})\frac{\Delta S_{n+1}^+(\omega_1,\ldots,\omega_n,\omega_{n+1}^2)}{V_{n+1}(\omega_1,\ldots,\omega_n,\omega_{n+1}^1,\omega_{n+1}^2)}\times$$

$$\mu_{n+1}^{\omega_1\ldots\omega_n,\omega_{n+1}^1}(A)dP_{n+1}^0(\omega_{n+1}^1)dP_{n+1}^0(\omega_{n+1}^2)+$$

$$\int\limits_{\Omega_{n+1}^{0+}} \chi_{\Omega_{n+1}^+}(\omega_1,\ldots,\omega_n,\omega_{n+1}^2) \int\limits_{\Omega_{n+1}^{0-}} \chi_{\Omega_{n+1}^-}(\omega_1,\ldots,\omega_n,\omega_{n+1}^1)\times$$

$$\alpha_{n+1}(\{\omega_1^1,\ldots,\omega_{n+1}^1\};\{\omega_1^2,\ldots,\omega_{n+1}^2\})\frac{\Delta S_{n+1}^-(\omega_1,\ldots,\omega_n,\omega_{n+1}^1)}{V_{n+1}(\omega_1,\ldots,\omega_n,\omega_{n+1}^1,\omega_{n+1}^2)}\times$$
$$\mu_{n+1}^{\omega_1\ldots\omega_n,\omega_{n+1}^2}(A)dP_{n+1}^0(\omega_{n+1}^1)dP_{n+1}^0(\omega_{n+1}^2).$$

So, we obtained the recurrent relations

$$\mu_n^{\omega_1\ldots\omega_n}(A)=$$
$$\int\limits_{\Omega_{n+1}^{0-}}\int\limits_{\Omega_{n+1}^{0+}}\chi_{\Omega_{n+1}^-}(\omega_1,\ldots,\omega_n,\omega_{n+1}^1)\chi_{\Omega_{n+1}^+}(\omega_1,\ldots,\omega_n,\omega_{n+1}^2)\times$$
$$\alpha_{n+1}(\{\omega_1^1,\ldots,\omega_{n+1}^1\};\{\omega_1^2,\ldots,\omega_{n+1}^2\})\left[\frac{\Delta S_{n+1}^+(\omega_1,\ldots,\omega_n,\omega_{n+1}^2)}{V_{n+1}(\omega_1,\ldots,\omega_n,\omega_{n+1}^1,\omega_{n+1}^2)}\mu_{n+1}^{\omega_1\ldots\omega_n,\omega_{n+1}^1}(A)+\right.$$
$$\left.\frac{\Delta S_{n+1}^-(\omega_1,\ldots,\omega_n,\omega_{n+1}^1)}{V_{n+1}(\omega_1,\ldots,\omega_n,\omega_{n+1}^1,\omega_{n+1}^2)}\mu_{n+1}^{\omega_1\ldots\omega_n,\omega_{n+1}^2}(A)\right]dP_{n+1}^0(\omega_{n+1}^1)dP_{n+1}^0(\omega_{n+1}^2), \qquad (67)$$

which are the same as (65). Theorem 2 is proved.

Theorem 3. *Suppose that the conditions of Lemma 1 are true. Then, the set of strictly positive random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2), n=\overline{1,N}$, satisfying the conditions*

$$E^{\mu_0}|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)|=$$
$$\int\limits_{\Omega_N}\prod_{i=1}^N\psi_i^\alpha(\omega_1,\ldots,\omega_i)|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)|\prod_{i=1}^N dP_i^0(\omega_i)<\infty, \quad n=\overline{1,N}, \qquad (68)$$

is a non empty set for the measures $\mu_0(A)$, given by the formula (60). The measure $\mu_0(A)$, constructed by the strictly positive random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2), n=\overline{1,N}$, satisfying the conditions (68) is a martingale measure for the evolution of risky asset, given by the formula (1). Every measure, belonging to the convex linear span of such measures, is also martingale measure for the evolution of risky asset, given by the formula (1). They are equivalent to the measure P_N. The set of spot measures $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ is a set of martingale measures for the evolution of risky asset, given by the formula (1).

Proof. The first fact, that the set of random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2), n=\overline{1,N}$, satisfying the conditions (68), is a non empty one, follows from Lemma 1. From the representation for the set of measures $\mu_0(A)$, given by the formula (60), as in the proof of Theorem 1, it is proved that this set of measures is a set of martingale measures, being equivalent to the measure P_N.

Let us prove the last statement of Theorem 3. Since for the spot measure $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ the representation

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)=$$

$$\sum_{i_1=1}^{2}\cdots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\chi_A(\omega_1^{i_1},\ldots,\omega_N^{i_N}), \quad A\in\mathcal{F}_N, \tag{69}$$

is true, let us calculate $\sum\limits_{i_j=1}^{2}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})$. We have

$$\sum_{i_j=1}^{2}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j}) = \psi_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1) + \psi_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2) =$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\psi_j^1(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^1)+$$

$$\chi_{\Omega_n^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\psi_j^2(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^1)+$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\psi_j^1(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^2)+$$

$$\chi_{\Omega_n^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\psi_j^2(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^2) =$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)}+$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)} =$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)} =$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2) = 1,$$

if $\omega_j^1\in\Omega_j^{0-}, \omega_j^2\in\Omega_j^{0+},\ j=\overline{1,N}$.

So, for every point $\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_2^2\}\} \in \prod_{j=1}^{N} [\Omega_j^{0-} \times \Omega_j^{0+}]$ the spot measure (69) is nonzero probability measure on the σ-algebra $\mathcal{F}_N$. Further,

$$\sum_{i_j=1}^{2} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \Delta S_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) =$$

$$\psi_j(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^1) \Delta S_j(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^1) +$$
$$\psi_j(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^2) \Delta S_j(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^2) =$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^1) \chi_{\Omega_j^+}(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^2) \times$$

$$\left[-\frac{\Delta S_j^+(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^2)}{V_j(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^1, \omega_j^2)} \Delta S_j^-(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^1) + \right.$$

$$\left. \frac{\Delta S_j^-(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^1)}{V_j(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^1, \omega_j^1)} \Delta S_j^+(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^2) \right] = 0, \quad j = \overline{1, N}. \tag{70}$$

Let us prove that the set of measures $\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}(A)$ is a set of martingale measures. Really, for A, belonging to the σ-algebra $\mathcal{F}_{n-1}$ of the filtration, we have $A = B \times \prod_{i=n}^{N} \Omega_i^0$, where B belongs to σ-algebra $\mathcal{F}_{n-1}$ of the measurable space $\{\Omega_{n-1}, \mathcal{F}_{n-1}\}$. Then,

$$\int_A \Delta S_n(\omega_1, \ldots, \omega_n) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} =$$

$$\sum_{i_1=1}^{2} \cdots \sum_{i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \chi_B(\omega_1^{i_1}, \ldots, \omega_{n-1}^{i_{n-1}}) \Delta S_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) =$$

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{j=1}^{n} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \chi_B(\omega_1^{i_1}, \ldots, \omega_{n-1}^{i_{n-1}}) \Delta S_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) =$$

$$\sum_{i_1=1}^{2} \cdots \sum_{i_{n-1}=1}^{2} \prod_{j=1}^{n-1} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \chi_B(\omega_1^{i_1}, \ldots, \omega_{n-1}^{i_{n-1}}) \times$$

$$\sum_{i_n=1}^{2} \psi_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) \Delta S_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) = 0, \quad A \in \mathcal{F}_{n-1}. \tag{71}$$

The last means the needed statement. Theorem 3 is proved.

Below, in Theorem 4, we present the consequence of Theorems 2, 3.
Let us introduce the denotations

$$\gamma_n(\{\omega\}_N^1; \{\omega\}_N^2) = \prod_{i=n}^{N} \chi_{\Omega_i^-}(\{\omega_1^1, \ldots, \omega_i^1\})\chi_{\Omega_i^+}(\{\omega_1^2, \ldots, \omega_i^2\}), \quad n = \overline{1, N}. \tag{72}$$

From the assumptions (6) - (8), it follows that

$$\gamma_n(\{\omega\}_N^1; \{\omega\}_N^2) = \prod_{i=n}^{N} \chi_{\Omega_i^{0-}}(\omega_i^1)\chi_{\Omega_i^{0+}}(\omega_i^2), \quad n = \overline{1, N}. \tag{73}$$

We also use the denotations

$$\Gamma_N = \{\{\{\omega\}_N^1, \{\omega\}_N^2\} \in \prod_{i=1}^{N}[\Omega_i^0 \times \Omega_i^0], \gamma_1(\{\omega\}_N^1, \{\omega\}_N^2) = 1\}, \tag{74}$$

$$\mu_N = \{\{\{\omega\}_N^1, \{\omega\}_N^2\} \in \prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}], \ \mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(\Omega_N) = 1\}. \tag{75}$$

. From the construction of spot measures $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ and assumptions (6) - (8), it follows that these sets (74), (75) coincide.

Theorem 4. *Let the evolution of risky asset, given by the formula (1), satisfy the conditions (6) - (8). Suppose that the random value $\alpha_N(\{\omega\}_N^1; \{\omega\}_N^2)$, given on the measurable space $\{\prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}], \prod_{i=1}^{N}[\mathcal{F}_i^{0-} \times \mathcal{F}_i^{0+}]\}$, satisfies the conditions*

$$0 < c_N \leq \alpha_N(\{\omega_1^1, \ldots, \omega_N^1\}; \{\omega_1^2, \ldots, \omega_N^2\}) \leq C_N < \infty. \tag{76}$$

If

$$\int\limits_{\prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N(\{\omega_1^1, \ldots, \omega_N^1\}; \{\omega_1^2, \ldots, \omega_N^2\}) \times$$

$$\prod_{i=1}^{N} dP_i^0(\omega_i^1) dP_i^0(\omega_i^2) = 1, \tag{77}$$

then the measure $\mu_0(A)$, given by the formula (78)

$$\mu_0(A) = \int\limits_{\prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N(\{\omega_1^1, \ldots, \omega_N^1\}; \{\omega_1^2, \ldots, \omega_N^2\}) \times$$

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A) \prod_{i=1}^{N} d[P_i^0(\omega_i^1) \times P_i^0(\omega_i^2)], \tag{78}$$

is a martingale measure, being equivalent to the measure P_N.

Proof. It is evident that

$$\int\limits_{\Omega_n^0\times\Omega_n^0} \chi_{\Omega_n^-}(\{\omega_1^1,\dots,\omega_n^1\})\chi_{\Omega_n^+}(\{\omega_1^2,\dots,\omega_n^2\})\times$$

$$\alpha_n^1(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)=1,$$

where

$$\alpha_N^1(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})=$$

$$\frac{\alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\})}{\int\limits_{\prod_{i=N}^{N}[\Omega_i^0\times\Omega_i^0]}\gamma_N(\{\omega\}_N^1;\{\omega\}_N^2)\alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\})\prod_{i=N}^{N}dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)},$$

$$\alpha_n^1(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})=$$

$$\frac{\int\limits_{\prod_{i=n+1}^{N}[\Omega_i^0\times\Omega_i^0]}\gamma_{n+1}(\{\omega\}_N^1;\{\omega\}_N^2)\alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\})\prod_{i=n+1}^{N}dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)}{\int\limits_{\prod_{i=n}^{N}[\Omega_i^0\times\Omega_i^0]}\gamma_n(\{\omega\}_N^1;\{\omega\}_N^2)\alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\})\prod_{i=n}^{N}dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)},$$

$$n=\overline{1,N-1}. \tag{79}$$

The set of positive random values $\alpha_n^1(\{\omega\}_n^1;\{\omega\}_n^2), n=\overline{1,N}$, given by the formula (79), are bounded as from below and above. Really,

$$\alpha_n^1(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})\leq$$

$$\frac{C_N}{c_N}\frac{\int\limits_{\prod_{i=n+1}^{N}[\Omega_i^0\times\Omega_i^0]}\gamma_{n+1}(\{\omega\}_N^1;\{\omega\}_N^2)\prod_{i=n+1}^{N}dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)}{\int\limits_{\prod_{i=n}^{N}[\Omega_i^0\times\Omega_i^0]}\gamma_n(\{\omega\}_N^1;\{\omega\}_N^2)\prod_{i=n}^{N}dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)}=$$

$$\frac{C_N}{c_N P_n^0(\Omega_n^{0-})P_n^0(\Omega_n^{0+})}<\infty.$$

Further,

$$\alpha_n^1(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})\geq$$

$$\frac{c_N}{C_N}\frac{\int\limits_{\prod\limits_{i=n+1}^{N}[\Omega_i^0\times\Omega_i^0]}\gamma_{n+1}(\{\omega\}_N^1;\{\omega\}_N^2)\prod\limits_{i=n+1}^{N}dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)}{\int\limits_{\prod\limits_{i=n}^{N}[\Omega_i^0\times\Omega_i^0]}\gamma_n(\{\omega\}_N^1;\{\omega\}_N^2)\prod\limits_{i=n}^{N}dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)}=$$

$$\frac{c_N}{P_n^0(\Omega_n^{0-})P_n^0(\Omega_n^{0+})C_N}>0.$$

Therefore, they satisfy the conditions

$$E^{\mu_0}|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)|=$$

$$\int\limits_{\Omega_N}\prod_{i=1}^{N}\psi_i(\omega_1,\ldots,\omega_i)|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)|\prod_{i=1}^{N}dP_i^0(\omega_i)<\infty,\quad n=\overline{1,N}. \qquad (80)$$

The boundedness of random values $\alpha_n^1(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})$, $n=\overline{1,N}$, means that they satisfy the conditions (20) - (22). It is evident that

$$\alpha_N(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\})=\prod_{n=1}^{N}\alpha_n^1(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\}). \qquad (81)$$

Owing to Theorem 3, $\mu_0(A)$, given by the formula (78), is a martingale measure, being equivalent to the measure P_N. Theorem 4 is proved.

Theorem 5. *Let the conditions of Theorem 4 be true. If the contingent claim $f_N=f_N(\omega_1,\ldots,\omega_N)$ satisfies the condition*

$$\sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}}\int\limits_{\Omega_N}f_N(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}<\infty,$$

then the equalities

$$\inf_{P\in M_b}E^Pf_N=\inf_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}}\int\limits_{\Omega_N}f_N(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}, \qquad (82)$$

$$\sup_{P\in M_b}E^Pf_N=\sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}}\int\limits_{\Omega_N}f_N(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1\omega_N^2\}}, \qquad (83)$$

are true, where M_b is the set of all martingale measures, figuring in Theorem 4, with $\alpha_N(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\})$ running all nonnegative bounded as from below and above the random values.

Proof. The inequality

$$\inf_{P\in M_b} E^P f_N \leq (1-\alpha)E^Q f_N + \alpha \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}, \quad 0<\alpha<1,$$

is valid for all $0<\alpha<1$, since $(1-\alpha)Q+\alpha\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}$ is a martingale measure, being equivalent to P_N. Tending α to one, we have

$$\inf_{P\in M_b} E^P f_N \leq \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}},$$

or

$$\inf_{P\in M_b} E^P f_N \leq \inf_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}.$$

To prove the inverse inequality, we use the representation

$$E^Q f_N =$$

$$\int\limits_{\prod_{i=1}^N [\Omega_i^{0-}\times\Omega_i^{0+}]} \alpha_N(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\})\times$$

$$\int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \prod_{i=1}^N d[P_i^0(\omega_i^1)\times P_i^0(\omega_i^2)]. \quad (84)$$

Using the representation (84), we obtain the inequality

$$E^Q f_N \geq \inf_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}. \quad (85)$$

Taking into account the inequality (85), we obtain the inequality

$$\inf_{Q\in M_b} E^Q f_N \geq \inf_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}. \quad (86)$$

This proves the equality (82). As before, $(1-\alpha)Q+\alpha\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}$ is a martingale measure, being equivalent to P_N, therefore the inequality

$$\sup_{P\in M_b} E^P f_N \geq (1-\alpha)E^Q f_N + \alpha \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}, \quad 0<\alpha<1,$$

is true. Tending α to one, we have

$$\sup_{P\in M_b} E^P f_N \geq \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}},$$

or

$$\sup_{P \in M_b} E^P f_N \geq \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\} \in \mu_N\}} \int_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}.$$

To prove the inverse inequality, we use the representation

$$E^Q f_N =$$

$$\int_{\prod_{i=1}^N [\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\}) \times$$

$$\int_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \prod_{i=1}^N d[P_i^0(\omega_i^1) \times P_i^0(\omega_i^2)]. \tag{87}$$

From (87) we have

$$E^Q f_N \leq \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\} \in \mu_N\}} \int_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}. \tag{88}$$

Taking into account the inequality (88), we obtain the inequality

$$\sup_{Q \in M_b} E^Q f_N \leq \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\} \in \mu_N\}} \int_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}. \tag{89}$$

This proves the equality (83). Theorem 5 is proved.

Let us introduce into the set of measure M_b the norm. If $P_1, P_2 \in M_b$, where

$$P_1(A) = \int_{\prod_{i=1}^N [\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N^1(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\}) \times$$

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A) \prod_{i=1}^N d[P_i^0(\omega_i^1) \times P_i^0(\omega_i^2)], \tag{90}$$

$$P_2(A) = \int_{\prod_{i=1}^N [\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N^2(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\}) \times$$

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A) \prod_{i=1}^N d[P_i^0(\omega_i^1) \times P_i^0(\omega_i^2)], \tag{91}$$

then we put

$$||P_1 - P_2|| = \int\limits_{\prod\limits_{i=1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]} |\alpha_N^1(\{\omega\}_N^1;\{\omega\}_N^2) - \alpha_N^2(\{\omega\}_N^1;\{\omega\}_N^2)|\times \prod_{i=1}^{N} d[P_i^0(\omega_i^1)\times P_i^0(\omega_i^2)]. \quad (92)$$

Denote M_0 the completion of the set M_b in the introduced metrics.

Theorem 6. *Let the conditions of Theorem 5 be true. Then, the equalities*

$$\inf_{P\in M_0} E^P f_N = \inf_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}, \quad (93)$$

$$\sup_{P\in M_0} E^P f_N = \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \quad (94)$$

are valid.

Proof. For arbitrary small $\varepsilon > 0$ there exists a measure $P_0 \in M_0$ such that $|\inf\limits_{P\in M_0} E^P f_N - E^{P_0} f_N| < \varepsilon$. Since $|E^{P_1} f_N - E^{P_2} f_N| \leq ||P_1 - P_2||$, then there exists a measure $P_n \in M_b$ such that $|E^{P_n} f_N - E^{P_0} f_N| \leq ||P_n - P_0|| < \varepsilon$. Due to the above inequalities, we have

$$\inf_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} = \inf_{P\in M_b} E^P f_N \geq$$

$$\inf_{P\in M_0} E^P f_N \geq -\varepsilon + E^{P_0} f_N \geq -2\varepsilon + E^{P_n} f_N \geq$$

$$-2\varepsilon + \inf_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}.$$

Since $\varepsilon > 0$ is arbitrary small we have the proof of (93).

Analogously, for arbitrary small $\varepsilon > 0$ there exists a measure $P_0 \in M_0$ such that $|\sup\limits_{P\in M_0} E^P f_N - E^{P_0} f_N| < \varepsilon$. Since $|E^{P_1} f_N - E^{P_2} f_N| \leq ||P_1 - P_2||$, then there exists a measure $P_n \in M_b$ such that $|E^{P_n} f_N - E^{P_0} f_N| \leq ||P_n - P_0|| < \varepsilon$. Due to the above inequalities, we have

$$\sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} = \sup_{P\in M_b} E^P f_N \leq$$

$$\sup_{P\in M_0} E^P f_N \leq \varepsilon + E^{P_0} f_N \leq 2\varepsilon + E^{P_n} f_N \leq$$

$$2\varepsilon + \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}.$$

Since $\varepsilon > 0$ is arbitrary small we have the proof of (94).

Denote $M \subseteq M_0$ the subset of all martingale measures from the set M_0, which are equivalent to P_N. As a consequence of Theorem 6, we obtain

Theorem 7. *Let the conditions of Theorem 5 be valid. Then, the equalities*

$$\inf_{P \in M} E^P f_N = \inf_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\} \in \mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}, \tag{95}$$

$$\sup_{P \in M} E^P f_N = \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\} \in \mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \tag{96}$$

are true.

Proof. The proof of Theorem 7 follows from the inclusions $M_b \subseteq M \subseteq M_0$ and Theorems 5, 6.

Theorem 8. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_n^0, \mathcal{F}_n^0, P_n^0\}$, let the evolution of risky asset be given by the formula (12), with $a_n(\omega_1,\ldots,\omega_n\ ) = b_n(\omega_1,\ldots,\omega_{n-1})$, $n = \overline{1,N}$,* where *the random variables $f_n(\omega_1,\ldots,\omega_n)$, $b_n(\omega_1,\ldots,\omega_{n-1})$, $\eta_n(\omega_n)$ satisfy the inequalities*

$$b_n(\omega_1,\ldots,\omega_{i-1},\omega_n) > 0, \quad f_n(\omega_1,\ldots,\omega_{n-1},\omega_n) > 0, \quad \sup_{\omega_n \in \Omega_n^0, \eta_n^-(\omega_n)>0} \eta_n^-(\omega_n) < \infty,$$

$$\sup_{\{\omega_1, \quad \ldots,\omega_n\} \in \Omega_n} b_n(\omega_1, \quad \ldots,\omega_{n-1},\omega_n) <$$

$$\frac{1}{\sup\limits_{\{\omega_1, \quad \ldots,\omega_n\} \in \Omega_n} f_n(\omega_1,\ldots,\omega_{n-1},\omega_n) \sup\limits_{\omega_n \in \Omega_n^0, \eta_n^-(\omega_i)>0} \eta_n^-(\omega_n)}, \quad n = \overline{1,N}. \tag{97}$$

For such an evolution, the family of martingale measures (78) described in Theorem 4 does not depend on the random variables $b_n(\omega_1,\ldots,\omega_{n-1})$, $n = \overline{1,N}$.

Proof. Due to the representation (78) for the measure $\mu_0(A)$ in Theorem 4, to prove Theorem 8, it needs to prove that all spot measures $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ do not depend on the random variables $b_n(\omega_1,\ldots,\omega_{n-1})$, $n = \overline{1,N}$. For this purpose, it is need to prove that $\psi_n(\omega_1,\ldots,\omega_n)$ do not depend on the random variables $b_n(\omega_1,\ldots,\omega_{n-1})$, $n = \overline{1,N}$, where

$$\psi_n(\omega_1,\ldots,\omega_n) = \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n)\, \psi_n^1(\omega_1,\ldots,\omega_n) +$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n)\, \psi_n^2(\omega_1,\ldots,\omega_n), \tag{98}$$

$$\psi_n^1(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2) \frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}, \tag{99}$$

$$\psi_n^2(\omega_1, \ldots, \omega_{n-1}, \omega_n) =$$

$$\chi_{\Omega_n^-}(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) \frac{\Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)}, \quad (\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1}. \tag{100}$$

It is evident that $\chi_{\Omega_n^-}(\omega_1, \ldots, \omega_{n-1}, \omega_n)$ and $\chi_{\Omega_n^+}(\omega_1, \ldots, \omega_{n-1}, \omega_n)$ do not depend on the random variables $b_n(\omega_1, \ldots, \omega_{n-1}),\ n = \overline{1, N}$, where
Since,

$$\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) =$$

$$S_{n-1}(\omega_1, \ldots, \omega_{n-1}) b_n(\omega_1, \ldots, \omega_{n-1}) f_n(\omega_1, \ldots, \omega_n^2) \eta_n^+(\omega_n^2), \tag{101}$$

$$\Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) =$$

$$S_{n-1}(\omega_1, \ldots, \omega_{n-1}) b_n(\omega_1, \ldots, \omega_{n-1}) f_n(\omega_1, \ldots, \omega_n^1) \eta_n^-(\omega_n^1), \tag{102}$$

we have

$$\frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} =$$

$$\frac{f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \eta_n^+(\omega_n^2)}{f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \eta_n^+(\omega_n^2) + f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) \eta_n^-(\omega_n^1)}, \tag{103}$$

$$\frac{\Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} =$$

$$\frac{f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) \eta_n^-(\omega_n^1)}{f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \eta_n^+(\omega_n^2) + f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) \eta_n^-(\omega_n^1)}, \tag{104}$$

$$(\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1}.$$

The equalities (103), (104) prove Theorem 8.

V. ASSESSMENT OF CONTINGENT CLAIM

In this section, we prove Theorems, giving us the formula for the fair price of super-hedge for the evolution of risky asset, given by the formula

$$S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) = S_0 \prod_{i=1}^{n} \left(1 + a_i \left(e^{\sigma_i(\omega_1, \ldots, \omega_{i-1}) \varepsilon_i(\omega_i)} - 1\right)\right), \quad n = \overline{1, N}, \tag{105}$$

where the random value $\varepsilon_i(\omega_i),\ \omega_i \in \Omega_i^0, i = \overline{1, N}$, takes all real values from R^1, $S_0 > 0$. The random values $\sigma_i(\omega_1, \ldots, \omega_{i-1})$ satisfy the inequalities $\sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0,\ 0 < a_i \leq 1, i = \overline{1, N}$. Due to Theorem 8, the set of equivalent martingale measures constructed by the evolution of risky asset, given by the formula (105), do not depend on parameters $0 < a_i \leq 1, i = \overline{1, N}$. The proposed parametric model based on the canonical model of the evolution of risky asset (9), which takes into account both memory and clustering, takes into account the fact that the price of a risky asset cannot fall to zero.

Theorem 9. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (105) and the conditions of Theorem 7 are true. Suppose that* $0 \leq a_i \leq 1$, $\infty > \sigma_i > \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $i = \overline{1, N}$. *If the nonnegative continuous payoff function* $f(x)$, $x \in [0, \infty)$, *satisfies the conditions:*
1) $f(0) = 0$, $f(x) \leq ax$, $\lim\limits_{x \to \infty} \frac{f(x)}{x} = a$, $a > 0$, *then the inequalities*

$$f\left(S_0 \prod_{i=1}^{N}(1 - a_i)\right) + aS_0\left(1 - \prod_{i=1}^{N}(1 - a_i)\right) \leq \sup_{P \in M} E^P f(S_N) \leq aS_0 \tag{106}$$

are true. If, in addition, the nonnegative payoff function $f(x)$ *is a convex down one, then*

$$\inf_{P \in M} E^P f(S_N) = f(S_0), \tag{107}$$

where M *is a set of equivalent martingale measures for the evolution of risky asset, given by the formula (105).*

Proof. Due to Theorem 7,

$$\sup_{P \in M} E^P f(S_N) = \sup_{\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\} \in \mu_N\}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}.$$

So, we have

$$aS_0 \geq \sup_{\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\} \in \mu_N\}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} =$$

$$\sup_{\{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, \ i = \overline{1,N}\}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} =$$

$$\sup_{\{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, \ i = \overline{1,N}\}} \sum_{i_1 = 1, \ldots, i_N = 1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N}\left(1 + a_s\left(e^{\sigma_s(\omega_1^{i_1}, \ldots, \omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right). \tag{108}$$

Further,

$$\sup_{\{\omega_N^1 \in \Omega_N^{0-}, \omega_N^2 \in \Omega_N^{0+}\}} \sum_{i_N = 1}^{2} \psi_N(\omega_1^{i_1}, \ldots, \omega_N^{i_N}) \times$$

$$f\left(S_0 \prod_{s=1}^{N}\left(1 + a_s\left(e^{\sigma_s(\omega_1^{i_1}, \ldots, \omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right) =$$

$$\sup_{\{\omega_N^1 \in \Omega_N^{0-}, \omega_N^2 \in \Omega_N^{0+}\}} \left[\frac{\Delta S_N^+(\omega_1^{i_1}, \ldots, \omega_{N-1}^{i_{N-1}}, \omega_N^2)}{V_N(\omega_1^{i_1}, \ldots, \omega_{N-1}^{i_{N-1}}, \omega_N^1, \omega_N^2)} \times\right.$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)+$$

$$\left.\frac{\Delta S_N^-(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)\right]\geq$$

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\left[\frac{\Delta S_N^+(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^2)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}\times\right.$$
$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)+$$

$$\left.\frac{\Delta S_N^-(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)\right]=$$

$$f(S_{N-1}(1-a_N))+aa_NS_{N-1}, \tag{109}$$

where we put

$$S_{N-1}=S_0\prod_{s=1}^{N-1}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right). \tag{110}$$

Really,

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\Delta S_N^+(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^2)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}\times$$
$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)=$$

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)}{\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}\right)}\times$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)=f(S_{N-1}(1-a_N)).$$

Further,

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\Delta S_N^-(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}\times$$
$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)=$$

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\left(1-e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}\right)}{\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}\right)}\times$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)=aa_NS_{N-1}.$$

Substituting the inequality (109) into (108), we obtain the inequality

$$\sup_{\{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},\ i=\overline{1,N}\}}\sum_{i_1=1,\dots,i_N=1}^{2}\sum_{i_1=1,\dots,i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)\geq$$

$$\sup_{\{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},\ i=\overline{1,N-1}}\sum_{i_1=1,\dots,i_{N-1}=1}^{2}\prod_{j=1}^{N-1}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\times$$

$$f\left(S_0(1-a_N)\prod_{s=1}^{N-1}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)+aa_NS_0. \tag{111}$$

Applying $(N-1)$ times the inequality (111), we obtain the inequality

$$\sup_{Q\in M}\int_{\Omega}f(S_N)dQ\geq f(S_0\prod_{i=1}^{N}(1-a_i))+aS_0\sum_{i=1}^{N}a_i\prod_{s=i+1}^{N}(1-a_s)=$$

$$f\left(S_0\prod_{i=1}^{N}(1-a_i)\right)+aS_0\left(1-\prod_{i=1}^{N}(1-a_i)\right). \tag{112}$$

Let us prove the equality (107). Using the Jensen inequality, we obtain

$$\inf_{P\in M}E^Pf(S_N)\geq f(S_0). \tag{113}$$

Let us prove the inverse inequality. The inequality

$$\sum_{i_1=1,\dots,i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)\geq\inf_{P\in M}E^Pf(S_N) \tag{114}$$

is true. If to put $\varepsilon_s(\omega_s^1)=0,\ s=\overline{1,N}$, then the inequality (114) turns into the inequality

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^2,\dots,\omega_{s-1}^2)\varepsilon_s(\omega_s^2)}-1\right)\right)\right)\geq\inf_{P\in M}E^Pf(S_N). \tag{115}$$

In the considered case $\Omega_i^{0-}=\{\omega_i\in\Omega_i^0,\varepsilon_i(\omega_i)\leq 0\}$, $\Omega_i^{0+}=\{\omega_i\in\Omega_i^0,\varepsilon_i(\omega_i)>0\}$. Since the value $\varepsilon_s(\omega_s^2)>0$ can be made as small as it needs for $\omega_s^2\in\Omega_s^{0+}$, then we can do the left side of the inequality (115) as close to $f(S_0)$ as it needs, since $\sigma_s(\omega_1^2,\dots,\omega_{s-1}^2)$ is bounded and $f(x)$ is a continuous one. The last proves the needed inequality. Theorem 9 is proved.

Theorem 10. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105) and the conditions of Theorem 7 are true. Suppose that $0 \leq a_i \leq 1$, $\infty > \sigma_i > \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $\quad i = \overline{1, N}$. If the nonnegative continuous payoff function $f(x)$, $x \in [0, \infty)$, satisfies the conditions:*
1) $f(0) = K$, $f(x) \leq K$, then

$$f\left(S_0 \prod_{i=1}^{N}(1-a_i)\right) \leq \sup_{P \in M} E^P f(S_N) \leq K. \tag{116}$$

If, in addition, the nonnegative payoff function $f(x)$ is a convex down one, then

$$\inf_{P \in M} E^P f(S_N) = f(S_0), \tag{117}$$

where M is the set of equivalent martingale measures for the evolution of risky asset, given by the formula (105).

Proof. Let us obtain the estimate from below. Due to Theorem 7,

$$\sup_{P \in M} E^P f_N = \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\} \in \mu_N\}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}.$$

So, we have

$$K \geq \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\} \in \mu_N\}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} =$$

$$\sup_{\{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+},\ i=\overline{1,N}\}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} =$$

$$\sup_{\{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+},\ i=\overline{1,N}\}} \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N}\left(1 + a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right). \tag{118}$$

Further,

$$\sup_{\{\omega_N^1 \in \Omega_N^{0-}, \omega_N^2 \in \Omega_N^{0+}\}} \sum_{i_N=1}^{2} \psi_N(\omega_1^{i_1}, \ldots, \omega_N^{i_N}) \times$$

$$f\left(S_0 \prod_{s=1}^{N}\left(1 + a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right) =$$

$$\sup_{\{\omega_N^1 \in \Omega_N^{0-}, \omega_N^2 \in \Omega_N^{0+}\}} \left[\frac{\Delta S_N^+(\omega_1^{i_1}, \ldots, \omega_{N-1}^{i_{N-1}}, \omega_N^2)}{V_N(\omega_1^{i_1}, \ldots, \omega_{N-1}^{i_{N-1}}, \omega_N^1, \omega_N^2)} \times\right.$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)+$$

$$\frac{\Delta S_N^-(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)\Bigg]\geq$$

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\left[\frac{\Delta S_N^+(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^2)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}\times\right.$$
$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)+$$

$$\frac{\Delta S_N^-(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)\Bigg]=$$

$$f(S_{N-1}(1-a_N)), \tag{119}$$

where we put

$$S_{N-1}=S_0\prod_{s=1}^{N-1}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right). \tag{120}$$

Really,

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\Delta S_N^+(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^2)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}\times$$
$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)=$$

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)}{\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}\right)}\times$$
$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)=f(S_{N-1}(1-a_N)).$$

Further,

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\Delta S_N^-(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}\times$$
$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)=$$

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\left(1-e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}\right)}{\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}\right)}\times$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)=0.$$

Substituting the inequality (119) into (118), we obtain the inequality

$$\sup_{\{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},\ i=\overline{1,N}\}}\sum_{i_1=1,\ldots,i_N=1}^{2}\sum_{i_1=1,\ldots,i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)\geq$$

$$\sup_{\{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},\ i=\overline{1,N-1}\}}\sum_{i_1=1,\ldots,i_{N-1}=1}^{2}\prod_{j=1}^{N-1}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\times$$

$$f\left(S_0(1-a_N)\prod_{s=1}^{N-1}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right). \tag{121}$$

Applying $(N-1)$ times the inequality (121), we obtain the inequality

$$\sup_{Q\in M}\int_{\Omega} f(S_N)dQ\geq f(S_0\prod_{i=1}^{N}(1-a_i)). \tag{122}$$

Let us prove the equality (117). Using the Jensen inequality, we obtain

$$\inf_{P\in M}E^Pf(S_N)\geq f(S_0). \tag{123}$$

Let us prove the inverse inequality. It is evident that the inequality

$$\sum_{i_1=1,\ldots,i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)\geq\inf_{P\in M}E^Pf(S_N) \tag{124}$$

is valid. If to put $\varepsilon_s(\omega_s^1)=0,\ \ s=\overline{1,N}$, then the inequality (124) turns into the inequality

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^2,\ldots,\omega_{s-1}^2)\varepsilon_s(\omega_s^2)}-1\right)\right)\right)\geq\inf_{P\in M}E^Pf(S_N). \tag{125}$$

In the considered case $\Omega_i^{0-}=\{\omega_i\in\Omega_i^0,\varepsilon_i(\omega_i)\leq 0\}$, $\Omega_i^{0+}=\{\omega_i\in\Omega_i^0,\varepsilon_i(\omega_i)>0\}$. Since the value $\varepsilon_s(\omega_s^2)>0$ can be made as small as it needs for $\omega_s^2\in\Omega_s^{0+}$, we can do the left side of the inequality (125) as close to $f(S_0)$ as it needs, since $\sigma_s(\omega_1^2,\ldots,\omega_{s-1}^2)$ is bounded and $f(x)$ is a continuous one. The last proves the needed inequality. Theorem 10 is proved.

Theorem 11. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105) and the conditions of Theorem 7 are true. Suppose that $0 \leq a_i \leq 1$, $\infty > \sigma_i > \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $i = \overline{1,N}$. For the payoff function $f(x) = (x-K)^+$, $x \in (0,\infty)$, $K > 0$, the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if} \quad S_0 \prod_{i=1}^{N} (1-a_i)) \geq K, \\ S_0 \left(1 - \prod_{i=1}^{N}(1-a_i)\right), & \text{if} \quad S_0 \prod_{i=1}^{N}(1-a_i) < K. \end{cases} \tag{126}$$

For $S_0 \prod_{i=1}^{N}(1-a_i)) \geq K$, the set of non arbitrage prices coincides with the point $(S_0 - K)^+$, in case if $S_0 \prod_{i=1}^{N}(1-a_i) < K$ the set of non arbitrage prices coincides with the set $\left((S_0-K)^+, S_0\left(1 - \prod_{i=1}^{N}(1-a_i)\right)\right)$.

Proof. Let us introduce the denotations

$$I_N = \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N}\left(1 + a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right), \tag{127}$$

$$I_N^1 = \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f_1\left(S_0 \prod_{s=1}^{N}\left(1 + a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right), \tag{128}$$

$$I_N^0 = \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N}\left(1 + a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right), \tag{129}$$

where we put $f_1(x) = (K-x)^+$. Let us estimate from above the value I_N. For this, we use the equality

$$I_N = I_N^1 + S_0 - K, \tag{130}$$

which follows from the identity: $f(x) = f_1(x) + x - K,\ \ x \geq 0$. Since

$$f_1\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right) \leq f_1\left(S_0\prod_{s=1}^{N}(1-a_s)\right), \tag{131}$$

we obtain the inequality

$$I_N \leq S_0 - K + f_1\left(S_0\prod_{s=1}^{N}(1-a_s)\right). \tag{132}$$

From the inequality (132), we have

$$I_N^0 \leq S_0 - K + f_1\left(S_0\prod_{s=1}^{N}(1-a_s))\right) =$$

$$\begin{cases} (S_0-K)^+, & \text{if} \quad S_0\prod\limits_{i=1}^{N}(1-a_i)) \geq K, \\ S_0\left(1-\prod\limits_{i=1}^{N}(1-a_i)\right), & \text{if} \quad S_0\prod\limits_{i=1}^{N}(1-a_i) < K. \end{cases} \tag{133}$$

From the inequality (106) of Theorem 9

$$I_N^0 \geq f\left(S_0\prod_{i=1}^{N}(1-a_i)\right) + S_0\left(1-\prod_{i=1}^{N}(1-a_i)\right) \tag{134}$$

and the inequality

$$I_N^0 \geq (S_0-K)^+, \tag{135}$$

which follows from the Jensen inequality, we have

$$I_N^0 \geq \max\left\{S_0-K)^+, f\left(S_0\prod_{i=1}^{N}(1-a_i)\right) + S_0\left(1-\prod_{i=1}^{N}(1-a_i)\right)\right\} =$$

$$\begin{cases} (S_0-K)^+, & \text{if} \quad S_0\prod\limits_{i=1}^{N}(1-a_i)) \geq K, \\ S_0\left(1-\prod\limits_{i=1}^{N}(1-a_i)\right), & \text{if} \quad S_0\prod\limits_{i=1}^{N}(1-a_i) < K. \end{cases} \tag{136}$$

This proves Theorem 11.

Theorem 12. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105) and the conditions of Theorem 7 are true. Suppose that* $0 \leq a_i \leq 1,\ \infty > \sigma_i > \sigma_i(\omega_1,\ldots,\omega_{i-1}) \geq \sigma_i^0 > 0,\quad i = \overline{1,N}$. *For the payoff* *function* $f_1(x) = (K-x)^+,\ x \in (0,\infty),\ K>0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_N) = f_1\left(S_0 \prod_{i=1}^{N}(1-a_i)\right). \tag{137}$$

The set of non arbitrage prices coincides with the interval $\left((K-S_0)^+, f_1\left(S_0 \prod_{i=1}^{N}(1-a_i)\right)\right)$.

Proof. The inequality

$$I_N^1 = \sum_{i_1=1,\dots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j}) \times$$

$$f_1\left(S_0 \prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right) \leq f_1\left(S_0 \prod_{i=1}^{N}(1-a_i)\right) \tag{138}$$

is true. Taking into account the inequality (116) of Theorem 10, we prove Theorem 12.

Theorem 13. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (105) and the conditions of Theorem 7 are true. Suppose that* $0 \leq a_i \leq 1,\ \infty > \sigma_i > \sigma_i(\omega_1,\dots,\omega_{i-1}) \geq \sigma_i^0 > 0, \quad i=\overline{1,N}$. *For the payoff function* $f_1(S_0,S_1,\dots,S_N) = \left(K - \frac{\sum_{i=0}^{N} S_i}{N+1}\right)^+,\ K>0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_0,S_1,\dots,S_N) = \left(K - \frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_s)}{N+1}\right)^+. \tag{139}$$

The set of non arbitrage prices coincides with the interval

$$\left((K-S_0)^+, \left(K - \frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_s)}{N+1}\right)^+\right).$$

Proof. Let us denote

$$S_n(\omega_1^1,\dots,\omega_n^1) = S_0 \prod_{s=1}^{n}\left(1+a_s\left(e^{\sigma_s(\omega_1^1,\dots,\omega_{s-1}^1)\varepsilon_s(\omega_s^1)}-1\right)\right), \quad n=\overline{1,N},$$

$$t_N(\omega_1^1,\dots,\omega_N^1) = \prod_{s=1}^{N} \frac{\left(e^{\sigma_s(\omega_1^1,\dots,\omega_{s-1}^1)\varepsilon_s(\omega_s^2)}-1\right)}{\left(e^{\sigma_s(\omega_1^1,\dots,\omega_{s-1}^1)\varepsilon_s(\omega_s^2)} - e^{\sigma_s(\omega_1^1,\dots,\omega_{s-1}^1)\varepsilon_s(\omega_s^1)}\right)}. \tag{140}$$

It is evident that

$$I_N^2 = \sup_{\{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \sum_{i_1=1,\dots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\times$$

$$f_1\left(S_0, S_1(\omega_1^{i_1}),\dots,S_N(\omega_1^{i_1},\dots,\omega_N^{i_N})\right) \geq \qquad (141)$$

$$\lim_{\varepsilon_s(\omega_s^1)=-\infty,\ \varepsilon_s(\omega_s^2)\to\infty, s=\overline{1,N}} f_1\left(S_0, S_1(\omega_1^1),\dots,S_N(\omega_1^1,\dots,\omega_N^1)\right)\times$$

$$t_N(\omega_1^1,\dots,\omega_N^1) = f_1\left(S_0, S_0(1-a_1),\dots,S_0\prod_{s=1}^{N}(1-a_s)\right).$$

So, we obtain the inequality

$$I_N^2 \geq f_1\left(S_0, S_0(1-a_1),\dots,S_0\prod_{s=1}^{N}(1-a_s)\right) = \left(K - \frac{S_0\sum_{i=0}^{N}\prod_{s=1}^{i}(1-a_s)}{N+1}\right)^+. \qquad (142)$$

Let us prove the inverse inequality. We have

$$I_N^2 \leq \sup_{\{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \sum_{i_1=1,\dots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\times$$

$$f_1\left(S_0, S_0(1-a_1),\dots,S_0\prod_{s=1}^{N}(1-a_s)\right) =$$

$$f_1\left(S_0, S_0(1-a_1),\dots,S_0\prod_{s=1}^{N}(1-a_s)\right) = \left(K - \frac{S_0\sum_{i=0}^{N}\prod_{s=1}^{N}(1-a_s)}{N+1}\right)^+. \qquad (143)$$

Therefore,

$$I_N^2 \leq \left(K - \frac{S_0\sum_{i=0}^{N}\prod_{s=1}^{i}(1-a_s)}{N+1}\right)^+. \qquad (144)$$

The inequalities (142), (144) prove Theorem 13.

Theorem 14. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (105) and the conditions of Theorem 7 are true. Suppose that* $0 \leq a_i \leq 1$, $\infty > \sigma_i > \sigma_i(\omega_1,\dots,\omega_{i-1}) \geq \sigma_i^0 > 0$, $\quad i = \overline{1,N}$. *For the payoff* *function* $f(S_0, S_1, \ldots, S_N) = \left(\frac{\sum_{i=0}^{N} S_i}{N+1} - K\right)^+$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_0, S_1, \ldots, S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if} \quad \frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_i)}{N+1} \geq K, \\ S_0\left(1 - \frac{\sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_s)}{N+1}\right), & \text{if} \quad S_0 \frac{\sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_s)}{N+1} < K. \end{cases} \tag{145}$$

The set of non arbitrage prices coincides with the point $(S_0 - K)^+$ *for* $\frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_i)}{N+1} \geq K$, *in case if* $S_0 \frac{\sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_s)}{N+1} < K$ *the set of non arbitrage prices coincides with the interval* $\left((S_0 - K)^+, S_0\left(1 - \frac{\sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_s)}{N+1}\right)\right)$.

Proof. Let us introduce the denotation

$$V_N = \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in \mu_N\}} \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f\left(S_0, S_1(\omega_1^{i_1}), \ldots, S_N(\omega_1^{i_1}, \ldots, \omega_N^{i_N})\right). \tag{146}$$

Then, we have

$$V_N = \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in \mu_N\}} \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f_1\left(S_0, S_1(\omega_1^{i_1}), \ldots, S_N(\omega_1^{i_1}, \ldots, \omega_N^{i_N})\right) + S_0 - K. \tag{147}$$

Due to Theorem 13,

$$V_N = (S_0 - K) + \left(K - \frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_s)}{N+1}\right)^+ =$$

$$\begin{cases} (S_0 - K)^+, & \text{if} \quad \frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_i)}{N+1} \geq K, \\ S_0\left(1 - \frac{\sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_s)}{N+1}\right), & \text{if} \quad S_0 \frac{\sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_s)}{N+1} < K. \end{cases} \tag{148}$$

In the formula (147) we introduced the denotation

$$f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{\sum\limits_{i=0}^{N} S_i}{N+1} \right)^+ . \tag{149}$$

Theorem 14 is proved.

If $S_0, \ldots, S_N$ is a sample of the process (105), let us denote the order statistic $S_{(0)}, \ldots, S_{(N)}$ of this sample.

Theorem 15. *Suppose that $S_0, \ldots, S_N$ is a sample of the random process (105). Then, for the parameters $a_1, \ldots, a_N$ the estimations*

$$a_i = 1 - \frac{S_{(N-i)}}{S_{(N-i+1)}}, \quad i = \overline{1, N}, \tag{150}$$

are valid. Under such estimations $0 < a_i < 1$, $i = \overline{1, N}$, the equalities

$$\prod_{i=1}^{N-k}(1 - a_i) = \frac{S_{(k)}}{S_{(N)}}, \quad i = \overline{0, N-1}, \tag{151}$$

are true.

Proof. The estimation of the parameters $a_1, \ldots, a_N$ we do using the representation of random process S_n, $n = \overline{1, N}$. The smallest value of the random variable S_n is equal $S_0 \prod\limits_{i=1}^{n}(1 - a_i)$, $n = \overline{1, N}$. Let us determine the parameters a_i from the relations

$$S_0 \prod_{i=1}^{N}(1 - a_i) = \tau S_{(0)}, \ldots, S_0 \prod_{i=1}^{N-k}(1 - a_i) = \tau S_{(k)},$$

$$S_0 \prod_{i=1}^{N-k-1}(1 - a_i) = \tau S_{(k+1)}, \ldots, S_0(1 - a_1) = \tau S_{(N-1)}, \tag{152}$$

where $\tau > 0$. Taking into account the relations (152), we obtain

$$S_0(1 - a_1) = \tau S_{(N-1)},$$

$$(1 - a_{N-k}) = \frac{S_{(k)}}{S_{(k+1)}} \quad k = \overline{0, N-1}. \tag{153}$$

Solving the relations (153), we have

$$a_1 = 1 - \frac{\tau}{S_0} S_{(N-1)}, \quad a_{N-k} = 1 - \frac{S_{(k)}}{S_{(k+1)}}, \quad k = \overline{1, N-2}. \tag{154}$$

It is evident that if to put $\tau = \frac{S_0}{S_{(N)}}$, then $1 - a_1 = \frac{S_{(N-1)}}{S_{(N)}}$. Therefore, $\prod\limits_{i=1}^{N-k}(1 - a_i) = \frac{S_{(k)}}{S_{(N)}}$, $k = \overline{0, N-1}$. Theorem 15 is proved.

Theorem 16. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105), with parameters a_i, $i = \overline{1,N}$, given by the formula (150). For the payoff function $f(x) = (x-K)^+$, $x \in (0,\infty)$, $K > 0$, the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if} \quad S_0 \frac{S_{(0)}}{S_{(N)}} \geq K, \\ S_0 \left(1 - \frac{S_{(0)}}{S_{(N)}}\right), & \text{if} \quad S_0 \frac{S_{(0)}}{S_{(N)}} < K. \end{cases} \tag{155}$$

If $S_0 \frac{S_{(0)}}{S_{(N)}} \geq K$, then the set of non arbitrage prices coincides with the point $(S_0 - K)^+$, in case if $S_0 \frac{S_{(0)}}{S_{(N)}} < K$ the set of non arbitrage prices coincides with the set $\left((S_0 - K)^+, S_0 \left(1 - \frac{S_{(0)}}{S_{(N)}}\right)\right)$.

Corollary 1. *Suppose that the strike price $K = S_0 \frac{S_{(0)}}{S_{(N)}}$, then the set of non arbitrage prices consists of one point $S_0 \left(1 - \frac{S_{(0)}}{S_{(N)}}\right)$. It is a fair price of a standard call option of European type with the payoff function $(S_N - K)^+$.*

This corollary is very important for practical application. The fair price of a standard call option of European type is proportional to the initial spot price of the underlying asset multiplied by the value of the relative swing of the market in the given horizon.

Theorem 17. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105) with the parameters a_i, $i = \overline{1,N}$, given by the formula (150). For the payoff function $f_1(x) = (K - x)^+$, $x \in (0,\infty)$, $K > 0$, the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_N) = f_1\left(S_0 \frac{S_{(0)}}{S_{(N)}}\right). \tag{156}$$

The set of non arbitrage prices coincides with the interval $\left((K - S_0)^+, f_1\left(S_0 \frac{S_{(0)}}{S_{(N)}}\right)\right)$.

Theorem 18. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105) with the parameters a_i, $i = \overline{1,N}$, given by the formula (150). For the payoff function $f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{\sum_{i=0}^{N} S_i}{N+1}\right)^+$, $K > 0$, the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{S_0 + S_0 \sum_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)}\right)^+. \tag{157}$$

The set of non arbitrage prices coincides with the interval

$$\left((K-S_0)^+, \left(K - \frac{S_0+S_0\sum\limits_{i=0}^{N-1}\frac{S_{(i)}}{S_{(N)}}}{(N+1)}\right)^+\right).$$

Theorem 19. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105) with the parameters a_i, $i=\overline{1,N}$, given by the formula (150). For the payoff function $f(S_0, S_1, \ldots, S_N) = \left(\frac{\sum\limits_{i=0}^{N} S_i}{N+1} - K\right)^+$, $K>0$, the fair price of super-hedge is given by the formula*

$$\sup_{Q\in M} E^Q f(S_0, S_1, \ldots, S_N) =$$

$$\begin{cases} (S_0-K)^+, & \text{if } \frac{S_0+S_0\sum\limits_{i=0}^{N-1}\frac{S_{(i)}}{S_{(N)}}}{(N+1)} \geq K, \\ \left(S_0 - \frac{S_0+S_0\sum\limits_{i=0}^{N-1}\frac{S_{(i)}}{S_{(N)}}}{(N+1)}\right), & \text{if } \frac{S_0+S_0\sum\limits_{i=0}^{N-1}\frac{S_{(i)}}{S_{(N)}}}{(N+1)} < K. \end{cases} \quad (158)$$

If $\frac{S_0+S_0\sum\limits_{i=0}^{N-1}\frac{S_{(i)}}{S_{(N)}}}{(N+1)} \geq K$, *then the set of non arbitrage prices coincides with the point* $(S_0-K)^+$, *in case if* $\frac{S_0+S_0\sum\limits_{i=0}^{N-1}\frac{S_{(i)}}{S_{(N)}}}{(N+1)} < K$ *the set of non arbitrage prices coincides with the interval* $\left((S_0-K)^+, \left(S_0 - \frac{S_0+S_0\sum\limits_{i=0}^{N-1}\frac{S_{(i)}}{S_{(N)}}}{(N+1)}\right)\right)$.

VI. DISCOUNTED EVOLUTION OF RISKY ASSET

In this section, we formulate Theorems, giving us the formula for the fair price of super-hedge for the evolution of risky asset, given by the formula

$$S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) = S_0 \prod_{i=1}^{n}\left(1 + a_i\left(\frac{e^{\sigma_i(\omega_1,\ldots,\omega_{i-1})\varepsilon_i(\omega_i)}}{e^{r_i}} - 1\right)\right), \quad n=\overline{1,N}. \quad (159)$$

where the random value $\varepsilon_i(\omega_i)$, $\omega_i \in \Omega_i^0, i=\overline{1,N}$, takes all real values from R^1, $S_0>0$. The random values $\sigma_i(\omega_1,\ldots,\omega_{i-1})$ satisfy the inequalities $\sigma_i(\omega_1,\ldots,\omega_{i-1}) \geq \sigma_i^0 > 0$ and $0 \leq a_i \leq 1$, $0 < r_i < \infty$. This parametric evolution (159) is built on the discounted evolution of risky asset (17) for which the representation (12) is valid. From this representation, it follows that for such a discounted evolution, all proven Theorems regarding the existence of a family of martingale measures are valid, since the representations (18), (19) is true. Due to Theorem 8, the set of martingale measures do not depend on parameters $0 \leq a_i \leq 1$. The proof of Theorems formulated below is analogous to the proof of Theorems 9 - 14.

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Theorem 20. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (159). Suppose that* $0 \leq a_i \leq 1$, $\infty > \sigma_i \geq \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $0 < r_i < \infty$, $i = \overline{1, N}$. *If the nonnegative continuous payoff function* $f(x)$, $x \in [0, \infty)$, *satisfies the conditions:*

1) $f(0) = 0$, $f(x) \leq ax$, $\lim\limits_{x \to \infty} \frac{f(x)}{x} = a$, $a > 0$, *then the inequalities*

$$f\left(S_0 \prod_{i=1}^{N}(1 - a_i)\right) + aS_0\left(1 - \prod_{i=1}^{N}(1 - a_i)\right) \leq \sup_{P \in M} E^P f(S_N) \leq aS_0 \qquad (160)$$

are true. If, in addition, the nonnegative payoff function $f(x)$ is a convex down one, then

$$\inf_{P \in M} E^P f(S_N) = f(S_0), \qquad (161)$$

where M is the set of equivalent martingale measures for the evolution of risky asset, given by the formula (159).

Theorem 21. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (159). Suppose that* $0 \leq a_i \leq 1$, $\infty > \sigma_i \geq \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $0 < r_i < \infty$, $i = \overline{1, N}$. *If the nonnegative continuous payoff function* $f(x)$, $x \in [0, \infty)$, *satisfies the conditions:*
1) $f(0) = K$, $f(x) \leq K$, *then*

$$f\left(S_0 \prod_{i=1}^{N}(1 - a_i)\right) \leq \sup_{P \in M} E^P f(S_N) \leq K. \qquad (162)$$

If, in addition, the nonnegative payoff function $f(x)$ is a convex down one, then

$$\inf_{P \in M} E^P f(S_N) = f(S_0), \qquad (163)$$

where M is the set of equivalent martingale measures for the evolution of risky asset, given by the formula (159).

Theorem 22. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (159). Suppose that* $0 \leq a_i \leq 1$, $\infty > \sigma_i \geq \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $0 < r_i < \infty$, $i = \overline{1, N}$. *For the payoff function* $f(x) = (x - K)^+$, $x \in (0, \infty)$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if} \quad S_0 \prod_{i=1}^{N}(1 - a_i)) \geq K, \\ S_0\left(1 - \prod_{i=1}^{N}(1 - a_i)\right), & \text{if} \quad S_0 \prod_{i=1}^{N}(1 - a_i) < K. \end{cases} \qquad (164)$$

For $S_0 \prod_{i=1}^{N}(1 - a_i)) \geq K$, *the set of non arbitrage prices coincides with the point* $(S_0 - K)^+$, *in case if* $S_0 \prod_{i=1}^{N}(1 - a_i) < K$ *the set of non arbitrage prices coincides with the set* $\left((S_0 - K)^+, S_0\left(1 - \prod_{i=1}^{N}(1 - a_i)\right)\right)$.

Theorem 23. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (159). Suppose that $0 \leq a_i \leq 1$, $\infty > \sigma_i \geq \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $0 < r_i < \infty$, $i = \overline{1,N}$. For the payoff function $f_1(x) = (K - x)^+$, $x \in (0, \infty)$, $K > 0$, the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_N) = f_1\left(S_0 \prod_{i=1}^{N}(1 - a_i)\right). \tag{165}$$

The set of non arbitrage prices coincides with the interval

$$\left((K - S_0)^+, f_1\left(S_0 \prod_{i=1}^{N}(1 - a_i)\right)\right).$$

Theorem 24. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (159). Suppose that $0 \leq a_i \leq 1$, $\infty > \sigma_i \geq \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $0 < r_i < \infty$, $i = \overline{1,N}$. For the payoff function $f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{\sum_{i=0}^{N} S_i}{N+1}\right)^+$, $K > 0$, the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{S_0 + S_0 \sum_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)}\right)^+. \tag{166}$$

The set of non arbitrage prices coincides with the interval

$$\left((K - S_0)^+, \left(K - \frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i}(1 - a_s)}{N+1}\right)^+\right).$$

Theorem 25. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (159). Suppose that $0 < a_i \leq 1$, $\infty > \sigma_i \geq \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $0 < r_i < \infty$, $i = \overline{1,N}$. For the payoff function $f(S_0, S_1, \ldots, S_N) = \left(\frac{\sum_{i=0}^{N} S_i}{N+1} - K\right)^+$, $K > 0$, the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_0, S_1, \ldots, S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if } \frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i}(1 - a_i)}{N+1} \geq K, \\ S_0\left(1 - \frac{\sum_{i=0}^{N} \prod_{s=1}^{i}(1 - a_s)}{N+1}\right), & \text{if } S_0 \frac{\sum_{i=0}^{N} \prod_{s=1}^{i}(1 - a_s)}{N+1} < K. \end{cases} \tag{167}$$

The set of non arbitrage prices coincides with the point $(S_0 - K)^+$ for $\frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i}(1 - a_i)}{N+1} \geq$

K, in case if $S_0 \frac{\sum\limits_{i=0}^{N} \prod\limits_{s=1}^{i}(1-a_s)}{N+1} < K$ *the set of non arbitrage prices coincides with the interval* $\left((S_0-K)^+, S_0\left(1-\frac{\sum\limits_{i=0}^{N} \prod\limits_{s=1}^{i}(1-a_s)}{N+1}\right)\right)$.

Theorem 26. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (159), with parameters* a_i, $i=\overline{1,N}$, *given by the formula (150). For the payoff function* $f(x)=(x-K)^+$, $x\in(0,\infty)$, $K>0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q\in M} E^Q f(S_N) =$$

$$\begin{cases} (S_0-K)^+, & \text{if} \quad S_0\frac{S_{(0)}}{S_{(N)}} \geq K, \\ S_0\left(1-\frac{S_{(0)}}{S_{(N)}}\right), & \text{if} \quad S_0\frac{S_{(0)}}{S_{(N)}} < K. \end{cases} \tag{168}$$

If $S_0\frac{S_{(0)}}{S_{(N)}} \geq K$, *then the set of non arbitrage prices coincides with the point* $(S_0-K)^+$, *in case if* $S_0\frac{S_{(0)}}{S_{(N)}} < K$ *the set of non arbitrage prices coincides with the set* $\left((S_0-K)^+, S_0\left(1-\frac{S_{(0)}}{S_{(N)}}\right)\right)$.

Corollary 2. *Suppose that the strike price* $K=S_0\frac{S_{(0)}}{S_{(N)}}$, *then the set of non arbitrage prices consists of one point* $S_0\left(1-\frac{S_{(0)}}{S_{(N)}}\right)$. *It is a fair price of a standard call option of European type with the payoff function* $(S_N-K)^+$.

This corollary is very important for practical application. The fair price of a standard call option of European type is proportional to the initial spot price of the underlying asset multiplied by the value of the relative swing of the market in the given horizon.

Theorem 27. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (159) with the parameters* a_i, $i=\overline{1,N}$, *given by the formula (150). For the payoff function* $f_1(x)=(K-x)^+$, $x\in(0,\infty)$, $K>0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q\in M} E^Q f_1(S_N) = f_1\left(S_0\frac{S_{(0)}}{S_{(N)}}\right). \tag{169}$$

The set of non arbitrage prices coincides with the interval $\left((K-S_0)^+, f_1\left(S_0\frac{S_{(0)}}{S_{(N)}}\right)\right)$.

Theorem 28. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (159) with the parameters* a_i, $i=\overline{1,N}$, *given by the formula (150). For the payoff function* $f_1(S_0,S_1,\ldots,S_N)=\left(K-\frac{\sum\limits_{i=0}^{N} S_i}{N+1}\right)^+$, $K>0$, *the fair* *price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{S_0 + S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)} \right)^+ . \qquad (170)$$

The set of non arbitrage prices coincides with the interval

$$\left((K - S_0)^+, \left(K - \frac{S_0 + S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)} \right)^+ \right).$$

Theorem 29. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (159) with the parameters* a_i, $i = \overline{1, N}$, *given by the formula (150). For the payoff function* $f(S_0, S_1, \ldots, S_N) = \left(\frac{\sum\limits_{i=0}^{N} S_i}{N+1} - K \right)^+$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_0, S_1, \ldots, S_N) =$$

$$\begin{cases} (S_0 - K)^+, & if \quad \frac{S_0 + S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)} \geq K, \\ \left(S_0 - \frac{S_0 + S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)} \right), & if \quad \frac{S_0 + S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)} < K. \end{cases} \qquad (171)$$

If $\frac{S_0 + S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)} \geq K$, *then the set of non arbitrage prices coincides with the point* $(S_0 - K)^+$, *in case if* $\frac{S_0 + S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)} < K$ *the set of non arbitrage prices coincides with the interval* $\left((S_0 - K)^+, \left(S_0 - \frac{S_0 + S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)} \right) \right)$.

VII. UNIQUENESS OF THE MARTINGALE MEASURE

In this section, the necessary and sufficient conditions of the uniqueness of martingale measure in terms of the evolution of risky assets are obtained. Under the fairly wide assumptions about the evolution of risky assets, an expression for a single martingale measure is found. Based on the explicit construction of the martingale measure and its invariance with respect to a certain type of evolutions, it is possible to construct the models of non arbitrage markets, both complete and incomplete.

In this and section 8, we put that $\Omega_i^0 = \{\omega_i^1, \omega_i^2\}$. Denote by $\mathcal{F}_i^0$ the σ-algebra of all subsets of the set Ω_i^0. Let P_i^0 be a probability measure on $\mathcal{F}_i^0$. We assume that $P_i^0(\omega_i^s) > 0$, $i = \overline{1, N}$, $s = \overline{1, 2}$. As before, we put that the probability space

$\{\Omega_N, \mathcal{F}_N, P_N\}$ is a direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, $i = \overline{1, N}$, and we put $N < \infty$. We also consider the probability spaces $\{\Omega_n, \mathcal{F}_n, P_n\}, n = \overline{1, N}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, $i = \overline{1, n}$. We assume that the evolution of a risky asset is given by the formula

$$S_n(\omega_1, \ldots, \omega_n) =$$

$$S_0 \prod_{i=1}^{n}(1 + a_i(\omega_1, \ldots, \omega_i)\eta_i(\omega_i)), \ \{\omega_1, \ldots, \omega_{n-1}, \omega_n\} \in \Omega_n, \ n = \overline{1, N}, \quad S_0 > 0, \tag{172}$$

where the random values $a_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)$, $\eta_n(\omega_n)$, $n = \overline{1, N}$, given on the probability space $\{\Omega_n, \mathcal{F}_n, P_n\}$, satisfy the conditions

$$a_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) > 0, \quad \max_{\{\omega_1, \ldots, \omega_{n-1}\} \in \Omega_{n-1}} a_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) < \frac{1}{\eta_n^-(\omega_n^1)},$$

$$\eta_n(\omega_n^2) > 0, \quad \eta_n(\omega_n^1) < 0. \tag{173}$$

So, for $\Delta S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)$, $n = \overline{1, N}$, the representation

$$\Delta S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) =$$

$$S_{n-1}(\omega_1, \ldots, \omega_{n-1}) a_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)\eta_n(\omega_n) =$$

$$d_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)\eta_n(\omega_n), \quad d_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) > 0, \quad n = \overline{1, N}, \tag{174}$$

is true, From these conditions, we obtain $\Omega_n^- = \Omega_{n-1} \times \Omega_n^{0-}$, $\Omega_n^+ = \Omega_{n-1} \times \Omega_n^{0+}$, where $\Omega_n^{0-} = \{\omega_n \in \Omega_n^0, \eta_n(\omega_n) < 0\}$, $\Omega_n^{0+} = \{\omega_n \in \Omega_n^0, \eta_n(\omega_n) > 0\}$.

Further, we assume that $P_n^0(\Omega_n^{0-}) > 0$, $P_n^0(\Omega_n^{0+}) > 0$. The measure P_n^{0-} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0-} = \Omega_n^{0-} \cap \mathcal{F}_n^0$, P_n^{0+} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0+} = \Omega_n^{0+} \cap \mathcal{F}_n^0$.

Let us introduce the following denotations. For every point $\{\omega_1, \ldots, \omega_{n-1}, \omega_n\} \in \Omega_n$, we introduce the set $A(\omega_1, \ldots, \omega_{n-1}, \omega_n) \in \Omega_N$, where

$$A(\omega_1, \ldots, \omega_{n-1}, \omega_n) = \bigcup_{i_{n+1}=1, \ldots, i_N=1}^{2} \{\omega_1, \ldots, \omega_{n-1}, \omega_n, \omega_{n+1}^{i_{n+1}}, \ldots, \omega_N^{i_N}\}.$$

Sometimes, for fixed indexes $i_1, \ldots, i_n$ we also use the denotation

$$A(\omega_1^{i_1}, \ldots, \omega_{n-1}^{i_{n-1}}, \omega_n^{i_n}) = A^{i_1, \ldots, i_n}.$$

It is evident that every set $A^{i_1, \ldots, i_n}$ has the form

$$A^{i_1, \ldots, i_n} = \bigcup_{i_{n+1}=1, \ldots, i_N=1}^{2} \{\omega_1^{i_1}, \ldots, \omega_n^{i_n}, \omega_{n+1}^{i_{n+1}}, \ldots, \omega_N^{i_N}\},$$

where indexes i_s, $s = \overline{1,N}$, take values from the set $\{1,2\}$. Then, $A^{i_1,\dots,i_{n-1}} = A^{i_1,\dots,i_{n-1},1} \cup A^{i_1,\dots,i_{n-1},2} \in \mathcal{F}_{n-1}$, where

$$A^{i_1,\dots,i_{n-1},1} = \bigcup_{i_{n+1}=1,\dots,i_N=1}^{2} \{\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^1,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\} \in \mathcal{F}_n,$$

$$A^{i_1,\dots,i_{n-1},2} = \bigcup_{i_{n+1}=1,\dots,i_N=1}^{2} \{\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^2,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\}\} \in \mathcal{F}_n.$$

If P_N is a measure on $\mathcal{F}_N$, then

$$P_N(A(\omega_1,\dots,\omega_{n-1},\omega_n)) = \sum_{i_{n+1}=1,\dots,i_N=1}^{2} P_N(\{\omega_1,\dots,\omega_{n-1},\omega_n,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\}).$$

We give an evident construction of martingale measure for risky asset evolution, given by the formula (172). Let us put $P_n^0(\omega_n^1) = p_n$, $P_n^0(\omega_n^2) = 1 - p_n$, where $0 < p_n < 1$. Then, to satisfy the conditions (14) - (16), (see [2]) we need to put that

$$\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1) < \infty, \quad (\omega_1,\dots,\omega_{n-1},\omega_n^1) \in \Omega_n^-,$$

$$\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2) < \infty, \quad (\omega_1,\dots,\omega_{n-1},\omega_n^2) \in \Omega_n^+. \qquad (175)$$

The next Lemma 3 is a consequence of results in [2].

Lemma 3. *On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, for the evolution of risky asset given by the formula (172) only one spot measure $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ exists, where $\{\omega_i^1,\omega_i^2\} \in \Omega_i^0$, $i = \overline{1,N}$. For it the representation*

$$\mu_0(A) = \mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A) =$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\chi_A(\omega_1^{i_1},\dots,\omega_N^{i_N}), \quad A \in \mathcal{F}_N, \qquad (176)$$

is true. This measure is a martingale one for the considered evolution of risky asset, where

$$\psi_n(\omega_1,\dots,\omega_n) = \chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n)\psi_n^1(\omega_1,\dots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n)\psi_n^2(\omega_1,\dots,\omega_n), \qquad (177)$$

$$\psi_n^1(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$\chi_{\Omega_n^{0+}}(\omega_n^2)\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\dots,\omega_{n-1}) \in \Omega_{n-1}, \quad \omega_n^2 \in \Omega_n^{0+}, \qquad (178)$$

$$V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2) = \Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1) + \Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2),$$

$$\psi_n^2(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$\chi_{\Omega_n^{0-}}(\omega_n^1)\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\dots,\omega_{n-1}) \in \Omega_{n-1}, \quad \omega_n^1 \in \Omega_n^{0-}. \qquad (179)$$

Next Theorem 30 appeared first in [24] (Theorem 1.4.1), where it was proved under the less general conditions.

Theorem 30. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, suppose that the evolution of risky asset $\{S_n(\omega_1, \ldots, \omega_n)\}_{n=1}^N$ is given by the formula (172). The necessary and sufficient conditions of the uniqueness of martingale measure $\mu_0(A)$, $A \in \mathcal{F}_N$, are the inequalities*

$$S_n(\omega_1^{i_1}, \ldots, \omega_{n-1}^{i_{n-1}}, \omega_n^1) \neq S_n(\omega_1^{i_1}, \ldots, \omega_{n-1}^{i_{n-1}}, \omega_n^2), \quad n = \overline{1, N}, \tag{180}$$

for every set of indexes $i_1, \ldots, i_{n-1}$. For any martingale $\{m_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)\}_{n=0}^N$ relative to the unique measure $\mu_0(A)$ the representation

$$m_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) =$$

$$\sum_{k=1}^{n} C_k(\omega_1, \ldots, \omega_{k-1})[S_k(\omega_1, \ldots, \omega_{k-1}, \omega_i) - S_{k-1}(\omega_1, \ldots, \omega_{k-1})] +$$

$$m_0, \quad n = \overline{1, N}, \tag{181}$$

is true, where

$$C_k(\omega_1, \ldots, \omega_{k-1}) = \sum_{i_1=1, \ldots, i_{k-1}=1}^{2} d_{i_1, \ldots, i_{k-1}} \chi_{A^{i_1, \ldots, i_{k-1}}}(\omega_1, \ldots, \omega_{k-1}), \tag{182}$$

$$d_{i_1, \ldots, i_{k-1}} =$$

$$\frac{m_k(\omega_1^{i_1}, \ldots, \omega_{k-1}^{i_{k-1}}, \omega_k^1) - m_k(\omega_1^{i_1}, \ldots, \omega_{k-1}^{i_{k-1}}, \omega_k^2)}{S_k(\omega_1^{i_1}, \ldots, \omega_{k-1}^{i_{k-1}}, \omega_k^1) - S_k(\omega_1^{i_1}, \ldots, \omega_{k-1}^{i_{k-1}}, \omega_k^2)}, \quad k = \overline{1, N}. \tag{183}$$

Proof. The necessity. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution $\{S_n(\omega_1, \ldots, \omega_n)\}_{n=1}^N$ of risky asset be such that the martingale measure $\mu_0(A)$, $A \in \mathcal{F}_N$, being equivalent to the measure P_N, is unique. Then, for every contingent liability $m_N(\omega_1, \ldots, \omega_N)$ the representation (181) is true [13] for some $\mathcal{F}_{k-1}$-measurable finite valued random value $C_k(\omega_1, \ldots, \omega_{k-1})$, $k = \overline{1, N}$, where $m_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) = E^{\mu_0}\{m_N(\omega_1, \ldots, \omega_N)|\mathcal{F}_n\}$. For $m_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)$ and $S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)$ the representations

$$m_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) =$$

$$\sum_{i_1=1, \ldots, i_n=1}^{2} \frac{\chi_{A^{i_1, \ldots, i_{n-1}, i_n}}(\omega_1, \ldots, \omega_n)}{\mu_0(A^{i_1, \ldots, i_{n-1}, i_n})} \int\limits_{A^{i_1, \ldots, i_{n-1}, i_n}} m_N(\omega_1, \ldots, \omega_N) d\mu_0, \ n = \overline{1, N}, \tag{184}$$

$$S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) =$$

$$\sum_{i_1=1, \ldots, i_n=1}^{2} \frac{\chi_{A^{i_1, \ldots, i_{n-1}, i_n}}(\omega_1, \ldots, \omega_n)}{\mu_0(A^{i_1, \ldots, i_{n-1}, i_n})} \int\limits_{A^{i_1, \ldots, i_{n-1}, i_n}} S_N(\omega_1, \ldots, \omega_N) d\mu_0, \ n = \overline{1, N}, \tag{185}$$

are true. From the representation (181) and the equality (182) for $\{\omega_1,\ldots,\omega_{n-1}\} \in A^{i_1,\ldots,i_{n-1}}$ we obtain the equality

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1},1}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},1})}\int\limits_{A^{i_1,\ldots,i_{n-1},1}} m_N(\omega_1,\ldots,\omega_N)d\mu_0+$$

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1},2}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},2})}\int\limits_{A^{i_1,\ldots,i_{n-1},2}} m_N(\omega_1,\ldots,\omega_N)d\mu_0-$$

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1}}}(\omega_1,\ldots,\omega_{n-1})}{\mu_0(A^{i_1,\ldots,i_{n-1}})}\int\limits_{A^{i_1,\ldots,i_{n-1}}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 =$$

$$d_{i_1,\ldots,i_{n-1}}\chi_{A^{i_1,\ldots,i_{n-1}}}(\omega_1,\ldots,\omega_{n-1})\times$$

$$\left[\frac{\chi_{A^{i_1,\ldots,i_{n-1},1}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},1})}\int\limits_{A^{i_1,\ldots,i_{n-1},1}} S_N(\omega_1,\ldots,\omega_N)d\mu_0+\right.$$

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1},2}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},2})}\int\limits_{A^{i_1,\ldots,i_{n-1},2}} S_N(\omega_1,\ldots,\omega_N)d\mu_0-$$

$$\left.\frac{\chi_{A^{i_1,\ldots,i_{n-1}}}(\omega_1,\ldots,\omega_{n-1})}{\mu_0(A^{i_1,\ldots,i_{n-1}})}\int\limits_{A^{i_1,\ldots,i_{n-1}}} S_N(\omega_1,\ldots,\omega_N)d\mu_0\right], \qquad (186)$$

where $d_{i_1,\ldots,i_{n-1}}$ is finite. Since

$$\int\limits_{A^{i_1,\ldots,i_{n-1}}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 =$$

$$\int\limits_{A^{i_1,\ldots,i_{n-1},1}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 + \int\limits_{A^{i_1,\ldots,i_{n-1},2}} m_N(\omega_1,\ldots,\omega_N)d\mu_0, \qquad (187)$$

we have

$$\mu_0(A^{i_1,\ldots,i_{n-1}})\int\limits_{A^{i_1,\ldots,i_{n-1},1}} m_N(\omega_1,\ldots,\omega_N)d\mu_0-$$

$$\mu_0(A^{i_1,\ldots,i_{n-1},1})\int\limits_{A^{i_1,\ldots,i_{n-1}}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 =$$

$$[\mu_0(A^{i_1,\ldots,i_{n-1},1})+\mu_0(A^{i_1,\ldots,i_{n-1},2})]\int\limits_{A^{i_1,\ldots,i_{n-1},1}} m_N(\omega_1,\ldots,\omega_N)d\mu_0-$$

$$\mu_0(A^{i_1,\dots,i_{n-1},1})\left[\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0 + \int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0\right] =$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2})\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\dots,i_{n-1},1})\int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0. \quad (188)$$

Further,

$$\mu_0(A^{i_1,\dots,i_{n-1}})\int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2})\int\limits_{A^{i_1,\dots,i_{n-1}}} m_N(\omega_1,\dots,\omega_N)d\mu_0 =$$

$$[\mu_0(A^{i_1,\dots,i_{n-1},1}) + \mu_0(A^{i_1,\dots,i_{n-1},2})]\int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2})\left[\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0 + \int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0\right] =$$

$$-\left[\mu_0(A^{i_1,\dots,i_{n-1},2})\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0 -\right.$$

$$\left.\mu_0(A^{i_1,\dots,i_{n-1},1})\int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0\right]. \quad (189)$$

If to put

$$R_1^m(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}}) = \mu_0(A^{i_1,\dots,i_{n-1},1})\int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2})\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0, \quad (190)$$

$$R_1^{S_N}(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}}) = \mu_0(A^{i_1,\dots,i_{n-1},1})\int\limits_{A^{i_1,\dots,i_{n-1},2}} S_N(\omega_1,\dots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2}) \int\limits_{A^{i_1,\dots,i_{n-1},1}} S_N(\omega_1,\dots,\omega_N)d\mu_0, \tag{191}$$

then the equality (186) is transformed into the equality

$$R_1^m(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}}) = d_{i_1,\dots,i_{n-1}} R_1^{S_N}(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}}). \tag{192}$$

Due to that $S_n(\omega_1,\dots,\omega_n)$ and $m_n(\omega_1,\dots,\omega_n)$ are martingales relative to the measure μ_0 and $A^{i_1,\dots,i_{n-1},1}$, $A^{i_1,\dots,i_{n-1},2} \in \mathcal{F}_n$ we have

$$\int\limits_{A^{i_1,\dots,i_{n-1},1}} S_N(\omega_1,\dots,\omega_N)d\mu_0 = \int\limits_{A^{i_1,\dots,i_{n-1},1}} S_n(\omega_1,\dots,\omega_n)d\mu_0 =$$

$$\mu_0(A^{i_1,\dots,i_{n-1},1})S_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^1), \tag{193}$$

$$\int\limits_{A^{i_1,\dots,i_{n-1},2}} S_N(\omega_1,\dots,\omega_N)d\mu_0 = \int\limits_{A^{i_1,\dots,i_{n-1},2}} S_n(\omega_1,\dots,\omega_n)d\mu_0 =$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2})S_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^2), \tag{194}$$

$$\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0 = \int\limits_{A^{i_1,\dots,i_{n-1},1}} m_n(\omega_1,\dots,\omega_n)d\mu_0 =$$

$$\mu_0(A^{i_1,\dots,i_{n-1},1})m_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^1), \tag{195}$$

$$\int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0 = \int\limits_{A^{i_1,\dots,i_{n-1},2}} m_n(\omega_1,\dots,\omega_n)d\mu_0 =$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2})m_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^2). \tag{196}$$

Since $d_{i_1,\dots,i_{n-1}}$ is finite and $m_N(\omega_1,\dots,\omega_N)$ is arbitrary, then $R_1^{S_N}(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}}) \neq 0$. The last means that inequality (180) takes place. This proves the equality

$$d_{i_1,\dots,i_{n-1}} = \tag{197}$$

$$\frac{m_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^1) - m_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^2)}{S_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^1) - S_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^2)},$$

$$n = \overline{1, N},$$

which means that (183) is true, where we introduced the denotations

$$m_n(\omega_1,\dots,\omega_n) = E^{\mu_0}\{m_N(\omega_1,\dots,\omega_N)|\mathcal{F}_n\} =$$

$$\frac{\sum\limits_{i_{n+1}=1,\dots,i_N=1}^{2} m(\omega_1,\dots,\omega_n,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N})\mu_0(\{\omega_1,\dots,\omega_n,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\})}{\sum\limits_{i_{n+1}=1,\dots,i_N=1}^{2} \mu_0(\{\omega_1,\dots,\omega_n,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\})}, \quad (198)$$

$$S_n(\omega_1,\dots,\omega_n) = E^{\mu_0}\{S_N(\omega_1,\dots,\omega_N)|\mathcal{F}_n\} =$$

$$\frac{\sum\limits_{i_{n+1}=1,\dots,i_N=1}^{2} S_N(\omega_1,\dots,\omega_n,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N})\mu_0(\{\omega_1,\dots,\omega_n,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\})}{\sum\limits_{i_{n+1}=1,\dots,i_N=1}^{2} \mu_0(\{\omega_1,\dots,\omega_n,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\})}. \quad (199)$$

This proves the necessity.

Proof of the sufficiency. Suppose that the inequalities (180) are true. Let us prove that the martingale measure μ_0 is unique. For this purpose, we prove that for every martingale the representation (181) is true with validity of equalities (182), (183).

Let us note that the equality (186) is true if for $d_{i_1,\dots,i_{n-1}}$ to choose the right hand side of the equality (197), since the equalities

$$\left[\frac{\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1},1})} - \frac{\int\limits_{A^{i_1,\dots,i_{n-1}}} m_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1}})}\right] \times$$

$$\left[\frac{\int\limits_{A^{i_1,\dots,i_{n-1},1}} S_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1},1})} - \frac{\int\limits_{A^{i_1,\dots,i_{n-1}}} S_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1}})}\right]^{-1} =$$

$$\left[\frac{\int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1},2})} - \frac{\int\limits_{A^{i_1,\dots,i_{n-1}}} m_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1}})}\right] \times$$

$$\left[\frac{\int\limits_{A^{i_1,\dots,i_{n-1},2}} S_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1},2})} - \frac{\int\limits_{A^{i_1,\dots,i_{n-1}}} S_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1}})}\right]^{-1} =$$

$$d_{i_1,\dots,i_{n-1}} \quad (200)$$

are valid. Taking into account the equality (186) and the equalities

$$d_{i_1,\dots,i_{n-1}}\chi_{A^{i_1,\dots,i_{n-1}}}(\omega_1,\dots,\omega_{n-1})\times$$

$$\left[\frac{\chi_{A^{i_1,\dots,i_{n-1},1}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{i_1,\dots,i_{n-1},1})}\int\limits_{A^{i_1,\dots,i_{n-1},1}} S_N(\omega_1,\dots,\omega_N)d\mu_0+\right.$$

$$\frac{\chi_{A^{i_1,\dots,i_{n-1},2}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{i_1,\dots,i_{n-1},2})}\int\limits_{A^{i_1,\dots,i_{n-1},2}} S_N(\omega_1,\dots,\omega_N)d\mu_0-$$

$$\left.\frac{\chi_{A^{i_1,\dots,i_{n-1}}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{i_1,\dots,i_{n-1}})}\int\limits_{A^{i_1,\dots,i_{n-1}}} S_N(\omega_1,\dots,\omega_N)d\mu_0\right] =$$

$$d_{i_1,\dots,i_{n-1}}\chi_{A^{i_1,\dots,i_{n-1}}}(\omega_1,\dots,\omega_{n-1})\times$$

$$\sum_{j_1=1,\dots j_{n-1}=1}^{2}\left[\frac{\chi_{A^{j_1,\dots,j_{n-1},1}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{j_1,\dots,j_{n-1},1})}\int\limits_{A^{j_1,\dots,j_{n-1},1}} S_N(\omega_1,\dots,\omega_N)d\mu_0+\right.$$

$$\frac{\chi_{A^{j_1,\dots,j_{n-1},2}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{j_1,\dots,j_{n-1},2})}\int\limits_{A^{j_1,\dots,j_{n-1},2}} S_N(\omega_1,\dots,\omega_N)d\mu_0-$$

$$\left.\frac{\chi_{A^{j_1,\dots,j_{n-1}}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{j_1,\dots,j_{n-1}})}\int\limits_{A^{j_1,\dots,j_{n-1}}} S_N(\omega_1,\dots,\omega_N)d\mu_0\right] = \tag{201}$$

$$d_{i_1,\dots,i_{n-1}}\chi_{A^{i_1,\dots,i_{n-1}}}(\omega_1,\dots,\omega_n)[S_n(\omega_1,\dots,\omega_{n-1},\omega_n) - S_{n-1}(\omega_1,\dots,\omega_{n-1})],$$

we have

$$\frac{\chi_{A^{i_1,\dots,i_{n-1},1}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{i_1,\dots,i_{n-1},1})}\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0+$$

$$\frac{\chi_{A^{i_1,\dots,i_{n-1},2}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{i_1,\dots,i_{n-1},2})}\int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0-$$

$$\frac{\chi_{A^{i_1,\dots,i_{n-1}}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{i_1,\dots,i_{n-1}})}\int\limits_{A^{i_1,\dots,i_{n-1}}} m_N(\omega_1,\dots,\omega_N)d\mu_0 =$$

$$d_{i_1,\dots,i_{n-1}}\chi_{A^{i_1,\dots,i_{n-1}}}(\omega_1,\dots,\omega_n)[S_n(\omega_1,\dots,\omega_{n-1},\omega_n) - S_{n-1}(\omega_1,\dots,\omega_{n-1})]. \tag{202}$$

Summing over all indexes $i_1,\dots,i_{n-1}$ the left and right hand sides of the equality (202), we obtain the equalities

$$m_n(\omega_1,\dots,\omega_n) - m_{n-1}(\omega_1,\dots,\omega_{n-1}) =$$

$$C_n(\omega_1,\dots,\omega_{n-1})[S_n(\omega_1,\dots,\omega_{n-1},\omega_n) - S_{n-1}(\omega_1,\dots,\omega_{n-1})], \tag{203}$$

$$C_n(\omega_1,\dots,\omega_{n-1}) = \sum_{i_1=1,\dots,i_{n-1}=1}^{2} d_{i_1,\dots,i_{n-1}}\chi_{A^{i_1,\dots,i_{n-1}}}(\omega_1,\dots,\omega_{n-1}). \tag{204}$$

We proved that for every martingale $\{m_n(\omega_1,\dots,\omega_n)\}_{n=0}^N$ relative to measure μ_0 the representation (181) is true, due to the conditions (180). Let us prove that the martingale measure is unique. Suppose that there are at most two martingale measures μ_0^1 and μ_0^2. If to put $m(\omega_1,\dots,\omega_N)=\chi_A(\omega_1,\dots,\omega_N)$, then

$$\chi_A(\omega_1,\dots,\omega_N)=$$

$$\sum_{n=1}^{N}C_n(\omega_1,\dots,\omega_{n-1})[S_n(\omega_1,\dots,\omega_{n-1},\omega_n)-S_{n-1}(\omega_1,\dots,\omega_{n-1})]+c_0. \quad (205)$$

From this representation, we obtain the equalities $\mu_0^1(A)=\mu_0^2(A)=c_0,\ A\in\mathcal{F}_N$. Contradiction. The last proves Theorem 30.

Next Theorem is concerned the case as the set of martingale measures consists of one measure.

Theorem 31. *On the probability space* $\{\Omega_N,\mathcal{F}_N,P_N\}$, *being the direct product of the probability spaces* $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, *suppose that the evolution of risky asset is given by the formula (172), then the set of martingale measures, being equivalent to the measure* P_N, *consists of one point*

$$\mu_0(A)=$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\chi_A(\omega_1^{i_1},\dots,\omega_N^{i_N}),\quad A\in\mathcal{F}_N. \quad (206)$$

The fair price φ_0 *of European type option with the payoff function* $\varphi(\omega_1,\dots,\omega_N)$ *is given by the formula*

$$\varphi_0=\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\varphi(\omega_1^{i_1},\dots,\omega_N^{i_N}), \quad (207)$$

where the number of shares is determined by the formula (208) and the number of bonds is determined by the formula (209)

$$\gamma_k(\omega_1,\dots,\omega_{k-1})=$$

$$\frac{m_k(\omega_1,\dots,\omega_{k-1},\omega_k^1)-m_k(\omega_1,\dots,\omega_{k-1},\omega_k^2)}{S_k(\omega_1,\dots,\omega_{k-1},\omega_k^1)-S_k(\omega_1,\dots,\omega_{k-1},\omega_k^2)},\quad k=\overline{1,N}, \quad (208)$$

$$\beta_k(\omega_1,\dots,\omega_{k-1})=$$

$$m_{k-1}(\omega_1,\dots,\omega_{k-1})-\gamma_k(\omega_1,\dots,\omega_{k-1})S_{k-1}(\omega_1,\dots,\omega_{k-1}),\quad k=\overline{1,N}, \quad (209)$$

where

$$m_k(\omega_1,\dots,\omega_k)=E^{\mu_0}\{\varphi(\omega_1,\dots,\omega_N)|\mathcal{F}_k\}=$$

$$\frac{\sum\limits_{i_{k+1}=1,\dots,i_N=1}^{2}\varphi(\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{k+1}},\dots,\omega_N^{i_N})\mu_0(\{\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{n+1}},\dots,\omega_N^{i_N}\})}{\sum\limits_{i_{k+1}=1,\dots,i_N=1}^{2}\mu_0(\{\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{k+1}},\dots,\omega_N^{i_N}\})},$$

$$\psi_n(\omega_1,\ldots,\omega_n) = \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n)\,\psi_n^1(\omega_1,\ldots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n)\,\psi_n^2(\omega_1,\ldots,\omega_n), \quad (210)$$

$$\psi_n^1(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\chi_{\Omega_n^{0+}}(\omega_n^2)\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}, \quad \omega_n^2\in\Omega_n^{0+}, \quad (211)$$

$$\psi_n^2(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\chi_{\Omega_n^{0-}}(\omega_n^1)\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}, \quad \omega_n^1\in\Omega_n^{0-}. \quad (212)$$

Proof. Since

$$\psi_n^1(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)} > 0, \quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}, \quad \omega_n^2\in\Omega_n^{0+}, \quad (213)$$

$$\psi_n^2(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)} > 0, \quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}, \quad \omega_n^1\in\Omega_n^{0-}, \quad (214)$$

we have

$$\psi_n(\omega_1,\ldots,\omega_n) = \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n)\,\psi_n^1(\omega_1,\ldots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n)\,\psi_n^2(\omega_1,\ldots,\omega_n) > 0, \quad (\omega_1,\ldots,\omega_n)\in\Omega_n. \quad (215)$$

From this, it follows that $\mu_0(A) > 0$ for every $A \in \mathcal{F}_N$. It means that $\mu_0(A)$ is equivalent to P_N. The inequality

$$S_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1) = \prod_{i=1}^{n-1}(1+a_i(\omega_1,\ldots,\omega_i)\eta_i(\omega_i))(1+a_n(\omega_1,\ldots,\omega_n^1)\eta_i(\omega_n^1)) \neq$$

$$S_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2) =$$

$$\prod_{i=1}^{n-1}(1+a_i(\omega_1,\ldots,\omega_i)\eta_i(\omega_i))(1+a_n(\omega_1,\ldots,\omega_n^2)\eta_i(\omega_n^2)), \quad n=\overline{1,N}, \quad (216)$$

is true, since

$$(1+a_n(\omega_1,\ldots,\omega_n^1)\eta_i(\omega_n^1)) \neq$$

$$(1 + a_n(\omega_1, \ldots, \omega_n^2)\eta_i(\omega_n^2)), \quad n = \overline{1, N}, \tag{217}$$

due to the suppositions relative to the evolutions of risky asset, given by the formula (172). Thanks to Theorem 30, the martingale measure μ_0 is unique.

To prove the rest statement of Theorem 31, we need to construct the self-financing strategy π such that the capital corresponding this strategy on (B, S) market satisfies the condition

$$X_0^\pi = E^{\mu_0}\varphi(\omega_1, \ldots, \omega_{n-1}, \omega_N), \quad X_N^\pi = \varphi(\omega_1, \ldots, \omega_{n-1}, \omega_N).$$

Let us consider the martingale

$$m_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) = E^{\mu_0}\{\varphi(\omega_1, \ldots, \omega_{n-1}, \omega_N)|\mathcal{F}_n\}.$$

Due to Theorem 30, for the finite martingale $\{m_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)\}_{n=0}^N$ relative to the the measure $\mu_0(A)$ the representation

$$m_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) =$$

$$\sum_{i=1}^n C_i(\omega_1, \ldots, \omega_{i-1})[S_i(\omega_1, \ldots, \omega_{i-1}, \omega_i) - S_{i-1}(\omega_1, \ldots, \omega_{i-1})] +$$

$$m_0, \quad n = \overline{1, N}, \tag{218}$$

is true, where $C_i(\omega_1, \ldots, \omega_{i-1})$ is $\mathcal{F}_{i-1}$ measurable random value, and $m_0 = E^{\mu_0}\varphi(\omega_1, \ldots, \omega_{n-1}, \omega_N)$. If to put $\pi = \{\beta_n, \gamma_n\}_{n=0}^N$, where

$$\gamma_n = C_n(\omega_1, \ldots, \omega_{n-1}), \quad \beta_n = m_{n-1}(\omega_1, \ldots, \omega_{n-1}) - \gamma_n S_{n-1}(\omega_1, \ldots, \omega_{n-1}),$$

then it easy to see that π is self-financed strategy. Really, since $B_n = 1, \ n = \overline{0, N}$, we have

$$\Delta\beta_n B_{n-1} + \Delta\gamma_n S_{n-1} = \Delta\beta_n + \Delta\gamma_n S_{n-1} =$$
$$m_{n-1} - \gamma_n S_{n-1} - m_{n-2} + \gamma_{n-1}S_{n-2} + (\gamma_n - \gamma_{n-1})S_{n-1} =$$
$$m_{n-1} - m_{n-2} - \gamma_{n-1}(S_{n-1} - S_{n-2}) = 0.$$

$\mathcal{F}_{n-1}$-measurability of (β_n, γ_n) is evident. It is easy to show that

$$X_n^\pi(\omega_1, \ldots, \omega_n) = \beta_n B_n + \gamma_n S_n = m_n(\omega_1, \ldots, \omega_n).$$

Therefore,

$$X_0^\pi = m_0 = E^{\mu_0}\varphi(\omega_1, \ldots, \omega_{n-1}, \omega_N), \quad X_N = \varphi(\omega_1, \ldots, \omega_{n-1}, \omega_N).$$

VIII. COMPLETE MARKET HEDGING

In this section, the securities market is constructed, the evolution of which occurs in accordance with formula (172). Possible for this was the observation that with respect to a certain class of evolutions of risky assets, the family of martingale measures is invariant. This fact turned out to be crucial for the construction of models of non-arbitrage markets. In papers [11], [13], such a possibility of the existence of non-arbitrage markets is established on the basis of the Hahn-Banach Theorem. This beautiful result has the disadvantage that it does not provide an algorithm for constructing models of non-arbitrage markets. How to build them having the evolution of risky assets is practically a difficult problem.

In Proposition 1, we establish the form of measurable transformations relative to which the only measure is invariant. Using that, a model of the securities market is built, which is complete. This result is constructive in contrast to the existence theorem from [11], [13]. Our denotations in this section are the same as in the previous section. We consider the evolution of risky asset, given by the formula (172), on the same probability space.

Proposition 1. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset be given by the formula (172), with $a_i(\omega_1, \ldots, \omega_i) = b_i(\omega_1, \ldots, \omega_{i-1}) f_i(\omega_1, \ldots, \omega_i)$, where the random variables $f_i(\omega_1, \ldots, \omega_i)$, $b_i(\omega_1, \ldots, \omega_{i-1})$, satisfy the inequalities*

$$f_i(\omega_1, \ldots, \omega_i) > 0, \quad b_i(\omega_1, \ldots, \omega_{i-1}) > 0, \quad \max_{\{\omega_1, \ldots, \omega_{i-1}\} \in \Omega_{i-1}} b_i(\omega_1, \ldots, \omega_{i-1}) <$$

$$\frac{1}{\max\limits_{\{\omega_1, \ldots, \omega_{i-1}\} \in \Omega_{i-1}} f_i(\omega_1, \ldots, \omega_{i-1}, \omega_i^1) \eta_i^-(\omega_i^1)}, \quad i = \overline{1, N}. \tag{219}$$

For such an evolution, the unique martingale measure μ_0 does not depend on the random variables $b_i(\omega_1, \ldots, \omega_{i-1})$, $i = \overline{1, N}$, and it is given by the formula

$$\mu_0(A) = \mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}(A) =$$

$$\sum_{i_1=1}^{2} \cdots \sum_{i_N=1}^{2} \prod_{n=1}^{N} \psi_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) \chi_A(\omega_1^{i_1}, \ldots, \omega_N^{i_N}), \quad A \in \mathcal{F}_N, \tag{220}$$

where

$$\psi_n(\omega_1, \ldots, \omega_n) = \chi_{\Omega_n^-}(\omega_1, \ldots, \omega_{n-1}, \omega_n) \, \psi_n^1(\omega_1, \ldots, \omega_n) +$$

$$\chi_{\Omega_n^+}(\omega_1, \ldots, \omega_{n-1}, \omega_n) \psi_n^2(\omega_1, \ldots, \omega_n), \tag{221}$$

$$\psi_n^1(\omega_1, \ldots, \omega_{n-1}, \omega_n) = \chi_{\Omega_n^{0+}}(\omega_n^2) \frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} =$$

$$\frac{f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \eta_n^+(\omega_n^2)}{f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \eta_n^+(\omega_n^2) + f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) \eta_n^-(\omega_n^1)}, \quad \omega_n^2 \in \Omega_n^{0+}, \tag{222}$$

$$\psi_n^2(\omega_1, \ldots, \omega_{n-1}, \omega_n) = \chi_{\Omega_n^{0-}}(\omega_n^1) \frac{\Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} =$$

$$\frac{f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) \eta_n^-(\omega_n^1)}{f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \eta_n^+(\omega_n^2) + f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) \eta_n^-(\omega_n^1)}, \quad \omega_n^1 \in \Omega_n^{0-}. \tag{223}$$

Proof. The proof of Proposition 1 is the same as proof of Theorem 8.

Suppose that the market consists of d assets the evolutions of which are given by the law

$$S_n((\omega_1,\dots,\omega_n) = \{S_n^1((\omega_1,\dots,\omega_n),\dots,S_n^d((\omega_1,\dots,\omega_n)\}, \quad n=\overline{1,N}, \tag{224}$$

where

$$S_n^k((\omega_1,\dots,\omega_n) = S_0^k\prod_{i=1}^{n}(1+b_i^k(\omega_1,\dots,\omega_{i-1})f_i(\omega_1,\dots,\omega_i)\eta_i(\omega_i)), \quad k=\overline{1,d}, \tag{225}$$

and the random values $\eta_i(\omega_i)$, $f_i(\omega_1,\dots,\omega_i)$, $i=\overline{1,N}$, does not depend on k, and satisfy inequalities

$$f_i(\omega_1,\dots,\omega_i)>0, \quad b_i^k(\omega_1,\dots,\omega_{i-1})>0, \quad \max_{\{\omega_1,\dots,\omega_{i-1}\}\in\Omega_{i-1}} b_i^k(\omega_1,\dots,\omega_{i-1}) <$$

$$\frac{1}{\max_{\{\omega_1,\dots,\omega_{i-1}\}\in\Omega_{i-1}} f_i(\omega_1,\dots,\omega_{i-1},\omega_i^1)\eta_i^-(\omega_i^1)}, \quad k=\overline{1,d}, \quad i=\overline{1,N}. \tag{226}$$

Proposition 2. *On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, if the evolution of d risky assets is given by the formula (224), (225), then such a market is complete non arbitrage one. The unique martingale measure does not depend on the random variables $b_i^k(\omega_1,\dots,\omega_{i-1}), k=\overline{1,d}, \quad i=\overline{1,N}$, and it is determined by the formula (220). For the contingent claims $\varphi_i(\omega_1,\dots,\omega_N), i=\overline{1,d}$, the fair prices φ_0^i are given by the formulas*

$$\varphi_0^i=\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\varphi_i(\omega_1^{i_1},\dots,\omega_N^{i_N}), \quad i=\overline{1,d}, \tag{227}$$

where the number of i-th shares is determined by the formula (228) and the number of i-th bonds is determined by the formula (229)

$$\gamma_k^i(\omega_1,\dots,\omega_{k-1})=$$

$$\frac{m_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^1)-m_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^2)}{S_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^1)-S_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^2)}, \quad k=\overline{1,N}, \tag{228}$$

$$\beta_k^i(\omega_1,\dots,\omega_{k-1})=$$

$$m_{k-1}^i(\omega_1,\dots,\omega_{k-1})-\gamma_k^i(\omega_1,\dots,\omega_{k-1})S_{k-1}^i(\omega_1,\dots,\omega_{k-1}), \quad k=\overline{1,N}, \tag{229}$$

$$m_k^i(\omega_1,\dots,\omega_k)=E^{\mu_0}\{\varphi_i(\omega_1,\dots,\omega_N)|\mathcal{F}_k\}=$$

$$\frac{\sum_{i_{k+1}=1,\dots,i_N=1}^{2}\varphi_i(\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{k+1}},\dots,\omega_N^{i_N})\mu_0(\{\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{n+1}},\dots,\omega_N^{i_N}\})}{\sum_{i_{k+1}=1,\dots,i_N=1}^{2}\mu_0(\{\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{k+1}},\dots,\omega_N^{i_N}\})},$$

Corollary 3. *(Cox, Ross, Rubinstein, see [25]) On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset is given by the formula*

$$S_n^1((\omega_1, \ldots, \omega_n) = S_0 \prod_{i=1}^{n}(1+\rho_i(\omega_i)), \quad n = \overline{1,N}, \tag{230}$$

where the random values $\rho_i(\omega_i)$, $i = \overline{1,N}$, are such that $\rho_i(\omega_i^1) = a$, $\rho_i(\omega_i^2) = b$, and let the bank account evolution be given by the formula

$$B_n = B_0(1+r)^n, \quad r > 0, \quad B_0 > 0, \quad n = \overline{1,N}. \tag{231}$$

Then, for the discount evolution of risky asset

$$S_n((\omega_1, \ldots, \omega_n) = \frac{S_0 \prod\limits_{i=1}^{n}(1+\rho_i(\omega_i))}{B_0(1+r)^n}, \quad n = \overline{1,N}, \tag{232}$$

the martingale measure μ_0 is unique if $a < r < b$. It is a direct product of measures $\mu_0^i(A)$, $A \in \mathcal{F}_i^0$, $i = \overline{1,N}$, given on the measurable space $\{\Omega_i^0, \mathcal{F}_i^0\}$, where $\mu_0^i(\omega_i^1) = \frac{b-r}{b-a}$, $\mu_0^i(\omega_i^2) = \frac{r-a}{b-a}$. The fair prices φ_0 of the contingent liability $\varphi_N(\omega_1, \ldots, \omega_N)$ is given by the formula

$$\varphi_0 = \int\limits_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\mu_0 =$$

$$\sum_{i_1=1}^{2} \ldots \sum_{i_N=1}^{2} \varphi_N(\omega_1^{i_1}, \ldots, \omega_N^{i_N}) \prod_{k=1}^{N} \mu_0^k(\omega_k^{i_k}), \tag{233}$$

where the number of shares is determined by the formula (234) and the number of bonds is determined by the formula (235)

$$\gamma_k(\omega_1, \ldots, \omega_{k-1}) =$$

$$\frac{m_k(\omega_1, \ldots, \omega_{k-1}, \omega_k^1) - m_k(\omega_1, \ldots, \omega_{k-1}, \omega_k^2)}{S_k(\omega_1, \ldots, \omega_{k-1}, \omega_k^1) - S_k(\omega_1, \ldots, \omega_{k-1}, \omega_k^2)}, \quad k = \overline{1,N}, \tag{234}$$

$$\beta_k(\omega_1, \ldots, \omega_{k-1}) =$$

$$m_{k-1}(\omega_1, \ldots, \omega_{k-1}) - \gamma_k(\omega_1, \ldots, \omega_{k-1}) S_{k-1}(\omega_1, \ldots, \omega_{k-1}), \tag{235}$$

$$m_k(\omega_1, \ldots, \omega_k) = E^{\mu_0}\{\varphi_N(\omega_1, \ldots, \omega_N) | \mathcal{F}_k\} =$$

$$\frac{\sum\limits_{i_{k+1}=1, \ldots, i_N=1}^{2} \varphi_N(\omega_1, \ldots, \omega_k, \omega_{k+1}^{i_{k+1}}, \ldots, \omega_N^{i_N}) \mu_0(\{\omega_1, \ldots, \omega_k, \omega_{k+1}^{i_{n+1}}, \ldots, \omega_N^{i_N}\})}{\sum\limits_{i_{k+1}=1, \ldots, i_N=1}^{2} \mu_0(\{\omega_1, \ldots, \omega_k, \omega_{k+1}^{i_{k+1}}, \ldots, \omega_N^{i_N}\})}.$$

Proof. For the discount evolution (232), the representation

$$S_n((\omega_1,\dots,\omega_n) = S_0\prod_{i=1}^{n}(1+\eta_i(\omega_i)), \quad n=\overline{1,N}, \tag{236}$$

is true, where $\eta_i(\omega_i) = \frac{\rho_i(\omega_i))-r}{(1+r)}$. Due to Theorems 30, 31, since $\eta_i(\omega_i^1) = \frac{a-r}{1+r} < 0$, $\eta_i(\omega_i^2) = \frac{b-r}{1+r} > 0$, then the measure μ_0 is unique. The rest statement of Corollary follows from Theorem 31.

Theorem 32. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset be given by the formula*

$$S_n^1((\omega_1,\dots,\omega_n) = S_0\prod_{i=1}^{n}(1+\rho_i(\omega_i)), \quad n=\overline{1,N}, \tag{237}$$

where the random values $\rho_i(\omega_i)$, $i=\overline{1,N}$, are such that $\rho_i(\omega_i^1) = b_i^1$, $\rho_i(\omega_i^2) = b_i^2$, $i=\overline{1,N}$, and let the bank account evolution be given by the formula

$$B_n = B_0\prod_{i=1}^{n}(1+r_{i-1}(\omega_{i-1})), \quad B_0>0, \quad n=\overline{1,N}, \tag{238}$$

where the random values $r_i(\omega_i)$, $i=\overline{1,N-1}$, are such that $r_i(\omega_i^1) = r_i^1$, $r_i(\omega_i^2) = r_i^2$, $i=\overline{1,N-1}$, $r_0>0$. Then, for the discount evolution of risky asset

$$S_n((\omega_1,\dots,\omega_n) = \frac{S_0\prod_{i=1}^{n}(1+\rho_i(\omega_i))}{B_0\prod_{i=1}^{n}(1+r_{i-1}(\omega_{i-1}))}, \quad n=\overline{1,N}, \tag{239}$$

the martingale measure μ_0 is unique, if $b_1^1 < r_0 < b_1^2$, $b_i^1 < r_{i-1}^1 < r_{i-1}^2 < b_i^2$, $i=\overline{2,N}$. It is determined by the formula (220) with

$$\eta_1(\omega_1) = \rho_1(\omega_1) - r_0, \quad \eta_i(\omega_i) = \rho_i(\omega_i) - r_{i-1}^2, \quad i=\overline{2,N},$$

$$f_1(\omega_1) = \frac{1}{1+r_0}, \quad f_i(\omega_1,\dots,\omega_i) =$$

$$\frac{\rho_i(\omega_i) - r_{i-1}(\omega_{i-1})}{(\rho_i(\omega_i) - r_{i-1}^2)(1+r_{i-1}(\omega_{i-1}))}, \quad i=\overline{2,N}. \tag{240}$$

The fair price φ_0 of the contingent liability $\varphi_N(\omega_1,\dots,\omega_N)$ is given by the formula

$$\varphi_0 = \int\limits_{\Omega_N}\varphi_N(\omega_1,\dots,\omega_N)d\mu_0 =$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\varphi_N(\omega_1^{i_1},\dots,\omega_N^{i_N}), \tag{241}$$

where the number of shares is determined by the formula (242) and the number of bonds is determined by the formula (243)

$$\gamma_k(\omega_1,\dots,\omega_{k-1}) =$$

$$\frac{m_k(\omega_1,\dots,\omega_{k-1},\omega_k^1) - m_k(\omega_1,\dots,\omega_{k-1},\omega_k^2)}{S_k(\omega_1,\dots,\omega_{k-1},\omega_k^1) - S_k(\omega_1,\dots,\omega_{k-1},\omega_k^2)}, \quad k=\overline{1,N}, \tag{242}$$

$$\beta_k(\omega_1,\dots,\omega_{k-1}) =$$

$$m_{k-1}(\omega_1,\dots,\omega_{k-1}) - \gamma_k(\omega_1,\dots,\omega_{k-1})S_{k-1}(\omega_1,\dots,\omega_{k-1}), \tag{243}$$

$$m_k(\omega_1,\dots,\omega_k) = E^{\mu_0}\{\varphi_N(\omega_1,\dots,\omega_N)|\mathcal{F}_k\} =$$

$$\frac{\sum\limits_{i_{k+1}=1,\dots,i_N=1}^{2}\varphi_N(\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{k+1}},\dots,\omega_N^{i_N})\mu_0(\{\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{n+1}},\dots,\omega_N^{i_N}\})}{\sum\limits_{i_{k+1}=1,\dots,i_N=1}^{2}\mu_0(\{\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{k+1}},\dots,\omega_N^{i_N}\})}.$$

Proof. To prove Theorem 32 it is necessary to prove the existence of unique spot measure. The discount evolution (239) can be represented in the form

$$S_n((\omega_1,\dots,\omega_n) =$$

$$\frac{S_0}{B_0}\prod_{i=1}^{n}(1+f_i(\omega_1,\dots,\omega_i)\eta_i(\omega_i)), \quad n=\overline{1,N}, \tag{244}$$

where

$$\eta_1(\omega_1) = \rho_1(\omega_1) - r_0, \quad \eta_i(\omega_i) = \rho_i(\omega_i) - r_{i-1}^2, \quad i=\overline{2,N},$$

$$f_1(\omega_1) = \frac{1}{1+r_0}, \quad f_i(\omega_1,\dots,\omega_i) =$$

$$\frac{\rho_i(\omega_i) - r_{i-1}(\omega_{i-1})}{(\rho_i(\omega_i) - r_{i-1}^2)(1+r_{i-1}(\omega_{i-1}))}, \quad i=\overline{2,N}. \tag{245}$$

It is evident that $\eta_i(\omega_i^1) < 0,\ \eta_i(\omega_i^2) > 0,\ f_i(\omega_1,\dots,\omega_i) > 0$. Therefore, from the representation (244), (245) it follows that we can construct only one spot measure, which is martingale measure, being equivalent to the initial measure P_N. In accordance with Theorem 30, since $S_n(\omega_1,\dots,\omega_n^1) \neq S_n(\omega_1,\dots,\omega_n^2),\ \{\omega_1,\dots,\omega_{n-1}\} \in \Omega_{n-1}$ such a measure is unique. Theorem 32 is proved.

Theorem 33. *On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, let the evolution of risky asset be given by the formula*

$$S_n^1((\omega_1,\dots,\omega_n) = S_0\prod_{i=1}^{n}e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)}, \quad n=\overline{1,N}, \tag{246}$$

where the random values $\varepsilon_i(\omega_i)$, $i = \overline{1, N}$, *are such that* $\varepsilon_i(\omega_i^1) < 0$, $\varepsilon_i(\omega_i^2) > 0$, $\sigma_i(\omega_1, \dots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $i = \overline{1, N}$, *and let the bank account evolution be given by the formula*

$$B_n = B_0 \prod_{i=1}^{n}(1 + r_{i-1}(\omega_{i-1})), \quad B_0 > 0, \quad n = \overline{1, N}, \tag{247}$$

where the random values $r_i(\omega_i)$, $i = \overline{1, N-1}$, *are such that* $r_i(\omega_i^1) = r_i^1$, $r_i(\omega_i^2) = r_i^2$, $i = \overline{1, N-1}$, $r_0 > 0$. *Then, for the discount evolution of risky asset*

$$S_n((\omega_1, \dots, \omega_n) = \frac{S_0 \prod_{i=1}^{n} e^{\sigma_i(\omega_1, \dots, \omega_{i-1})\varepsilon_i(\omega_i)}}{B_0 \prod_{i=1}^{n}(1 + r_{i-1}(\omega_{i-1}))}, \quad n = \overline{1, N}, \tag{248}$$

the martingale measure μ_0 *is unique, if*

$$\exp\{\sigma_1^0 \varepsilon_1(\omega_1^1)\} < r_0 < \exp\{\sigma_1^0 \varepsilon_1(\omega_1^2)\},$$

$$\exp\{\sigma_i^0 \varepsilon_i(\omega_i^1)\} < r_{i-1}^1 < r_{i-1}^2 < \exp\{\sigma_i^0 \varepsilon_i(\omega_i^2)\}, \; i = \overline{2, N}. \tag{249}$$

It is determined by the formula (220) with

$$\eta_1(\omega_1) = \exp\{\sigma_1^0 \varepsilon_1(\omega_1)\} - r_0, \quad f_1(\omega_1) = \frac{1}{1 + r_0},$$

$$\eta_i(\omega_i) = \exp\{\sigma_i^0 \varepsilon_i(\omega_i)\} - r_{i-1}^2, \quad f_i(\omega_1, \dots, \omega_i) =$$

$$\frac{e^{\sigma_i(\omega_1, \dots, \omega_{i-1})\varepsilon_i(\omega_i)} - r_{i-1}(\omega_{i-1})}{(\exp\{\sigma_i^0 \varepsilon_i(\omega_i)\} - r_{i-1}^2)(1 + r_{i-1}(\omega_{i-1}))}, \quad \{\omega_1, \dots, \omega_i\} \in \Omega_n, \quad i = \overline{2, N}. \tag{250}$$

The fair price φ_0 *of the contingent liability* $\varphi_N(\omega_1, \dots, \omega_N)$ *is given by the formula*

$$\varphi_0 = \int_{\Omega_N} \varphi_N(\omega_1, \dots, \omega_N) d\mu_0 =$$

$$\sum_{i_1=1}^{2} \dots \sum_{i_N=1}^{2} \prod_{n=1}^{N} \psi_n(\omega_1^{i_1}, \dots, \omega_n^{i_n}) \varphi_N(\omega_1^{i_1}, \dots, \omega_N^{i_N}), \tag{251}$$

where the number of shares is determined by the formula (252) and the number of bonds is determined by the formula (253)

$$\gamma_k(\omega_1, \dots, \omega_{k-1}) =$$

$$\frac{m_k(\omega_1, \dots, \omega_{k-1}, \omega_k^1) - m_k(\omega_1, \dots, \omega_{k-1}, \omega_k^2)}{S_k(\omega_1, \dots, \omega_{k-1}, \omega_k^1) - S_k(\omega_1, \dots, \omega_{k-1}, \omega_k^2)}, \quad k = \overline{1, N}, \tag{252}$$

$$\beta_k(\omega_1,\ldots,\omega_{k-1}) =$$

$$m_{k-1}(\omega_1,\ldots,\omega_{k-1}) - \gamma_k(\omega_1,\ldots,\omega_{k-1})S_{k-1}(\omega_1,\ldots,\omega_{k-1}), \tag{253}$$

$$m_k(\omega_1,\ldots,\omega_k) = E^{\mu_0}\{\varphi_N(\omega_1,\ldots,\omega_N)|\mathcal{F}_k\} =$$

$$\frac{\sum_{i_{k+1}=1,\ldots,i_N=1}^{2} \varphi_N(\omega_1,\ldots,\omega_k,\omega_{k+1}^{i_{k+1}},\ldots,\omega_N^{i_N})\mu_0(\{\omega_1,\ldots,\omega_k,\omega_{k+1}^{i_{n+1}},\ldots,\omega_N^{i_N}\})}{\sum_{i_{k+1}=1,\ldots,i_N=1}^{2} \mu_0(\{\omega_1,\ldots,\omega_k,\omega_{k+1}^{i_{k+1}},\ldots,\omega_N^{i_N}\})}.$$

Proof. For the discount evolution (248), the following representation

$$S_n((\omega_1,\ldots,\omega_n) =$$

$$\frac{S_0}{B_0}\prod_{i=1}^{n}\left(1+f_i(\omega_1,\ldots,\omega_i)\eta_i(\omega_i)\right), \quad n=\overline{1,N}, \tag{254}$$

is true, where

$$\eta_1(\omega_1) = \exp\{\sigma_1^0\varepsilon_1(\omega_1)\} - r_0, \quad f_1(\omega_1) = \frac{1}{1+r_0},$$

$$\eta_i(\omega_i) = \exp\{\sigma_i^0\varepsilon_i(\omega_i)\} - r_{i-1}^2, \quad f_i(\omega_1,\ldots,\omega_i) =$$

$$\frac{e^{\sigma_i(\omega_1,\ldots,\omega_{i-1})\varepsilon_i(\omega_i)} - r_{i-1}(\omega_{i-1})}{(\exp\{\sigma_i^0\varepsilon_i(\omega_i)\} - r_{i-1}^2)(1+r_{i-1}(\omega_{i-1}))}, \quad \{\omega_1,\ldots,\omega_i\}\in\Omega_n, \quad i=\overline{2,N}. \tag{255}$$

It is evident that $\eta_i(\omega_i^1) < 0,\ \eta_i(\omega_i^2) > 0,\ f_i(\omega_1,\ldots,\omega_i) > 0$. From this, we obtain that the spot measure exists and it is unique. Theorem 33 is proved.

On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, being the direct product of probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, suppose that the market consists of d assets the evolution of which is given by the law

$$S_n((\omega_1,\ldots,\omega_n) = \{S_n^1((\omega_1,\ldots,\omega_n),\ldots,S_n^d((\omega_1,\ldots,\omega_n)\}, \quad n=\overline{1,N}, \tag{256}$$

where

$$S_n^k((\omega_1,\ldots,\omega_n) = S_0^k\prod_{i=1}^{n}(1+a_i^k f_i(\omega_1,\ldots,\omega_i)\eta_i(\omega_i)), \quad k=\overline{1,d}, \tag{257}$$

and the random values $\eta_i(\omega_i),\ f_i(\omega_1,\ldots,\omega_i),\ i=\overline{1,N}$, and constants a_i^k satisfy the inequalities

$$\eta_i(\omega_i^1) < 0, \quad \eta_i(\omega_i^2) > 0, \quad f_i(\omega_1,\ldots,\omega_i) > 0,$$

$$0 < a_i^k < \frac{1}{\max_{\{\omega_1,\ldots,\omega_{i-1}\}\in\Omega_{i-1}} f_i(\omega_1,\ldots,\omega_i^1)\eta_i^-(\omega_i^1)}, \quad i=\overline{1,N}, \quad k=\overline{1,d}. \tag{258}$$

Proposition 3. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky assets be given by the formulas (256), (257), where constants a_i^k $i = \overline{1,N}$, $k = \overline{1,d}$, satisfy the inequalities (258). For such an evolution of risky asset the martingale measure μ_0 does not depend on a_i^k and is unique. It is determined by the formula (220). For the contingent claims $\varphi_N^i(\omega_1, \ldots, \omega_N)$, $i = \overline{1,d}$, the fair prices φ_0^i are given by the formulas*

$$\varphi_0^i = \sum_{i_1=1}^{2} \ldots \sum_{i_N=1}^{2} \prod_{n=1}^{N} \psi_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) \varphi_N^i(\omega_1^{i_1}, \ldots, \omega_N^{i_N}), \quad i = \overline{1,d}, \tag{259}$$

where the number of i-th shares is determined by the formula (260) and the number of i-th bonds is determined by the formula (261)

$$\gamma_k^i(\omega_1, \ldots, \omega_{k-1}) =$$

$$\frac{m_k^i(\omega_1, \ldots, \omega_{k-1}, \omega_k^1) - m_k^i(\omega_1, \ldots, \omega_{k-1}, \omega_k^2)}{S_k^i(\omega_1, \ldots, \omega_{k-1}, \omega_k^1) - S_k^i(\omega_1, \ldots, \omega_{k-1}, \omega_k^2)}, \quad k = \overline{1,N}, \tag{260}$$

$$\beta_k^i(\omega_1, \ldots, \omega_{k-1}) =$$

$$m_{k-1}^i(\omega_1, \ldots, \omega_{k-1}) - \gamma_k^i(\omega_1, \ldots, \omega_{k-1}) S_{k-1}^i(\omega_1, \ldots, \omega_{k-1}), \quad k = \overline{1,N}, \tag{261}$$

$$m_k^i(\omega_1, \ldots, \omega_k) = E^{\mu_0}\{\varphi_i(\omega_1, \ldots, \omega_N) | \mathcal{F}_k\} =$$

$$\frac{\sum_{i_{k+1}=1,\ldots,i_N=1}^{2} \varphi_i(\omega_1, \ldots, \omega_k, \omega_{k+1}^{i_{k+1}}, \ldots, \omega_N^{i_N}) \mu_0(\{\omega_1, \ldots, \omega_k, \omega_{k+1}^{i_{n+1}}, \ldots, \omega_N^{i_N}\})}{\sum_{i_{k+1}=1,\ldots,i_N=1}^{2} \mu_0(\{\omega_1, \ldots, \omega_k, \omega_{k+1}^{i_{k+1}}, \ldots, \omega_N^{i_N}\})}.$$

If $f_i(\omega_1, \ldots, \omega_i) = 1$, $i = \overline{1,N}$, the unique martingale measure is a direct product of measures $\mu_0^i(A)$, $A \in \mathcal{F}_i^0$, given on the measurable space $\{\Omega_i^0, \mathcal{F}_i^0\}$, $i = \overline{1,N}$, where

$$\mu_0^i(\omega_i^1) = \frac{\eta_i^+(\omega_i^2)}{(\eta_i^-(\omega_i^1) + \eta_i^+(\omega_i^2))}, \quad \mu_0^i(\omega_i^2) = \frac{\eta_i^-(\omega_i^1)}{(\eta_i^-(\omega_i^1) + \eta_i^+(\omega_i^2))}. \tag{262}$$

The fair prices φ_0^i, $i = \overline{1,N}$, of the contingent liability $\varphi_N^i(\omega_1, \ldots, \omega_N)$, $i = \overline{1,N}$, are given by the formula

$$\varphi_0^i = \int_{\Omega_N} \varphi_N^i(\omega_1, \ldots, \omega_N) d\mu_0 =$$

$$\sum_{i_1=1}^{2} \ldots \sum_{i_N=1}^{2} \varphi_N^i(\omega_1^{i_1}, \ldots, \omega_N^{i_N}) \prod_{k=1}^{N} \mu_0^k(\omega_k^{i_k}), \tag{263}$$

where the number of i-th shares is determined by the formula (264) and the number of i-th bonds is determined by the formula (265)

$$\gamma_k^i(\omega_1, \ldots, \omega_{k-1}) =$$

$$\frac{m_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^1) - m_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^2)}{S_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^1) - S_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^2)}, \quad k=\overline{1,N}, \tag{264}$$

$$\beta_k^i(\omega_1,\dots,\omega_{k-1}) =$$

$$m_{k-1}^i(\omega_1,\dots,\omega_{k-1}) - \gamma_k^i(\omega_1,\dots,\omega_{k-1})S_{k-1}^i(\omega_1,\dots,\omega_{k-1}), \quad k=\overline{1,N}, \tag{265}$$

$$m_k^i(\omega_1,\dots,\omega_k) = E^{\mu_0}\{\varphi_i(\omega_1,\dots,\omega_N)|\mathcal{F}_k\} =$$

$$\frac{\sum\limits_{i_{k+1}=1,\dots,i_N=1}^{2} \varphi_i(\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{k+1}},\dots,\omega_N^{i_N})\mu_0(\{\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{n+1}},\dots,\omega_N^{i_N}\})}{\sum\limits_{i_{k+1}=1,\dots,i_N=1}^{2} \mu_0(\{\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{k+1}},\dots,\omega_N^{i_N}\})}.$$

If $S_0^i, S_1^i, \dots, S_N^i$, $i=\overline{1,d}$, are the samples of the processes (256), (257) let us denote the order statistics $S_{(0)}^i, S_{(1)}^i, \dots, S_{(N)}^i$, $i=\overline{1,d}$, of this samples.

Proposition 4. *Suppose that $S_0^i, S_1^i, \dots, S_N^i$ is a sample of the random processes (256), (257). Then, for the parameters $a_1^i, \dots, a_N^i$ the estimation*

$$a_k^i = \frac{\left[1 - \frac{S_{(N-k)}^i}{S_{(N-k+1)}^i}\right]}{f_k \eta_k^-(\omega_k^1)}, \quad k=\overline{1,N}, \quad i=\overline{1,d}, \tag{266}$$

is valid.

In the formulas (266) we put that $f_k = \max\limits_{\{\omega_1,\dots,\omega_{k-1}\}\in\Omega_{k-1}} f_k(\omega_1,\dots,\omega_{k-1},\omega_k^1), \quad k = \overline{1,N}$.

Proof. The proof of Proposition 4 is the same as the proof of Theorem 15.

IX. MARTINGALE MEASURES ON DISCRETE PROBABILITY SPACE

This section presents all the necessary results for constructing a non-arbitrage incomplete market on a discrete probability space. The conditions under which the entire family of martingale measures is described for the considered class of evolution of risky assets are minimal. In particular, conditions are presented under which the family of martingale measures considered is equivalent to the original measure. They are minimal. The entire set of equivalent martingale measures is a convex combination of a finite number of spot martingale measures. On this basis, new formulas were found for the fair price of the super hedge.

In this section, we put that $\Omega_i^0 = \{\omega_i^1, \dots, \omega_i^M\}$, $i=\overline{1,N}$, and we assume that $2 < M < \infty$, the σ-algebra $\mathcal{F}_i^0$ consists from all subsets of Ω_i^0. We suppose that $P_i^0(\omega_i^k) > 0, \omega_i^k \in \Omega_i^0, k=\overline{1,M}$, $i=\overline{1,N}$. As before, the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$ is a direct product of probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}, i=\overline{1,N}$. Sometimes, any elementary event $\omega_i^k \in \Omega_i^0$ it is convenient to denote by ω_i not indicating the index k. Further, we use the both denotations. As in section 2, we introduce filtration $\mathcal{F}_n$ on the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$. As before, it is convenient to introduce the family of probability spaces $\{\Omega_n, \mathcal{F}_n, P_n\}, n = \overline{1, N}$, being a direct product of probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}, i = \overline{1, n}$.

The evolution of risky asset is given by the formula (1) with the assumptions given in the section 2. In this case

$$\Omega_n^- = \Omega_{n-1} \times \Omega_n^{0-}, \ \Omega_n^+ = \Omega_{n-1} \times \Omega_n^{0+}. \tag{267}$$

Further, we also use the measurable space with measure

$$\left\{ \prod_{i=1}^{N} [\Omega_i^{0-} \times \Omega_i^{0+}], \prod_{i=1}^{N} [\mathcal{F}_i^{0-} \times \mathcal{F}_i^{0+}], \prod_{i=1}^{N} [P_i^{0-} \times P_i^{0+}] \right\}. \tag{268}$$

The measure P_n^{0-} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0-} = \Omega_n^{0-} \cap \mathcal{F}_n^0$, P_n^{0+} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0+} = \Omega_n^{0+} \cap \mathcal{F}_n^0$. Additionally, we assume

$$P_n^0(\{\omega_n \in \Omega_n^0, |\eta_n(\omega_n)| < \infty\}) = 1. \tag{269}$$

Let us consider the random values

$$\psi_n(\omega_1, \ldots, \omega_n) = \chi_{\Omega_n^{0-}}(\omega_n)\, \psi_n^1(\omega_1, \ldots, \omega_n) +$$

$$\chi_{\Omega_n^{0+}}(\omega_n)\, \psi_n^2(\omega_1, \ldots, \omega_n), \quad n = \overline{1, N}, \tag{270}$$

where

$$\psi_n^1(\omega_1, \ldots, \omega_{n-1}, \omega_n) =$$

$$\chi_{\Omega_n^{0+}}(\omega_n^2) \frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)}, \quad (\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1}, \quad \omega_n^2 \in \Omega_n^{0+}, \tag{271}$$

$$\psi_n^2(\omega_1, \ldots, \omega_{n-1}, \omega_n) =$$

$$\chi_{\Omega_n^{0-}}(\omega_n^1) \frac{\Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)}, \quad (\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1}, \quad \omega_n^1 \in \Omega_n^{0-}. \tag{272}$$

Definition 1. *Let the evolution of risky asset be given by the formula (1). On the measurable space* $\{\Omega_N, \mathcal{F}_N\}$*, being the direct product of the measurable spaces* $\{\Omega_i^0, \mathcal{F}_i^0\}$*, for every point* $\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}\} \in \prod_{i=1}^{N} [\Omega_i^{0-} \times \Omega_i^{0+}]$ *let us introduce the spot measure*

$$\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}(A) =$$

$$\sum_{i_1=1}^{2} \ldots \sum_{i_N=1}^{2} \prod_{n=1}^{N} \psi_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) \chi_A(\omega_1^{i_1}, \ldots, \omega_N^{i_N}), \quad A \in \mathcal{F}_N, \tag{273}$$

where $\psi_n(\omega_1, \ldots, \omega_n)$ *is determined by the formulas (270) - (272).*

Lemma 4. *The spot measure* $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$*, given by the formula (273), is a martingale measure for the evolution of risky asset, given by the formula (1), for every point* $\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\} \in \prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}]$*. If the point* $\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}$ *is such that* $\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1) < 0,\ \Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2) > 0,\ \{\omega_1,\ldots,\omega_{n-1}\} \in \Omega_{n-1},\ n = \overline{1,N}$*, then the spot measure* $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ *is a martingale measure, being equivalent to the measure* P_N*.*

Proof. Let us prove that $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ is a probability measure. Let us calculate

$$\sum_{i_j=1}^{2} \psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j}) = \psi_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1) + \psi_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2) =$$

$$\chi_{\Omega_j^{0-}}(\omega_j^1)\,\psi_j^1(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^1)+$$

$$\chi_{\Omega_j^{0+}}(\omega_j^1)\psi_j^2(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^1)+$$

$$\chi_{\Omega_j^{0-}}(\omega_j^2)\psi_j^1(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^2)+$$

$$\chi_{\Omega_j^{0+}}(\omega_j^2)\,\psi_j^2(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^2) =$$

$$\chi_{\Omega_j^{0-}}(\omega_j^1)\chi_{\Omega_j^{0+}}(\omega_j^2)\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^{0+}}(\omega_j^1)\chi_{\Omega_j^{0-}}(\omega_j^1)\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)}+$$

$$\chi_{\Omega_j^{0-}}(\omega_j^2)\chi_{\Omega_j^{0+}}(\omega_j^2)\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^{0+}}(\omega_j^2)\chi_{\Omega_j^{0-}}(\omega_j^1)\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)} =$$

$$\chi_{\Omega_j^{0-}}(\omega_j^1)\chi_{\Omega_j^{0+}}(\omega_j^2)\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^{0+}}(\omega_j^2)\chi_{\Omega_j^{0-}}(\omega_j^1)\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)} = \chi_{\Omega_j^{0-}}(\omega_j^1)\chi_{\Omega_j^{0+}}(\omega_j^2) = 1.$$

The last equalities proves that $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(\Omega_N) = 1$ for every point $\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\} \in \prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}]$. Further,

$$\sum_{i_j=1}^{2} \psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\Delta S_j(\omega_1^{i_1},\ldots,\omega_j^{i_j}) =$$

$$\psi_j(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\Delta S_j(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^1)+$$
$$\psi_j(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\Delta S_j(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^2)=$$
$$\chi_{\Omega_j^{0-}}(\omega_j^1)\chi_{\Omega_j^{0+}}(\omega_j^2)\times$$
$$\left[-\frac{\Delta S_j^+(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}\Delta S_j^-(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^1)+\right.$$
$$\left.\frac{\Delta S_j^-(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)}\Delta S_j^+(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\right]=0, \quad j=\overline{1,N}. \tag{274}$$

Let us prove that the set of measures $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ is a set of martingale measures. Really, for A, belonging to the σ-algebra $\mathcal{F}_{n-1}$ of the filtration we have $A=B\times\prod_{i=n}^{N}\Omega_i^0$, where B belongs to σ-algebra $\mathcal{F}_{n-1}$ of the measurable space $\{\Omega_{n-1},\mathcal{F}_{n-1}\}$. Then,

$$\int_A \Delta S_n(\omega_1,\dots,\omega_n)d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}=$$
$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\chi_B(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}})\Delta S_n(\omega_1^{i_1},\dots,\omega_n^{i_n})=$$
$$\sum_{i_1=1}^{2}\dots\sum_{i_n=1}^{2}\prod_{j=1}^{n}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\chi_B(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}})\Delta S_n(\omega_1^{i_1},\dots,\omega_n^{i_n})=$$
$$\sum_{i_1=1}^{2}\dots\sum_{i_{n-1}=1}^{2}\prod_{j=1}^{n-1}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\chi_B(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}})\times$$
$$\sum_{i_n=1}^{2}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\Delta S_n(\omega_1^{i_1},\dots,\omega_n^{i_n})=0, \quad A\in\mathcal{F}_{n-1}. \tag{275}$$

To prove the last statement it needs to prove that $\psi_n(\omega_1,\dots,\omega_n)>0$, $n=\overline{1,N}$. But,

$$\psi_n(\omega_1,\dots,\omega_n)=\chi_{\Omega_n^{0-}}(\omega_n)\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}+$$
$$\chi_{\Omega_n^{0+}}(\omega_n)\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}>0, \quad n=\overline{1,N}. \tag{276}$$

The last means the needed statement.

Suppose that the random values $a_i(\omega_1, \dots, \omega_i)$, $\eta_i(\omega_i)$ satisfy the inequalities

$$a_i(\omega_1, \dots, \omega_i) > 0, \qquad \sup_{\{\omega_1,\dots,\omega_i\}\in\Omega_i} a_i(\omega_1, \dots, \omega_i) < \frac{1}{\sup\limits_{\omega_i\in\Omega_i^0, \eta_i(\omega_i)<0} \eta_i^-(\omega_i)},$$

$$P_i^0(\eta_i(\omega_i) < 0) > 0, \quad P_i^0(\eta_i(\omega_i) > 0) > 0, \quad i = \overline{1, N}, \tag{277}$$

the evolution $S_n(\omega_1, \dots, \omega_n)$ is given by the formula

$$S_n(\omega_1, \dots, \omega_n) = S_0 \prod_{i=1}^{n} (1 + a_i(\omega_1, \dots, \omega_i)\eta_i(\omega_i)), \quad n = \overline{1, N}, \quad S_0 > 0. \tag{278}$$

Below, we describe the convex set of equivalent martingale measures.
We use for $\alpha_N(\{\omega_1^1, \dots, \omega_N^1\}; \{\omega_1^2, \dots, \omega_N^2\})$ the denotation $\alpha_N(\{\omega\}_N^1; \{\omega\}_N^2)$.

Theorem 34. *Let the evolution of risky asset be given by the formula (278). On the measurable space with measure* $\{\prod_{i=1}^N [\Omega_i^{0-} \times \Omega_i^{0+}], \prod_{i=1}^N [\mathcal{F}_i^{0-} \times \mathcal{F}_i^{0+}], \prod_{i=1}^N [P_i^{0-} \times P_i^{0+}]\}$, *suppose that the random value* $\alpha_N(\{\omega\}_N^1; \{\omega\}_N^2)$ *satisfies the conditions*

$$\alpha_N(\{\omega\}_N^1; \{\omega\}_N^2) > 0, \quad \{\omega_1^1, \omega_1^2\}, \dots, \{\omega_N^1, \omega_N^2\} \in \prod_{i=1}^N [\Omega_i^{0-} \times \Omega_i^{0+}], \tag{279}$$

$$\int\limits_{\prod_{i=1}^N [\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N(\{\omega_1^1, \dots, \omega_N^1\}; \{\omega_1^2, \dots, \omega_N^2\}) \prod_{i=1}^N dP_i^0(\omega_i^1) dP_i^0(\omega_i^2) = 1. \tag{280}$$

The measure $\mu_0(A)$, given by the formula

$$\mu_0(A) =$$

$$\int\limits_{\prod_{i=1}^N [\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N(\{\omega\}_N^1; \{\omega\}_N^2) \mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A) d\prod_{i=1}^N [P_i^{0-} \times P_i^{0+}], \tag{281}$$

is a martingale measure, being equivalent to the measure P_N.

Proof. Introduce the denotations

$$\alpha_n(\{\omega_1^1, \dots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \dots, \omega_{n-1}^2, \omega_n^2\}) =$$

$$\frac{\int\limits_{\prod_{i=n+1}^N [\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N(\{\omega_1^1, \dots, \omega_N^1\}; \{\omega_1^2, \dots, \omega_N^2\}) \prod\limits_{i=n+1}^N dP_i^0(\omega_i^1) dP_i^0(\omega_i^2)}{\int\limits_{\prod_{i=n}^N [\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N(\{\omega_1^1, \dots, \omega_N^1\}; \{\omega_1^2, \dots, \omega_N^2\}) \prod\limits_{i=n}^N dP_i^0(\omega_i^1) dP_i^0(\omega_i^2)}, \ n = \overline{1, N-1},$$

$$\alpha_N(\{\omega_1^1,\ldots,\omega_{N-1}^1,\omega_N^1\};\{\omega_1^2,\ldots,\omega_{N-1}^2,\omega_N^2\}) =$$

$$\frac{\alpha_N(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\})}{\int\limits_{\Omega_N^{0-}\times\Omega_N^{0+}} \alpha_N(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\})dP_N^0(\omega_N^1)dP_N^0(\omega_N^2)}. \quad (282)$$

It is not difficult to note that

$$\prod_{n=1}^N \alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\}) = \alpha_N(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\}).$$

Since the random values $\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})$ are finite valued, then

$$\int\limits_{\Omega_n^{0-}\times\Omega_n^{0+}} \alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1)dP_n^0(\omega_n^2) < \infty,$$

$$(\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}. \quad (283)$$

It is evident that the set of strictly positive finite valued random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2), n=\overline{1,N}$, given by the formula (282), satisfy the conditions

$$E^{\mu_0}|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)| =$$

$$\int\limits_{\Omega_N}\prod_{i=1}^N \psi_i(\omega_1,\ldots,\omega_i)|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)|\prod_{i=1}^N dP_i^0(\omega_i) < \infty, \quad n=\overline{1,N}. \quad (284)$$

Moreover, for the measure (281) the representation (32) is true, meaning that it is equivalent to the measure P_N. The last proves Theorem 34.

Let us define the integral for the random value $f_N(\omega_1,\ldots,\omega_{N-1},\omega_N)$ relative to the measure $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ by the formula

$$\int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_{N-1},\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} =$$

$$\sum_{i_1=1}^2\ldots\sum_{i_N=1}^2\prod_{n=1}^N \psi_n(\omega_1^{i_1},\ldots,\omega_n^{i_n})f_N(\omega_1^{i_1},\ldots,\omega_N^{i_N}). \quad (285)$$

Theorem 35. *Let the evolution of risky asset be given by the formula (278). If the conditions of Theorem 34 are true, then the fair price of super-hedge f_0 for the nonnegative payoff function $f(x)$ is given by the formula*

$$f_0 = \sup_{P\in M} E^P f(S_N) = \max_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}. \quad (286)$$

Moreover,

$$\inf_{P\in M} E^P f(S_N) = \min_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}. \tag{287}$$

Proof. Let us prove the formula (286). Denote M the set of all martingale measure, being equivalent to P_N. If an equivalent martingale measure $P_0 \in M$, then $\alpha P_0 + (1-\alpha)\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \in M$ for arbitrary $0<\alpha\leq 1$. We have the inequality

$$\alpha E^{P_0} f(S_N) + (1-\alpha)\int\limits_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \leq \sup_{P\in M} E^P f(S_N).$$

Since $\alpha>0$ is arbitrary, we obtain the inequality

$$\int\limits_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \leq \sup_{P\in M} E^P f(S_N).$$

From here, we obtain the inequality

$$\max_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \leq \sup_{P\in M} E^P f(S_N).$$

The inverse inequality follows from the representation (281) for any martingale measure, being equivalent to the measure P_N. Really,

$$E^P f_N = \int\limits_{\prod_{i=1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]} \alpha_N(\{\omega\}_N^1;\{\omega\}_N^2)\times$$

$$\int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_{N-1},\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} d\prod_{i=1}^{N}[P_i^{0-}\times P_i^{0+}]. \tag{288}$$

From the formula (288) it follows the inequality

$$E^P f_N \leq \max_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}.$$

Or,

$$\sup_{P\in M} E^P f_N \leq \max_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}.$$

The proof of (287) is analogous. We have the inequality

$$\alpha E^{P_0} f(S_N) + (1-\alpha)\int\limits_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \geq \inf_{P\in M} E^P f(S_N).$$

Tending α to zero and taking the minimum all over the spot measures we obtain

$$\min_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}}\int\limits_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \geq \inf_{P\in M} E^P f(S_N).$$

Using the representation (288) we have

$$E^P f_N \geq \min_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}}\int\limits_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}.$$

Taking the infimum all over the martingale measures we obtain

$$\inf_{P\in M} E^P f_N \geq \min_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}}\int\limits_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}.$$

Theorem 35 is proved.

X. MODELS OF NON-ARBITRAGE INCOMPLETE FINANCIAL MARKETS

Using the construction of the family of spot measures introduced in the previous section, this section presents the conditions under which the considered family of spot measures is invariant with respect to a certain class of evolutions of risky assets. For a certain class of contingent liabilities including a standard call option, the fair price of the super hedge is shown to be less than the spot price of the underlying asset. Specific applications of the results obtained for the previously known evolutions of risky assets are considered. New formulas is found for the non-arbitrage price range. A model of a non-arbitrage incomplete market is proposed and estimates are obtained in the case of a multi-parameter model of a non-arbitrage market.

On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let us assume that the random values $b_i(\omega_1,\ldots,\omega_{i-1})$, $f_i(\omega_1,\ldots,\omega_i)$, $\eta_i(\omega_i)$, $i=\overline{1,N}$, satisfy the inequalities

$$b_i(\omega_1,\ldots,\omega_{i-1})>0, \quad f_i(\omega_1,\ldots,\omega_i)>0,$$

$$\max_{\{\omega_1,\ldots,\omega_{i-1}\}\in\Omega_{i-1}} b_i(\omega_1,\ldots,\omega_{i-1}) <$$

$$\frac{1}{\max\limits_{\{\omega_1,\ldots,\omega_{i-1}\}\in\Omega_{i-1}}\ \max\limits_{\{\omega_i,\eta_i(\omega_i)<0\}} f_i(\omega_1,\ldots,\omega_i)\eta_i^-(\omega_i)},$$

$$P_i^0(\eta_i(\omega_i)<0)>0, \quad P_i^0(\eta_i(\omega_i)>0)>0, \quad i=\overline{1,N}. \tag{289}$$

As before, we put $\Omega_i^{0-}=\{\omega_i\in\Omega_i^0,\ \eta_i(\omega_i)\leq 0\}$, $\Omega_i^{0+}=\{\omega_i\in\Omega_i^0,\ \eta_i(\omega_i)>0\}$. We assume that the evolution $S_n(\omega_1,\ldots,\omega_n)$ of risky asset is given by the formula

$$S_n(\omega_1,\ldots,\omega_n)=S_0\prod_{i=1}^n(1+b_i(\omega_1,\ldots,\omega_{i-1})f_i(\omega_1,\ldots,\omega_i)\eta_i(\omega_i)), \quad n=\overline{1,N}. \tag{290}$$

With every point $v = \{(\omega_1^1, \omega_1^2), \ldots, (\omega_N^1, \omega_N^2)\} \in \mathcal{V}$, where $\mathcal{V} = \prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}]$, we connect the spot measure

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A) =$$

$$\sum_{i_1=1}^{2} \cdots \sum_{i_N=1}^{2} \prod_{n=1}^{N} \psi_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) \chi_A(\omega_1^{i_1}, \ldots, \omega_N^{i_N}), \quad A \in \mathcal{F}_N. \tag{291}$$

Let us denote $\nu_v(A) = \prod_{i=1}^{N} \nu_{\omega_i^1,\omega_i^2}(A_i)$, $A = \prod_{i=1}^{N} A_i, \in \mathcal{F}_N$, the direct product of the measures $\nu_{\omega_i^1,\omega_i^2}(A_i)$, $A_i \in \mathcal{F}_i^0$, $i = \overline{1,N}$, where $v = \{(\omega_1^1, \omega_1^2), \ldots, (\omega_N^1, \omega_N^2)\} \in \mathcal{V}$, $\mathcal{V} = \prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}]$, and

$$\nu_{\omega_i^1,\omega_i^2}(A_i) = \chi_{A_i}(\omega_i^1)\frac{\eta_i^+(\omega_i^2)}{\eta_i^-(\omega_i^1) + \eta_i^+(\omega_i^2)} + \chi_{A_i}(\omega_i^2)\frac{\eta_i^-(\omega_i^1)}{\eta_i^-(\omega_i^1) + \eta_i^+(\omega_i^2)}, \tag{292}$$

for $\omega_i^1 \in \Omega_i^{0-}$, $\omega_i^2 \in \Omega_i^{0+}$, $A_i \in \mathcal{F}_i^0$. Then, there exists a countable additive function $\nu_v(A), A \in \mathcal{F}_N$, on the σ-algebra $\mathcal{F}_N$ for every $v \in \mathcal{V}$.

Theorem 36. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset be given by the formula (290). For every point $v = \{(\omega_1^1, \omega_1^2), \ldots, (\omega_N^1, \omega_N^2)\} \in \mathcal{V}$, the spot measure $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ given by the formula (291) does not depend on the random values $b_i(\omega_1, \ldots, \omega_{i-1})$, $i = \overline{1,N}$. In the case as $f_i(\omega_1, \ldots, \omega_i) = 1$, $i = \overline{1,N}$, the formula*

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A) = \nu_v(A) \tag{293}$$

is true. For the evolution of risky asset (290), the set of martingale measures being equivalent to the measure P_N does not depend on the random values $b_i(\omega_1, \ldots, \omega_{i-1})$, $i = \overline{1,N}$.

Proof. The proof of Theorem 36 is the same as proof of the Theorem 8.

Theorem 37. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset be given by the formula (290). Suppose that the nonnegative convex down payoff function $f(x)$ on the set $0 \leq x < \infty$ satisfies the inequality $0 \leq f(x) < x$. Then, the inequalities*

$$f(S_0) \leq \sup_{P \in M} E^P f(S_N) =$$

$$\max_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} < S_0 \tag{294}$$

are true.

Proof. Since the set of points $v = \{(\omega_1^1, \omega_1^2), \dots, (\omega_N^1, \omega_N^2)\}$ in the set $\mathcal{V}$ is finite then the minimum in the formula

$$\min_{\omega_1,\dots,\omega_N} [S_N(\omega_1, \dots, \omega_N) - f(S_N(\omega_1, \dots, \omega_N))] = d > 0 \tag{295}$$

is reached at a certain point $v_0 = \{(\omega_1^{1,0}, \omega_1^{2,0}), \dots, (\omega_N^{1,0}, \omega_N^{2,0})\}$. Therefore, the inequality

$$S_N(\omega_1, \dots, \omega_N) - f(S_N(\omega_1, \dots, \omega_N)) \geq d, \quad \{\omega_1, \dots, \omega_N\} \in \Omega_N, \tag{296}$$

is true

Integrating left and right parts of inequality over the measure $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$, we have

$$\int\limits_{\Omega_N} S_N(\omega_1, \dots, \omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}} -$$

$$\int\limits_{\Omega_N} d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}} f(S_N(\omega_1, \dots, \omega_N)) \geq d. \tag{297}$$

Since

$$\int\limits_{\Omega_N} S_N(\omega_1, \dots, \omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}} = S_0 \tag{298}$$

we obtain the needed. It is evident that from the convexity down of payoff function $f(x)$ and Jensen inequality we obtain the inequality

$$\int\limits_{\Omega_N} f(S_N(\omega_1, \dots, \omega_N)) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}} \geq f(S_0). \tag{299}$$

Theorem 37 is proved.

Let us note that the interval of non arbitrage prices for a certain processes was found in the papers [26], [27].

Corollary 4. *For the standard call option of European type with payoff function $f(x) = (x - K)^+, K > 0$, the conditions of Theorem 37 are true. Therefore, the inequalities (294) are valid.*

Theorem 38. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset be given by the formula (290). Suppose that the nonnegative convex down payoff function $f(x)$ on the set $0 \leq x < \infty$ satisfies the inequality $0 \leq f(x) \leq K$, $K > 0$. Then, the inequalities*

$$f(S_0) \leq \sup_{P \in M} E^P f(S_N) = \max_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}} \leq K \tag{300}$$

are true.

Proof. The proof is evident.

Corollary 5. *For the standard put option of European type with payoff function $f(x) = (K - x)^+, K > 0$, the conditions of Theorem 38 are true. Therefore, the inequalities (300) are valid.*

Corollary 6 For the standard call option of European type with payoff function $f(x) = (x - K)^+, K > 0$, *the interval of non arbitrage prices coincide with the interval*

$$\left(\min_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}, \right.$$

$$\left. \max_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} \right). \tag{301}$$

Corollary 7. *For the standard put option of European type with payoff function $f(x) = (K - x)^+, K > 0$, the interval of non arbitrage prices coincide with the interval*

$$\left(\min_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}, \right.$$

$$\left. \max_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} \right). \tag{302}$$

Corollary 8. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset is given by the formula*

$$S_n^1((\omega_1, \ldots, \omega_n) = S_0 \prod_{i=1}^{n} (1 + \rho_i(\omega_i)), \quad n = \overline{1,N}, \quad S_0 > 0, \tag{303}$$

where the random value $\rho_i(\omega_i)$ is given on the probability space $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}, i = \overline{1,N}$, and let the bank account evolution be given by the formula

$$B_n = B_0(1 + r)^n, \quad r > 0, \quad B_0 > 0, \quad n = \overline{1,N}. \tag{304}$$

Then, for the discount evolution of risky asset

$$S_n((\omega_1, \ldots, \omega_n) = \frac{S_0 \prod_{i=1}^{n} (1 + \rho_i(\omega_i))}{B_0(1 + r)^n}, \quad n = \overline{1,N}, \tag{305}$$

the set of martingale measure is nonempty one if the following conditions are true

$$P_i^0(\rho_i(\omega_i) - r < 0) > 0, \; P_i^0(\rho_i(\omega_i) - r > 0) > 0,$$

$$P_i^0(\rho_i(\omega_i) - r < 0) + P_i^0(\rho_i(\omega_i) - r > 0) = 1, \quad i = \overline{1,N}.$$

For every point $v = \{(\omega_1^1, \omega_1^2), \ldots, (\omega_N^1, \omega_N^2)\}$ *in the set* $\mathcal{V}$ *the spot measure* $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ *is a direct product of measures* $\mu_0^i(A_i)$, $A_i \in \mathcal{F}_i^0$, $i = \overline{1,N}$, *given on the measurable space* $\{\Omega_i^0, \mathcal{F}_i^0\}$, *where* $\mu_0^i(A_i) = \nu_{\omega_i^1,\omega_i^2}(A_i)$, *and* $\nu_{\omega_i^1,\omega_i^2}(A_i)$ *is given by the formula (292) with* $\eta_i(\omega_i) = \frac{\rho_i(\omega_i)-r}{1+r}$, $i = \overline{1,N}$. *The fair price* φ_0 *of super-hedge of the nonnegative contingent liability* $\varphi_N(\omega_1, \ldots, \omega_N)$ *is given by the formula*

$$\varphi_0 = \max_{v \in \mathcal{V}} \int_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\nu_v.$$

The interval of non-arbitrage prices is written in the form

$$\left(\min_{v \in \mathcal{V}} \int_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\nu_v, \ \max_{v \in \mathcal{V}} \int_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\nu_v \right).$$

Theorem 39. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *being the direct product of the probability spaces* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, *let the evolution of risky asset be given by the formula*

$$S_n^1((\omega_1, \ldots, \omega_n) = S_0 \prod_{i=1}^n (1 + \rho_i(\omega_i)), \quad n = \overline{1,N}, \tag{306}$$

where the random value $\rho_i(\omega_i)$, *is given on the probability space* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, $P_i^0(\{\rho_i(\omega_i) < 0\}) > 0$, $P_i^0(\{\rho_i(\omega_i) > 0\}) > 0$, $i = \overline{1,N}$, *and let the bank account evolution be given by the formula*

$$B_n = B_0 \prod_{i=1}^n (1 + r_{i-1}(\omega_{i-1})), \quad B_0 > 0, \quad n = \overline{1,N}, \tag{307}$$

where the strictly positive random values $r_i(\omega_i)$ *are given on the probability* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}, i = \overline{1,N}$. *Then, for the discount evolution of risky asset*

$$S_n((\omega_1, \ldots, \omega_n) = \frac{S_0 \prod_{i=1}^n (1 + \rho_i(\omega_i))}{B_0 \prod_{i=1}^n (1 + r_{i-1}(\omega_{i-1}))}, \quad n = \overline{1,N}, \tag{308}$$

the set of martingale measure is nonempty one if the following conditions are true

$$\max_{\omega_{i-1} \in \Omega_{i-1}} r_{i-1}(\omega_{i-1}) < \min_{\omega_i \in \Omega_i, \rho_i(\omega_i) > 0} \rho_i(\omega_i),$$

$$\min_{\omega_{i-1} \in \Omega_{i-1}} r_{i-1}(\omega_{i-1}) > 0, \ i = \overline{2,N}$$

$$0 < r_0 < \min_{\omega_1 \in \Omega_1, \rho_1(\omega_1) > 0} \rho_1(\omega_1). \tag{309}$$

The fair price φ_0 of super-hedge of the nonnegative contingent liability $\varphi_N(\omega_1,\dots,\omega_N)$ is given by the formula

$$\varphi_0 = \max_{\omega_i^1\in\Omega_i^{0-},\ \omega_i^2\in\Omega_i^{0+},\ i=\overline{1,N}} \int\limits_{\Omega_N} \varphi_N(\omega_1,\dots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}.$$

The interval of non-arbitrage prices is written in the form

$$\Bigg(\min_{\omega_i^1\in\Omega_i^{0-},\ \omega_i^2\in\Omega_i^{0+},\ i=\overline{1,N}} \int\limits_{\Omega_N} \varphi_N(\omega_1,\dots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}},$$

$$\max_{\omega_i^1\in\Omega_i^{0-},\ \omega_i^2\in\Omega_i^{0+},\ i=\overline{1,N}} \int\limits_{\Omega_N} \varphi_N(\omega_1,\dots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}\Bigg).$$

Proof. The discount evolution (308) can be represented in the form

$$S_n(\omega_1,\dots,\omega_n) =$$

$$\frac{S_0}{B_0}\left(1+\frac{(\rho_1(\omega_1)-r_0)}{1+r_0}\right)\prod_{i=2}^{n}\left(1+\frac{\rho_i(\omega_i)-r_{i-1}(\omega_{i-1})}{1+r_{i-1}(\omega_{i-1})}\right) =$$

$$\frac{S_0}{B_0}\prod_{i=1}^{n}\left(1+f_i(\omega_1,\dots,\omega_i)\eta_i(\omega_i)\right), \tag{310}$$

where

$$f_1(\omega_1) = \frac{1}{1+r_0}, \quad \eta_1(\omega_1) = \rho_1(\omega_1) - r_0, \tag{311}$$

$$f_i(\omega_1,\dots,\omega_i) = \frac{\rho_i(\omega_i)-r_{i-1}(\omega_{i-1})}{(\rho_i(\omega_i) - \min\limits_{\omega_{i-1}\in\Omega_{i-1}} r_{i-1}(\omega_{i-1})(1+r_{i-1}(\omega_{i-1}))},$$

$$\eta_i(\omega_i) = \rho_i(\omega_i) - \min_{\omega_{i-1}\in\Omega_{i-1}} r_{i-1}(\omega_{i-1}) \quad i=\overline{2,N}. \tag{312}$$

Since

$$f_i(\omega_1,\dots,\omega_i) > 0, \quad i=\overline{1,N}, \tag{313}$$

$$P_i^0(\eta_i(\omega_i)<0)>0, \quad P_i^0(\eta_i(\omega_i)>0)>0, \quad i=\overline{1,N}, \tag{314}$$

then it means that the set of martingale measures, being equivalent to R_N, is a nonempty set. Theorem 39 is proved.

Theorem 40. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset be given by the formula*

$$S_n^1((\omega_1, \ldots, \omega_n) = S_0 \prod_{i=1}^{n} e^{\sigma_i(\omega_1, \ldots, \omega_{i-1})\varepsilon_i(\omega_i)}, \quad n = \overline{1, N}, \tag{315}$$

where the random values $\varepsilon_i(\omega_i)$, $i = \overline{1, N}$, are such that

$$P_i^0(\varepsilon_i(\omega_i) < 0) > 0, \quad P_i^0(\varepsilon_i(\omega_i) > 0) > 0,$$

$$P_i^0(\varepsilon_i(\omega_i) < 0) + P_i^0(\varepsilon_i(\omega_i) > 0) = 1,$$

$$\sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0, \quad i = \overline{1, N},$$

and let the bank account evolution be given by the formula

$$B_n = B_0 \prod_{i=1}^{n} (1 + r_{i-1}(\omega_{i-1})), \quad B_0 > 0, \quad n = \overline{1, N}, \tag{316}$$

where the random values $r_i(\omega_i)$, $i = \overline{1, N-1}$, are strictly positive ones, $r_0 > 0$. Then, for the discount evolution of risky asset

$$S_n((\omega_1, \ldots, \omega_n) = \frac{S_0 \prod_{i=1}^{n} e^{\sigma_i(\omega_1, \ldots, \omega_{i-1})\varepsilon_i(\omega_i)}}{B_0 \prod_{i=1}^{n} (1 + r_{i-1}(\omega_{i-1}))}, \quad n = \overline{1, N}, \tag{317}$$

the set of martingale measure is nonempty one, if

$$\exp\{\sigma_1^0 \max_{\{\omega_1, \varepsilon_1(\omega_1) < 0\}} \varepsilon_1(\omega_1)\} < r_0 < \exp\{\sigma_1^0 \min_{\{\omega_1, \varepsilon_1(\omega_1) > 0\}} \varepsilon_1(\omega_1)\},$$

$$\exp\{\sigma_i^0 \max_{\{\omega_i, \varepsilon_i(\omega_i) < 0\}} \varepsilon_i(\omega_i)\} < \min_{\{\omega_{i-1} \in \Omega_{i-1}\}} r_{i-1}(\omega_{i-1}) <$$

$$\max_{\{\omega_{i-1} \in \Omega_{i-1}\}} r_{i-1}(\omega_{i-1}) < \exp\{\sigma_i^0 \min_{\{\omega_i, \varepsilon_i(\omega_i) > 0\}} \varepsilon_i(\omega_i)\}, \quad i = \overline{2, N}. \tag{318}$$

Then, the fair price of super-hedge φ_0 of the nonnegative contingent liability $\varphi_N(\omega_1, \ldots, \omega_N)$ is given by the formula

$$\varphi_0 = \max_{v \in \mathcal{V}} \int_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} =$$

$$\max_{v \in \mathcal{V}} \sum_{i_1=1}^{2} \ldots \sum_{i_N=1}^{2} \prod_{n=1}^{N} \psi_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) \varphi_N(\omega_1^{i_1}, \ldots, \omega_N^{i_N}). \tag{319}$$

Proof. For the discount evolution (317), the following representation

$$S_n((\omega_1,\dots,\omega_n) =$$

$$\frac{S_0}{B_0}\prod_{i=1}^{n}\left(1+f_i(\omega_1,\dots,\omega_i)\eta_i(\omega_i)\right), \quad n=\overline{1,N}, \tag{320}$$

is true, where

$$\eta_1(\omega_1)=\exp\{\sigma_1^0\varepsilon_1(\omega_1)\}-r_0, \quad f_1(\omega_1)=\frac{1}{1+r_0},$$

$$\eta_i(\omega_i)=\exp\{\sigma_i^0\varepsilon_i(\omega_i)\}-\max_{\{\omega_{i-1}\in\Omega_{i-1}\}} r_{i-1}(\omega_{i-1}),$$

$$f_i(\omega_1,\dots,\omega_i)=$$

$$\frac{e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)}-r_{i-1}(\omega_{i-1})}{(\exp\{\sigma_i^0\varepsilon_i(\omega_i)\}-\max\limits_{\{\omega_{i-1}\in\Omega_{i-1}\}} r_{i-1}(\omega_{i-1}))(1+r_{i-1}(\omega_{i-1}))}>0,$$

$$\{\omega_1,\dots,\omega_i\}\in\Omega_i, \quad i=\overline{2,N}. \tag{321}$$

In this case, the spot measures

$$\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)=$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\chi_A(\omega_1^{i_1},\dots,\omega_N^{i_N}), \quad A\in\mathcal{F}_N, \tag{322}$$

figuring in the formula (319), are determined by the formulas

$$\psi_n(\omega_1,\dots,\omega_n)=\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n)\,\psi_n^1(\omega_1,\dots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n)\,\psi_n^2(\omega_1,\dots,\omega_n), \tag{323}$$

$$\psi_n^1(\omega_1,\dots,\omega_{n-1},\omega_n)=$$

$$\chi_{\Omega_n^{0+}}(\omega_n^2)\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\dots,\omega_{n-1})\in\Omega_{n-1}, \quad \omega_n^2\in\Omega_n^{0+}, \tag{324}$$

$$\psi_n^2(\omega_1,\dots,\omega_{n-1},\omega_n)=$$

$$\chi_{\Omega_n^{0-}}(\omega_n^1)\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\dots,\omega_{n-1})\in\Omega_{n-1}, \quad \omega_n^1\in\Omega_n^{0-}, \tag{325}$$

where

$$\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)} =$$

$$\frac{f_n(\omega_1,\dots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)}{f_n(\omega_1,\dots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)+f_n(\omega_1,\dots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}, \qquad (326)$$

$$\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)} =$$

$$\frac{f_n(\omega_1,\dots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}{f_n(\omega_1,\dots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)+f_n(\omega_1,\dots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}, \qquad (327)$$

$$(\omega_1,\dots,\omega_{n-1}) \in \Omega_{n-1}.$$

and the random values $\eta_i(\omega_i)$, $f_i(\omega_1,\dots,\omega_i)$, $i=\overline{1,N}$, are given by the formulas (321). The obtained representation (320) proves Theorem 40.

Suppose that the random values $\eta_k(\omega_k)$, $f_k(\omega_1,\dots,\omega_k)$, $k=\overline{1,N}$, and constants a_k^i satisfy the inequalities

$$0 < a_k^i < \frac{1}{\max\limits_{\{\omega_1,\dots,\omega_{k-1}\}\in\Omega_{k-1}} \max\limits_{\{\omega_k,\eta_k(\omega_k)<0\}} f_k(\omega_1,\dots,\omega_k)\eta_k^-(\omega_k)}, \quad k=\overline{1,N}, \quad i=\overline{1,d},$$

$$f_i(\omega_1,\dots,\omega_i)>0, \quad P_i^0(\eta_i(\omega_i)<0)>0, \quad P_i^0(\eta_i(\omega_i)>0)>0, \quad i=\overline{1,N}. \qquad (328)$$

We assume that the evolutions of d risky assets $S_n(\omega_1,\dots,\omega_n)$ is given by the formula

$$S_n(\omega_1,\dots,\omega_n) = \{S_n^i(\omega_1,\dots,\omega_n)\}_{i=1}^d, \qquad (329)$$

where

$$S_n^i(\omega_1,\dots,\omega_n) = S_0^i\prod_{k=1}^n(1+a_k^i f_k(\omega_1,\dots,\omega_k)\eta_k(\omega_k)), \quad n=\overline{1,N}, \quad i=\overline{1,d}. \qquad (330)$$

Proposition 5. *On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, let the evolution of risky assets be given by the formulas (329), (330), where the random values $\eta_k(\omega_k)$, $f_k(\omega_1,\dots,\omega_k)$ and constants a_k^i, $k=\overline{1,N}$, $i=\overline{1,d}$ satisfy the inequalities (328). For such an evolution of risky assets the set of martingale measures μ_0 does not depend on a_k^i. The spot measures $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ are determined by the formulas (322) - (327). The fair price φ_0^i of super-hedge of the nonnegative contingent liability $\varphi_N^i(\omega_1,\dots,\omega_N)$ is given by the formula*

$$\varphi_0^i = \max_{v\in\mathcal{V}}\int\limits_{\Omega_N}\varphi_N^i(\omega_1,\dots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}, \quad i=\overline{1,d}.$$

The interval of non-arbitrage prices is written in the form

$$\left(\min_{v\in\mathcal{V}} \int\limits_{\Omega_N} \varphi_N^i(\omega_1,\dots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}, \right.$$

$$\left. \max_{v\in\mathcal{V}} \int\limits_{\Omega_N} \varphi_N^i(\omega_1,\dots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}} \right), \quad i=\overline{1,d}.$$

In the case $f_k(\omega_1,\dots,\omega_k)=1, k=\overline{1,N}$, *for every point* $v=\{(\omega_1^1,\omega_1^2),\dots,(\omega_N^1,\omega_N^2)\}$ *in the set* $\mathcal{V}$ *the spot measure* $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ *is a direct product of measures* $\mu_0^i(A_i),\ A_i\in\mathcal{F}_i^0,\ i=\overline{1,N}$, *given on the measurable space* $\{\Omega_i^0,\mathcal{F}_i^0\}$, *where* $\mu_0^i(A_i)=\nu_{\omega_i^1,\omega_i^2}(A_i)$, *and* $\nu_{\omega_i^1,\omega_i^2}(A_i)$ *is given by the formula (292).*

If $S_0^i,S_1^i,\dots,S_N^i,\ i=\overline{1,d}$, are the samples of the processes (329), (330), let us denote the order statistics $S_{(0)}^i,S_{(1)}^i,\dots,S_{(N)}^i,\ i=\overline{1,d}$, of this samples. Introduce the denotations

$$f_k^1=\max_{\{\omega_1,\dots,\omega_{k-1}\}\in\Omega_{k-1},\ \omega_k^1\in\Omega_1^{0-}} f_k(\omega_1,\dots,\omega_{k-1},\omega_k^1),\ k=\overline{1,N}.$$

Proposition 6. *Suppose that* $S_0^i,S_1^i,\dots,S_N^i$ *is a sample of the random processes (329), (330). Then, for the parameters* $a_1^i,\dots,a_N^i$ *the estimation*

$$a_k^i=\frac{\left[1-\frac{S_{(N-k)}^i}{S_{(N-k+1)}^i}\right]}{f_k^1\max\limits_{\omega_k^1\in\Omega_k^{0-}}\eta_k^-(\omega_k^1)}, \quad k=\overline{1,N}, \quad i=\overline{1,d}, \tag{331}$$

is valid.

XI. CONCLUSIONS

Section 1 provides an overview of the achievements and formulates the main problem that has been solved. Section 2 contains the formulation of conditions which must satisfy the evolution of risky asset. In Section 3, the conditions for the evolution of risky asset and random variables are formulated, on the basis of which a recursive method of constructing a family of martingale measures equivalent to the original measure is proposed. Lemma 1 gives a simple proof of the non-emptiness of the set of random variables satisfying conditions (20) - (22), in contrast to similar results in [2]. In Lemma 2, an integral representation is obtained for the measure constructed by the recursive method (28) - (30), from which it follows that it is equivalent to the original measure. In Theorem 1, the conditions under which the recursively constructed measure is martingale one and equivalent to the original measure are formulated.

The Section 4 introduces a family of spot measures and a measure built on the basis of these spot measures and a family of random variables. In Theorem 2, an integral representation is found for the introduced family of measures, which means that this family of measures is absolutely continuous to the original measure. Theorem 3 guarantees the conditions under which the constructed family of measures are martingale and equivalent to the original measure. Theorem 4 gives a complete description of martingale measures equivalent to the original measure. Theorems

5 and 6 are auxiliary. Theorem 7 guarantees conditions when the infimum and supremum of the average value of the payment function over the set of martingale measures coincide with the infimum and supremum of the average value of the payment function over the set of all spot measures. Theorem 8 proves that the family of martingale measures is invariant with respect to a certain class of transformations.

In Section 5, based on Theorem 8, a parametric family of evolutions of risky assets based on some evolution of risky asset is introduced. The proposed parametric model based on the canonical model of the evolution of risky assets, which takes into account both memory and price clustering, takes into account the fact that the price of a risky asset cannot fall to zero.

For a wide class of payment functions, in Theorem 9, an estimate is obtained both from above and from below for the supremum of the average value of the payment functions over the set of all martingale measures. A similar result as in Theorem 9 is obtained in Theorem 10 only for another class of payment functions. For the considered parametric evolution, in Theorem 11, a fair superhedge price is found for the payment function of a standard European-type call option. The same Theorem 11 specifies the interval of non-arbitrage prices. In Theorem 12 , for the considered parametric evolution of the risky asset, a fair superhedge price is found for the payment function of a standard European-type put option. In Theorems 13 and 14, similar results are obtained as in Theorems 11, 12 only for the payment functions of Asian call and put options. On the basis of the sample, in Theorem 15, the estimates of the parameters of the introduced parametric model of the evolution of risky assets are obtained.

In Theorems 16, 17 the fair price of the superhedge for the payment functions of the standard call and put options are given in terms of the obtained parameter estimates. Analogous results are given in Theorems 18 and 19 for fair superhedge prices for Asian-type call and put option payment functions.

Another parametric model of the evolution of risky assets is considered in Section 6. It differs from the previous one in that it considers the discounted evolution of risky asset. Theorems 20 - 21 are proved, in which estimates are obtained both from above and from below and established. Theorems 22 - 23 derive formulas for the fair price of a superhedge for the payment functions of call and put options, respectively. A similar result is obtained in Theorems 24 - 25 for the payment functions of Asian-type put and call options. In Theorems 26 - 29, based on the sample for the evolution of the risky asset, the formulas for the fair price of the superhedge through parameter estimation are presented.

Section 7 contains Theorems 30 and 31, which give the necessary and sufficient conditions for the evolution of risky assets for which the martingale measure is unique. Formulas for the fair price of option contracts and investor hedging strategies are found. A clear construction of such martingale measures and hedging strategies of the investor is given.

In section 8, Proposition 1 establishes the invariance of a single martingale measure with respect to a certain class of evolutions of risky asset. On this basis, proposition 2 builds a parametric model of the financial market and finds formulas for the fair price of an option contract and the investor's hedging strategies. In Corollary 3 and Theorems 32, 33 examples of various evolutions of risky asset are given and the conditions for the existence of a single martingale measure are established. An explicit construction of a single martingale measure is given and formulas for the fair price of an option contract and investor's hedging strategies are constructed.

Proposition 3 constructs a parametric securities market model with a single martingale measure and provides formulas for the fair prices of options contracts and investor hedging strategies. Proposition 4 provides an estimate of the parameters of the introduced parametric models through realizations of risky assets.

Section 9 contains models of incomplete financial markets in discrete probability space. Theorem 34 gives a complete description of all martingale measures equivalent to the original one. Theorem 35 establishes formulas for both the lower and upper limits of the interval of non-arbitrage prices for the evolution of risky assets through the minimum and maximum of the average value of the payment functions over a finite set of spot measures.

Section 10 considers models of the evolution of risky assets that are invariant with respect to a certain class of evolutions of risky assets. Theorem 37 establishes that for a certain class of payment functions and for a wide class of evolutions of risky assets, the fair price of the superhedge is strictly less than the price of the underlying asset. Among such payment functions is the payment function of the standard call option of the European type. Theorems 39, 40 give various examples of discounted evolutions of risky assets that satisfy the conditions of the proved theorems 35 - 37, and find the conditions under which the family of martingale measures is nonempty. Formulas for a fair superhedge price have been found. Proposition 5 contains the construction of a parametric model of an incomplete financial market, a family of martingale measures of which does not depend on the considered parameters. Proposition 6 provides an estimates of the parameters of the constructed models of incomplete markets through realizations of the considered evolutions of risky assets.

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A Proof of the Four-Color Problem based on a New Law of Logic—the Law of the Middle Term

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ABSTRACT

The Four-Color Conjecture, also known as the Four-Color Problem, was first proposed by Francis Guthrie, an Englishman, in 1852. The most famous previous proof of this problem was made by Kenneth Appel and Wolfgang Haken in the United States in 1976 using computers. Afterwards, there are still a considerable number of people hoping to find an artificial proof of this problem. My paper titled "A Logical Proof of the Four-Color Problem" was published in the Journal of Applied Mathematics and Physics in May 2020. Later, it was found that the key logical proof part can form a new logical law — the law of the middle term. This paper aims to give a proof of the Four-Color Problem based on the law of the middle term in logic proposed in this paper, so that the proof idea is clearer, the proof process is more rigorous, and more concise. While solving the problem of graph theory, also made a little contribution to the development of logic.

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Classification: FOR CODE: 090899

Language: English

London Journals Press

LJP Copyright ID: 925634
Print ISSN: 2631-8490
Online ISSN: 2631-8504

London Journal of Research in Science: Natural and Formal

Volume 23 | Issue 4 | Compilation 1.0

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I. INTRODUCTION

The Four-Color Conjecture (hereinafter referred to as 4CC), also known as the Four-Color Problem, was first proposed by Francis Guthrie, an Englishman, in 1852[1]. The most famous previous proof of this problem was made by Kenneth Appel and Wolfgang Haken in the United States in 1976 using computers [2]. Afterwards, there are still a considerable number of people hoping to find an artificial proof of this problem. My paper titled "A Logical Proof of the Four-Color Problem [3]" was published in the Journal of Applied Mathematics and Physics in May 2020. Later, it was found that the key logical proof part can form a new logical law — the law of the middle term. This paper aims to give a proof of the 4CC based on the law of the middle term in logic proposed in this paper, so that the proof idea is clearer, the proof process is more rigorous, and more concise. While solving the problem of graph theory, also made a little contribution to the development of logic.

II. METHODS

This paper is based on Kempe's work.

Kempe once tried to prove 4CC by means of reduction to absurdity. The main idea is that if there are five color maps, there will at least be a "minimal five color map" G_5 with the least number of countries.

Kempe first proved a conclusion about the planar graph: in any map, there must be a country whose number of neighbors is less than or equal to 5.

Next, Kempe looked at the country with the least number of neighbors in the minimal five color map G_5 — country u (he had proved that country u has no more than five neighbors). Suppose there are n countries in G_5. If there are no more than 3 neighbors of country u, it can be "removed" to form a map with only n-1 countries, which should be 4-colorable. The original three neighbors of country u used at most three colors, such as red, yellow and green. At this time, put the country u back and color it with the color unused by its neighbors, such as blue, so that the minimal five color map G_5 can be 4-colored again, see Figure 1.

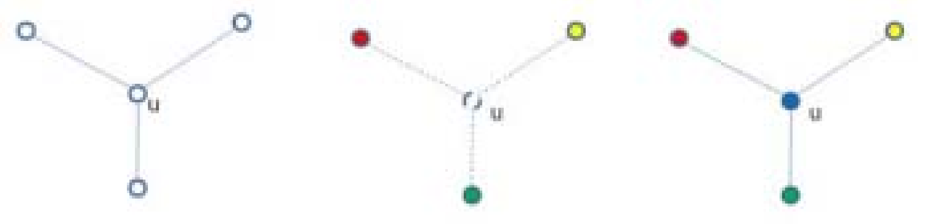

Figure 1: Country u owned three neighboring countries.

This kind of subgraph that can reduce the number of map colors by "removinG* and "restorinG* a country is later called "reducible configuration".

III. RESEARCH IDEA

Kempe's work putted forward two important concepts, which laid the foundation for further solving 4 CC in the future.

Kempe's first concept was "configuration". He first proved that there must be a country on any map whose number of neighbors is five or less. In other words, a set of "configurations" of one to five neighbors is inevitable on each map.

Another concept proposed by Kempe is "reducibility". Kempe found in his research that the chromatic number of relevant maps can be reduced by "removinG* and "restorinG* a country in some subgraphs. Since the introduction of the concepts of "configuration" and "reducibility", some standard methods for checking the configuration of a graph to determine whether it is reducible have been developed. Seeking the inevitable group of reducible configurations is an important way to prove 4CC. The first part of the proof of this paper is the same as Kempe's proof idea. It starts with the assumption that there is a minimal five color map (called 5-critical graph in this paper) G, then analyzes the logical relationship between graph G's related subgraphs when they are 4-coloring, and then uses the law of the middle term based on logic proved in this paper, it is proved that the necessary configurations composed of four or five neighbors in graph G are reducible, so 4CC is proved to be true by means of reduction to absurdity.

IV. LABELS AND CONCEPTS

In this paper, δ is used to represent the minimum degree of the vertices of a graph; use PA to express a proposition about something A; use PA → PB to represent the sufficient condition that PA is PB. If V is the set of all the vertices of a graph G and V' is a non-empty subset of V, then the induced subgraph of graph G induced by V' is represented by G[V'] (The so-called induced subgraph is a subgraph composed of some vertices in a certain graph and all the edges connecting these vertices in the original graph).

A coloring of a graph is to assign a set of colors to each vertex so that no two adjacent vertices have the same color. The set of all vertices with the same color is independent and is called a color group. An n-coloring of graph G is a coloring with n colors, according to this coloring, all its vertices are divided into n color groups.

Among all the colorings of a certain graph G, the color number of the coloring with the least color is called its chromatic number, denoted as χ (G). if χ (G) ≤ n, graph G is called n-colorable or n colorable graph; if χ (G) = n, G is called n-color or n-color graph.

A graph G is said to be critical if for all its vertices or edges v/e, χ(G-v/e)＜χ(G)；if χ (G) = n, Then G is called an n-critical or n-critical graph.

V. THE LAW OF THE MIDDLE TERM

The law of the middle term: if PA → PC, but PA acts on PC through and only through B, then there must be a PB such that PA → PB and PB → PC.

Proof: If this law does not hold, that is, if PA → PC, when PA acts on PC through and only through B, for any PB, it is all not "PA → PB and PB → PC", that is, neither of them is PA → PC, then obviously this would contradict the premise PA→PC.

VI. RESULTS

The Four Color Theorem: For all planar graph G, χ(G) ≤4.

Proof: Use the method of reduction to absurdity. If this theorem is not valid, then there should be 5-color graphs in planar graphs [4][5][6]. Let G is a 5-critical graph, and let u be the vertex with the smallest degree, that is, deg(u)=δ, it can be proved that δ=x(4≤x≤5) [7][8] in G.

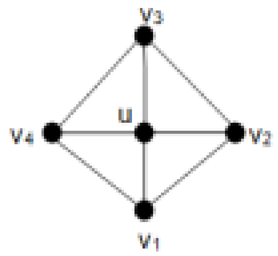

Figure 2: deg(u) = 4.

When deg(u) = 4, set the vertices adjacent to u as v_1, v_2, v_3, v_4, as shown in Figure 2. The reason why edges v_1v_2, v_2v_3, v_3v_4, v_4v_1 exist in G is that if anyone of them are missing, such as v_1v_2 is missing, then the graph obtained by combining v_1 and v_2 into v_{12} is G', as shown in Figure 3. Because of the number of edges of G' is less than G, G' should be a 4-colorable graph. In this case, as long as G' is changed back to G, we can get 4-colored G, which contradicts the hypothesis that G is a 5-critical graph.

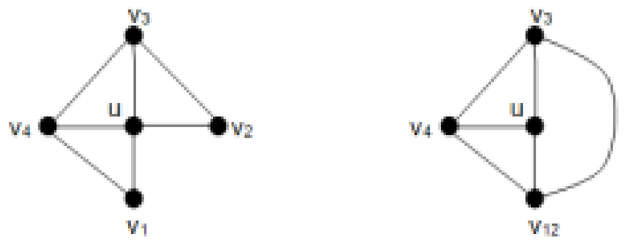

Figure 3: If the edge v_1v_2 is missing, the graph can become 4-colorable.

Let $G^*= G-uv_1$, $G_d = G^* [\{v_2, v_3, v_4\}]$, Since the number of edges of G^* is less than G, G^* should be a 4 color graph. It is easy to know that when we make 4-coloring for G^*, u and v_1 must always be colored the same color, otherwise, as long as we put uv_1 back between u and v_1, we can get a 4 colored G, which contradicts the hypothesis that G is a 5-critical graph, as shown in Figure 4. In other words, when using color group C composed of red, yellow, green and blue to make 4-coloring for G^*, If Pu is used to represent "u is red" and Pv_1 is used to represent " v_1 is red", first, $Pu \rightarrow Pv_1$. Otherwise, if Pu is true and Pv_1 is false, that is, u and v_1 are different in red, which will contradict the above inference that when we make 4-coloring for G^*, u and v_1 must always be colored the same color [9].

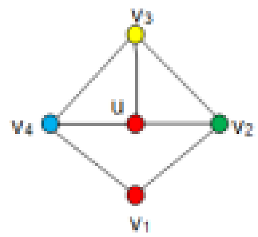

Figure 4: When we make 4-coloring for G^*, u and v_1 must always be colored the same color.

Secondly, when using color group C to color G^*, if Pu is true, that is, u is red, then from the above inference, Pv_1 will also be true, that is, v_1 will also be red with u. It is known from the law of the middle term and $Pu \rightarrow Pv_1$, and Pu acts on Pv_1 through and only through G_d that, at this time, for G_d, there must be a coloring PG_d, making $Pu \rightarrow PG_d$ and $PG_d \rightarrow Pv_1$. But in the aforementioned coloring process, G_d obviously can have "On all vertices of G_d have all the three colors of yellow, green and blue" and "On all the vertices of G_d have only some two colors of the three colors of yellow, green and blue". But PG_d obviously cannot including the latter case, otherwise it is only necessary to change the red of u to another color among the three colors of yellow, green and blue that are not used on all vertices of G_d, so that u and v_1 are different colors, so that it contradicts the inference that "when 4-coloring G^*, u and v_1 must be the same color". Thus, in this case, PG_d can obviously only be the former case, that is, on all vertices of G_d have all the three colors of yellow, green and blue. But this is obviously only possible if there are odd circles in G_d [10].

It follows from there is odd circle in G_d that v_2 must adjacent to v_4.

In the same way, it can also be inferred that v_1 must adjacent to v_3, so that there is a contradictory result of edge intersection in G, as shown in Figure 5.

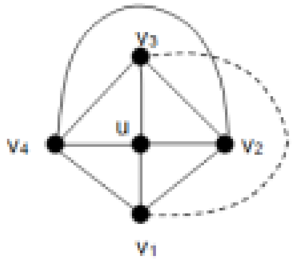

Figure 5: Shows the result of contradiction with intersecting edges in G.

When deg(u) = 5, let the vertices adjacent to u are v_1, v_2, v_3, v_4, v_5. Similar to the case of deg(u) =4, edges v_1v_2, v_2v_3, v_3v_4, v_4v_5 and v_5v_1 should exist, as shown in Figure 6.

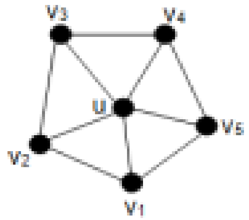

Figure 6: deg(u) = 5.

Let Gd=G*[{v_2, v_3, v_4, v_5}], it can also be proved by imitating the situation of deg(u) =4: there must be an odd cycle in Gd, therefore, either v_2 is adjacent to v_4, or v_3 is adjacent to v_5. If v_2 is adjacent to v_4, it can be deduced in the same way that in G, either v_1 is adjacent to v_4, or v_2 is adjacent to v_5. And if v_1 is adjacent to v_4, it can be deduced in the same way that in G, either v_1 is adjacent to v_3, or v_2 is adjacent to v_5, so that there is a contradictory result of edge intersection in G, see Figure 7.

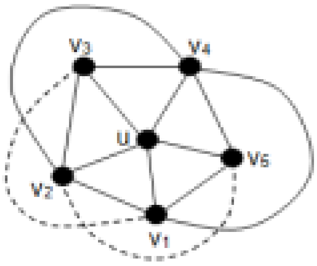

Figure 7: Shows the result of contradiction with intersecting edges in G.

Similarly, it can be proved that when v_2 is adjacent to v_4 and v_2 is adjacent to v_5. Similarly, it can be proved that when v_3 is adjacent to v_5. This proves theorem.

VI. CONCLUSIONS

On the basis of my previous relevant proofs, this paper refines the key logical proof part into a new logical law called the law of the middle term, which makes the proof thinking clearer, the proof process more rigorous, and more concise. While discussing difficult problems of graph, it also made a little contribution to the development of logic.

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