The Paradigm of Complex Probability and Prognostic using FORM

Abdo Abou Jaoudé

Notre Dame University - Louaize

ABSTRACT

The imaginary set of numbers is incorporated by extending the probability system of five axioms of Andrey Nikolaevich Kolmogorov set up in 1933 and this by adding three supplementary axioms. Therefore, any random experiment can thus be performed in the extended complex probability set $\mathcal{C}$ which is the sum of the real set $\mathcal{R}$ of real probabilities and the imaginary set $\mathcal{M}$ of imaginary probabilities. The aim here is to determine the complex probabilities by taking into consideration additional new imaginary dimensions to the event that occurs in the "real" laboratory. The outcome of the stochastic phenomenon in $\mathcal{C}$ can be foretold perfectly whatever the probability distribution of the input random variable in $\mathcal{R}$ is since the corresponding probability in the whole set $\mathcal{C}$ is permanently and constantly equal to one. Therefore, the consequence that follows indicates that randomness and chance in $\mathcal{R}$ is substituted now by absolute determinism in $\mathcal{C}$. This novel complex probability paradigm will be implemented to the field of prognostic based on reliability, hence to the concepts of the system remaining useful lifetime (RUL) and degradation. Additionally, the First-Order Reliability Method (FORM) analysis will be applied to Young's modulus to illustrate my original and innovative paradigm.

Keywords: Degradation, Remaining Useful Lifetime, Complex Probability, First-Order Reliability Method (FORM), Failure Probability, Young Modulus, Prognostic.

Classification: FOR Code: 010299

Language: English

London Journals Press

LJP Copyright ID: 925631
Print ISSN: 2631-8490
Online ISSN: 2631-8504

London Journal of Research in Science: Natural and Formal

Volume 20 | Issue 4 | Compilation 1.0

London Journal of Research in Science: Natural and Formal

The Paradigm of Complex Probability and Prognostic using FORM

Abdo Abou Jaoudé

Abstract

The imaginary set of numbers is incorporated by extending the probability system of five axioms of Andrey Nikolaevich Kolmogorov set up in 1933 and this by adding three supplementary axioms. Therefore, any random experiment can thus be performed in the extended complex probability set $\mathcal{C}$ which is the sum of the real set $\mathcal{R}$ of real probabilities and the imaginary set $\mathcal{M}$ of imaginary probabilities. The aim here is to determine the complex probabilities by taking into consideration additional new imaginary dimensions to the event that occurs in the "real" laboratory. The outcome of the stochastic phenomenon in $\mathcal{C}$ can be foretold perfectly whatever the probability distribution of the input random variable in $\mathcal{R}$ is since the corresponding probability in the whole set $\mathcal{C}$ is permanently and constantly equal to one. Therefore, the consequence that follows indicates that randomness and chance in $\mathcal{R}$ is substituted now by absolute determinism in $\mathcal{C}$. This novel complex probability paradigm will be implemented to the field of prognostic based on reliability, hence to the concepts of the system remaining useful lifetime (*RUL*) and degradation. Additionally, the First-Order Reliability Method (FORM) analysis will be applied to Young's modulus to illustrate my original and innovative paradigm.

Keywords: Degradation, Remaining Useful Lifetime, Complex Probability, First-Order Reliability Method (FORM), Failure Probability, Young Modulus, Prognostic.

Author: Department of Mathematics and Statistics, Faculty of Natural and Applied Sciences, Notre Dame University-Louaize, Lebanon. Email: abdoaj@idm.net.lb

Nomenclature

$\mathcal{R}$ = Real probability set of events
$\mathcal{M}$ = Imaginary probability set of events
$\mathcal{C}$ = Complex probability set of events
i = the imaginary number where $i=\sqrt{-1}$ or $i^2=-1$
EKA = Extended Kolmogorov's Axioms
CPP = Complex Probability Paradigm
P_{rob} = Probability of any event
P_r = Probability in the real set $\mathcal{R}$ = system failure probability
P_m = Probability in the imaginary set $\mathcal{M}$ corresponding to the real probability in $\mathcal{R}$ = system survival probability in $\mathcal{M}$
P_m/i = System survival probability in $\mathcal{R}$
Pc = Probability of an event in $\mathcal{R}$ with its associated event in $\mathcal{M}$, it is the probability in the complex set $\mathcal{C}$
Z = Complex probability number and vector, it is the sum of P_r and P_m
DOK = $|Z|^2$ = Degree of Our Knowledge of the random experiment and event, it is the square of the norm of Z.
Chf = Chaotic factor
MChf = Magnitude of the Chaotic factor

t = System cycles time
t_C = System cycles time till failure
E = Young modulus
E_C = Young modulus till system failure
$f(t)$ = Failure probability density function of t
$F(t)$ = Failure cumulative distribution function of t
$f(E)$ = Failure probability density function of E
$F(E)$ = Failure cumulative distribution function of E
ψ = Simulation magnifying factor
$1/\psi$ = The normalizing constant of P_r
D = Degradation indicator of a system
RUL = Remaining Useful Lifetime of a system
$P_{rob}[RUL(t)]$ = Probability of RUL after a system cycles time t
$P_{rob}[RUL(E)]$ = Probability of RUL after a Young modulus E

I. Introduction

The First-Order Reliability Method, (FORM), is a semi-probabilistic reliability analysis method devised to evaluate the reliability of a system. The accuracy of the method can be improved by averaging over many samples, which is known as Line Sampling [1,2].

The method is also known as the Hasofer-Lind Reliability Index, developed by Professor Michael Hasofer and Professor Neil Lind in 1974 [3]. The index has been recognized as an important step towards the development of contemporary methods to effectively and accurately estimate structural safety [4,5].

Moreover, reliability engineering is a sub-discipline of systems engineering that emphasizes dependability in the life cycle management of a product. Reliability, describes the ability of a system or component to function under stated conditions for a specified period of time [6]. Reliability is closely related to availability, which is typically described as the ability of a component or system to function at a specified moment or interval of time.

The Reliability function is theoretically defined as the probability of success (Reliability = 1 – Probability of Failure); as, $R(t)$, the probability of failure at time t; as a probability derived from reliability, availability, testability, and maintainability. Availability, testability, maintainability, and maintenance are often defined as a part of "reliability engineering" in reliability programs. Reliability plays a key role in the cost-effectiveness of systems for example cars have a higher resale value when they fail less often.

Reliability and quality are closely related. Normally quality focuses on the prevention of defects during the warranty phase whereas reliability looks at preventing failures during the useful lifetime of the product or system from commissioning to decommissioning.

Reliability engineering deals with the estimation, prevention, and management of high levels of "lifetime" engineering uncertainty and risks of failure. Although stochastic parameters define and affect reliability, reliability is not (solely) achieved by mathematics and statistics [7,8]. One cannot really find a root cause (needed to effectively prevent failures) by only looking at statistics. "Nearly all teaching and literature on the subject emphasize these aspects, and ignore the reality that the ranges of uncertainty involved largely invalidate quantitative methods for prediction and measurement" [9]. For example, it is easy to represent "probability of failure" as a symbol or value in an equation, but it is almost impossible to predict its true magnitude in practice, which is massively multivariate, so having the equation for reliability does not begin to equal having an accurate predictive measurement of reliability.

Reliability engineering relates closely to safety engineering and to system safety, in that they use common methods for their analysis and may require input from each other. Reliability engineering focuses on costs of failure caused by system downtime, cost of spares, repair equipment, personnel, and cost of warranty claims. Safety engineering normally focuses more on preserving life and nature than on cost, and therefore deals only with particularly dangerous system-failure modes. High reliability (safety factor) levels also result from good engineering and from attention to detail, and almost never from only reactive failure management (using reliability accounting and statistics) [10].

The word *reliability* can be traced back to 1816, and is first attested to the poet Samuel Taylor Coleridge [11]. Before World War II the term was linked mostly to repeatability; a test (in any type of science) was considered "reliable" if the same results would be obtained repeatedly. In the 1920s, product improvement through the use of statistical process control was promoted by Dr. Walter A. Shewhart at Bell Labs [12], around the time that Waloddi Weibull was working on statistical models for fatigue. The development of reliability engineering was here on a parallel path with quality. The modern use of the word reliability was defined by the U.S. military in the 1940s, characterizing a product that would operate when expected and for a specified period of time.

In World War II, many reliability issues were due to the inherent unreliability of electronic equipment available at the time, and to fatigue issues. In 1945, M.A. Miner published the seminal paper titled "Cumulative Damage in Fatigue" in an ASME journal. A main application for reliability engineering in the military was for the vacuum tube as used in radar systems and other electronics, for which reliability proved to be very problematic and costly. The IEEE formed the Reliability Society in 1948. In 1950, the United States Department of Defense formed group called the "Advisory Group on the Reliability of Electronic Equipment" (AGREE) to investigate reliability methods for military equipment [13]. This group recommended three main ways of working:

1) Improve component reliability.
2) Establish quality and reliability requirements for suppliers.
3) Collect field data and find root causes of failures.

Furthermore, in the 1960s, more emphasis was given to reliability testing on component and system level. The famous military standard MIL-STD-781 was created at that time. Around this period also the much-used predecessor to military handbook 217 was published by RCA and was used for the prediction of failure rates of electronic components. The emphasis on component reliability and empirical research (e.g. Mil Std 217) alone slowly decreased. More pragmatic approaches, as used in the consumer industries, were being used. In the 1980s, televisions were increasingly made up of solid-state semiconductors. Automobiles rapidly increased their use of semiconductors with a variety of microcomputers under the hood and in the dash. Large air conditioning systems developed electronic controllers, as had microwave ovens and a variety of other appliances. Communications systems began to adopt electronics to replace older mechanical switching systems. Bellcore issued the first consumer prediction methodology for telecommunications, and SAE developed a similar document SAE870050 for automotive applications. The nature of predictions evolved during the decade, and it became apparent that die complexity wasn't the only factor that determined failure rates for integrated circuits (ICs). Kam Wong published a paper questioning the bathtub curve [14] — one can refer also to reliability-centered maintenance. During this decade, the failure rate of many components dropped by a factor of 10. Software became important to the reliability of systems. By the 1990s, the pace of IC development was picking up. Wider use of stand-alone microcomputers was common, and the PC market helped keep IC densities following Moore's law and doubling about every 18 months.

Reliability engineering was now changing as it moved towards understanding the physics of failure. Failure rates for components kept dropping, but system-level issues became more prominent. Systems thinking became more and more important. For software, the CMM model (Capability Maturity Model) was developed, which gave a more qualitative approach to reliability. ISO 9000 added reliability measures as part of the design and development portion of certification. The expansion of the World-Wide Web created new challenges of security and trust. The older problem of too little reliability information available had now been replaced by too much information of questionable value. Consumer reliability problems could now be discussed online in real time using data. New technologies such as micro-electromechanical systems (MEMS), handheld GPS, and hand-held devices that combined cell phones and computers all represent challenges to maintain reliability. Product development time continued to shorten through this decade and what had been done in three years was being done in 18 months. This meant that reliability tools and tasks had to be more closely tied to the development process itself. In many ways, reliability became part of everyday life and consumer expectations.

Finally, and to recapitulate, this research paper is structured as follows: After the introduction in section I, the advantages and the purpose of the present paper are presented in section II. Next, in section III, we will illustrate and explain the paradigm of complex probability with its novel parameters and concepts. In section IV, we will do a review of reliability theory. In section V, we will apply the complex probability paradigm to prognostic based on reliability. Also, in section VI, we will apply FORM to prognostic. Furthermore, in section VII the new model will be applied to Young modulus. Additionally, in section VIII a comprehensive analysis will be achieved where we will clarify all the results and then display the equations of general prognostic. Finally, I conclude the paper by doing a complete summary in section IX, and then at the end cite the references supporting the current research work.

II. The Purpose and the Advantages of The Present Work

Computing probabilities is all our work in the classical theory of probability. Adding new dimensions to our stochastic experiment is the innovative idea in the current paradigm which will make the study absolutely deterministic. As a matter of fact, the theory of probability is a nondeterministic theory by essence that means that all the random events outcome is due to luck and chance. Hence, we make the study deterministic by adding new imaginary dimensions to the phenomenon occurring in the "real" laboratory which is $\mathcal{R}$, and therefore a stochastic experiment will have a certain outcome in the complex probabilities set $\mathcal{C}$. It is of great significance that random systems become completely predictable since we will be perfectly knowledgeable to predict the outcome of all stochastic and chaotic phenomena that occur in nature like for example in all stochastic processes, in statistical mechanics, or in the well-established prognostic field. Consequently, the work that should be done is to add the contributions of $\mathcal{M}$ which is the set of imaginary probabilities to the set of real probabilities $\mathcal{R}$ that will make the random phenomenon in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ completely deterministic. Since this paradigm is found to be fruitful, then a new theory in prognostic and stochastic sciences is established and this to understand deterministically those events that used to be stochastic events in $\mathcal{R}$. This is what I coined by the term "The Complex Probability Paradigm" that was elaborated and initiated in my fourteen previous papers [15-28].

Furthermore, although the prognostic laws are sometimes deterministic and well-known in general but there are chaotic and stochastic aspects (such as in engineering: geometry dimensions, humidity, water action, material nature, atmospheric pressure, applied load location, corrosion, soil pressure and friction, temperature, etc…) that influence the system and make its function of degradation deviate from its computed trajectory predicted by these deterministic laws. An updated follow-up of the behavior of degradation with cycle number or time, and which is under the influence of non-chaotic and chaotic effects, is done by what I named the system failure probability due to its definition that evaluates and calculates the jumps in the function of degradation D.

Moreover, my objective in this present paper is to link the complex probability paradigm to the system prognostic based on reliability by using FORM. In fact, the system failure probability derived from FORM will be included in and applied to the complex probability paradigm and then related to prognostic. This will lead to the novel and original prognostic model illustrated in this current work. Consequently, by calculating the new prognostic model parameters, we will be able to evaluate the degree of our knowledge, the magnitude of the chaotic factor, the system survival and failure probabilities, the complex probability, and the *RUL* probability, after that a simulation cycles time t or a Young modulus E has been applied to the studied system and which are all functions of the system degradation under the influence of random and chaotic influences. An application of the novel model to Young modulus will be done to illustrate the original idea and method.

Subsequently, to summarize, the advantages and the objectives of the current work are to:

1- Relate probability theory to the field of complex variables and analysis in mathematics and therefore to extend the theory of classical probability to the set of complex numbers. This task was elaborated and initiated in my fourteen previous papers.

2- Do an updated follow-up of the behavior of degradation D with cycle number or time or Young modulus E which is under the influence of chaos. This follow-up is achieved by the system real failure probability computed by FORM due to its definition that calculates the jumps in D; and thus, to relate a system degradation to probability theory in an innovative and a new way.

3- Extend the concepts of prognostic to the complex set $\mathcal{C}$ of probabilities by applying the novel probability paradigm and axioms to prognostic.

4- Demonstrate that any stochastic and random event and experiment can be expressed deterministically in the complex probabilities set $\mathcal{C}$.

5- Quantify both the chaos magnitude and the degree of our knowledge of the system remaining useful lifetime and its degradation.

6- Represent graphically and illustrate the parameters and functions of the original paradigm related to the system prognostic and to Young modulus.

7- Demonstrate that the classical concepts of stochastic remaining useful lifetime and degradation have a probability of occurring permanently equal to one in the complex set; consequently, no disorder, no ignorance, no unpredictability, no stochasticity, no randomness, no nondeterminism, and no chaos exist in:

$$\mathcal{C} \text{ (complex set)} = \mathcal{R} \text{ (real set)} + \mathcal{M} \text{ (imaginary set)}.$$

8- Show that we will be able to do prognostic in a deterministic way in the complex set $\mathcal{C}$ by adding new and supplementary imaginary dimensions to any stochastic system and random experiment.

9- Prepare to apply the novel paradigm to other topics in stochastic processes, in statistical mechanics, and to the field of prognostics in science and engineering. This will be the task in my following research work and publications.

The novel proposed mathematical prognostic paradigm will be applied to practical engineering and as a future work, it will be implemented in the study of a wide set of dynamic systems like offshore and buried petrochemical pipes and vehicle suspension systems which are under the influence of fatigue and in the nonlinear and linear cases of damage accumulation.

To recapitulate, compared with existing literature, the major contribution of the present work is to apply the novel complex probability paradigm to the concepts of stochastic remaining useful lifetime and degradation of a system therefore to the field of prognostic and to Young modulus.

The following figure recapitulates the objectives of the present work (Figure 1):

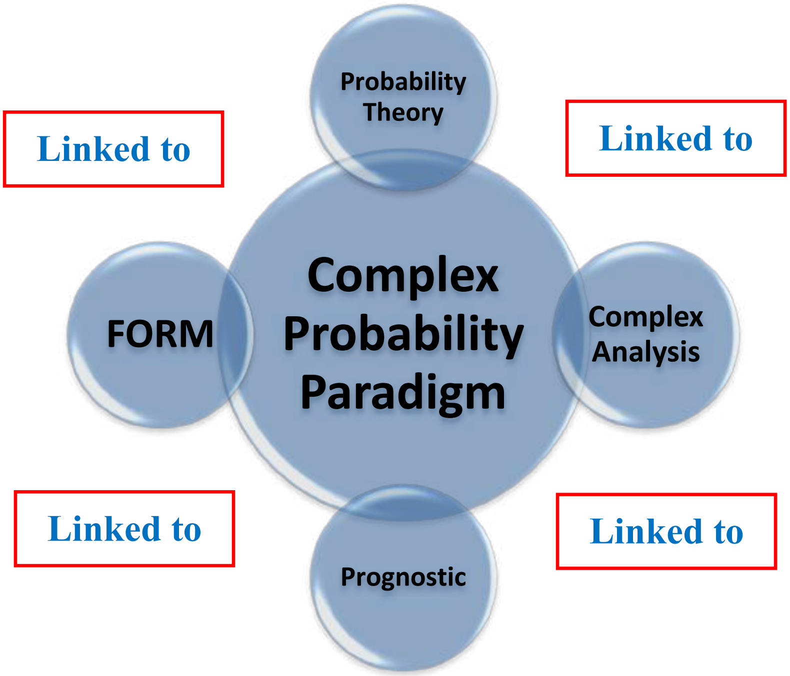


Figure 1: The major objectives of the Complex Probability Paradigm (*CPP*)

III. The Extended Set of Probability Axioms [29-72]

In this section, we will present the extended set of probability axioms of the complex probability paradigm.

3.1 The Original Andrey Nikolaevich Kolmogorov Set of Axioms

The simplicity of Kolmogorov's system of axioms may be surprising. Let E be a collection of elements $\{E_1, E_2, \ldots\}$ called elementary events and let F be a set of subsets of E called random events. The five axioms for a finite set E are [29-32]:

Axiom 1: F is a field of sets.
Axiom 2: F contains the set E.
Axiom 3: A non-negative real number $P_{rob}(A)$, called the probability of A, is assigned to each set A in F. We have always $0 \le P_{rob}(A) \le 1$.
Axiom 4: $P_{rob}(E)$ equals 1.
Axiom 5: If A and B have no elements in common, the number assigned to their union is:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B)$$

hence, we say that A and B are disjoint; otherwise, we have:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B) - P_{rob}(A \cap B)$$

And we say also that: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B / A) = P_{rob}(B) \times P_{rob}(A / B)$ which is the conditional probability. If both A and B are independent then: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B)$.

Moreover, we can generalize and say that for N disjoint (mutually exclusive) events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \le j \le N$), we have the following additivity rule:

$$P_{rob}\left(\bigcup_{j=1}^{N} A_j\right) = \sum_{j=1}^{N} P_{rob}\left(A_j\right)$$

And we say also that for N independent events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \le j \le N$), we have the following product rule:

$$P_{rob}\left(\bigcap_{j=1}^{N} A_j\right) = \prod_{j=1}^{N} P_{rob}\left(A_j\right)$$

3.2 Adding the Imaginary Part M

Now, we can add to this system of axioms an imaginary part such that:

Axiom 6: Let $P_m = i \times (1 - P_r)$ be the probability of an associated event in $\mathcal{M}$ (the imaginary part) to the event A in $\mathcal{R}$ (the real part). It follows that $P_r + P_m / i = 1$ where i is the imaginary number with $i = \sqrt{-1}$ or $i^2 = -1$.

Axiom 7: We construct the complex number or vector $Z = P_r + P_m = P_r + i(1 - P_r)$ having a norm $|Z|$ such that: $|Z|^2 = P_r^2 + (P_m / i)^2$.

Axiom 8: Let Pc denote the probability of an event in the complex probability universe $\mathcal{C}$ where $\mathcal{C} = \mathcal{R} + \mathcal{M}$. We say that Pc is the probability of an event A in $\mathcal{R}$ with its associated event in $\mathcal{M}$ such that:

$$Pc^2 = (P_r + P_m / i)^2 = |Z|^2 - 2iP_r P_m \text{ and is always equal to } 1.$$

We can see that the system of axioms defined by Kolmogorov could be hence expanded to take into consideration the set of imaginary probabilities by adding three novel axioms [33-45].

3.3 The Purpose of Extending the Axioms

It is clear from *CPP* extended set of axioms that adding to any real event an imaginary counterpart makes the event probability in the set $\mathcal{C}$ permanently equal to one. As a matter of fact, understanding will follow directly if we start to conceive the set of probabilities as divided into two complementary parts: one probability part is real and the other probability part is imaginary. The stochastic event that occurs in the real set $\mathcal{R}$ of probabilities (like getting a head when tossing a coin) has a corresponding probability P_r. Now we denote by $\mathcal{M}$ the set of imaginary probabilities and we denote by $|Z|^2$ the Degree of Our Knowledge (*DOK* for short) of this stochastic event. P_r is according to Kolmogorov's axioms, and as always, the probability of the random phenomenon.

A full ignorance of the probability set $\mathcal{M}$ leads to:

$P_r = 0.5$ and $|Z|^2 = DOK$ in this case is equal to: $1 - 2P_r(1 - P_r) = 1 - (2 \times 0.5) \times (1 - 0.5) = 0.5$

Conversely, a perfect knowledge of the set in $\mathcal{R}$ leads to:

London Journal of Research in Science: Natural and Formal

P_{rob}(event) $= P_r = 1$ and $P_m = P_{rob}$(imaginary part) = 0. Here we have $|Z|^2 = DOK = 1-(2\times 1)\times(1-1) = 1$ because the random event is totally and perfectly known, that is, all its variables and laws are fully determined; therefore, our degree of our knowledge of the stochastic event is 1 = 100%.

Now, if we are sure that an event is impossible and will never occur, that is, like 'getting nothing' (the empty set), P_r is accordingly = 0, that is the event will never happen in $\mathcal{R}$. Hence, P_m will be equal to: $i(1-P_r) = i(1-0) = i$, and $|Z|^2 = DOK = 1-(2\times 0)\times(1-0) = 1$, because we can tell that the event of getting nothing surely will never happen; thus, the Degree of Our Knowledge (*DOK*) of the stochastic event is 1 = 100%. [15]

We can infer that we have always:

$$0.5 \le |Z|^2 = DOK \le 1, \quad \forall P_r: \ 0 \le P_r \le 1 \text{ and}$$
$$|Z|^2 = DOK = P_r^2 + (P_m / i)^2 \text{, where } 0 \le P_r, P_m / i \le 1 \tag{1}$$

And what is truly significant and crucial is that we have in all cases:

$$Pc^2 = (P_r + P_m / i)^2 = |Z|^2 - 2iP_rP_m = [P_r + (1-P_r)]^2 = 1^2 = 1 \tag{2}$$

As a matter of fact, the game is a game of chance according to an experimenter in $\mathcal{R}$: the experimenter ignores the outcome of the random event. Accordingly, a probability P_r is assigned to each outcome and he will affirm that the output and result are nondeterministic. But an observer will be able to foretell the output of the game of chance in the probability universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$, since he considers the contribution of the probability set $\mathcal{M}$, so he states that:

$$Pc^2 = (P_r + P_m / i)^2$$

therefore Pc is permanently equal to one. Actually, the addition to our stochastic experiment of the imaginary probability set $\mathcal{M}$ leads to the abolition of indeterminism and ignorance. Subsequently, the study of this class of events in the set $\mathcal{C}$ is of great worth since we will be able to foretell with certainty the outputs of the conducted random experiments. As a matter of fact, conducting experiments in $\mathcal{R}$ leads to uncertainty and unpredictability. Consequently, and to study all random phenomena, we place ourselves in $\mathcal{C}$ instead of placing ourselves in $\mathcal{R}$ then study the random phenomena, since in $\mathcal{C}$ the contributions of $\mathcal{M}$ are considered and hence a deterministic study of the random events becomes conceivable. Conversely, when we consider the contribution of the probability set $\mathcal{M}$ we place ourselves in $\mathcal{C}$ and when we ignore $\mathcal{M}$ we restrict our study to nondeterministic and probabilistic events in $\mathcal{R}$. [46-56]

Furthermore, we can infer from the axioms and definitions stated above that [15]:

$$\begin{aligned} 2iP_rP_m &= 2i \times P_r \times i \times (1-P_r) \\ &= 2i^2 \times P_r \times (1-P_r) = -2P_r(1-P_r) \\ \Rightarrow 2iP_rP_m &= Chf \end{aligned} \tag{3}$$

$2iP_rP_m$ will be called the <u>Ch</u>aotic <u>f</u>actor in our random experiment and we will denote it accordingly: 'Chf'. We will understand now why we have called this term the chaotic factor. As a matter of fact:

In case $P_r = 1$, that is the case of an event which is certain, then the event chaotic factor is equal to 0.
In case $P_r = 0$, that is the case of an event which is impossible, then the event chaotic factor is equal to 0 also.
Therefore, in both two last cases, there is no chaos since the outcome is known in advance and is certain.

In case $P_r = 0.5$, that is in the case of complete ignorance, then the event chaotic factor is equal to –0.5. (Figures 2-4)

We can infer that:

$$-0.5 \leq Chf \leq 0, \quad \forall P_r : 0 \leq P_r \leq 1.$$

What is crucial here is that we have consequently quantified both the chaotic factor and the degree of our knowledge of any stochastic phenomenon and thus we state now:

$$Pc^2 = |Z|^2 - 2iP_rP_m = DOK - Chf \tag{4}$$

Therefore, we can conclude that:

$$Pc^2 = \text{Degree of our knowledge of the system} - \text{Chaotic factor} = 1,$$

Hence, $Pc = 1$ permanently.

This straightforwardly can be interpreted as follows: if we succeed to eliminate and subtract the chaotic factor in any stochastic phenomenon, like we have done in the equation above, then the outcome probability will be permanently equal to one. [57-72]

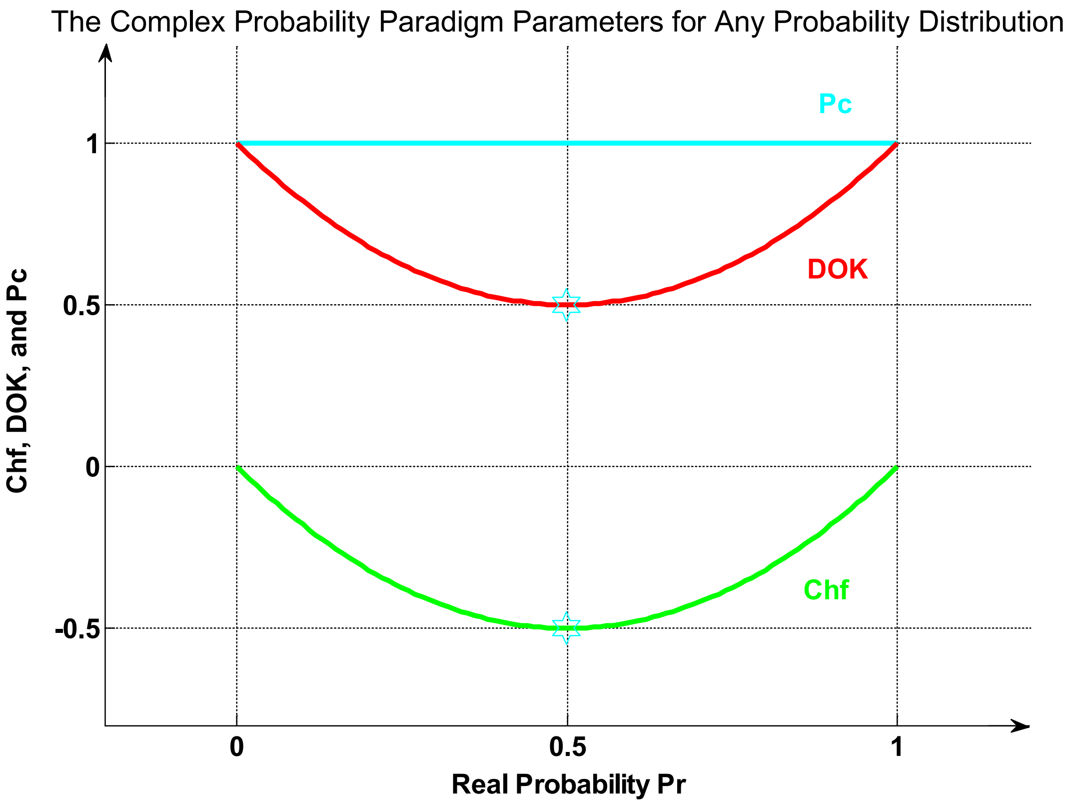


Figure 2: Chf, DOK, and Pc for any probability distribution in $2D$

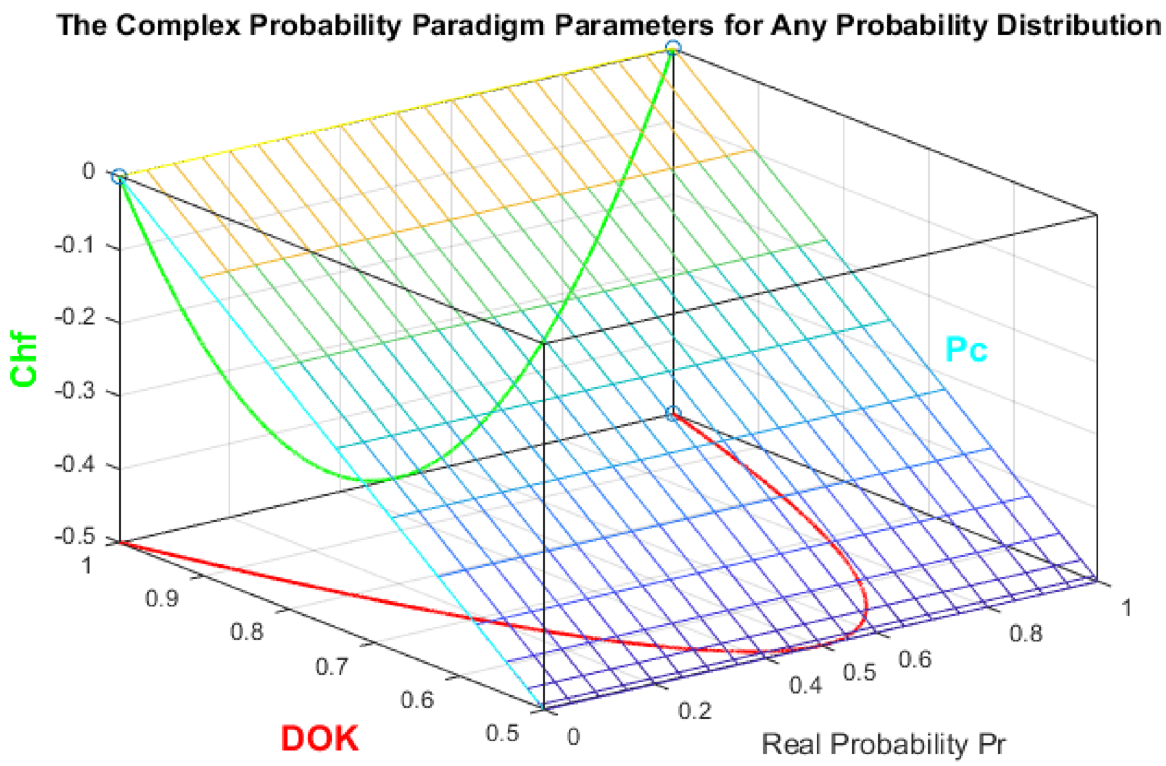


Figure 3: DOK, Chf, and Pc for any probability distribution in $3D$ with $Pc^2 = DOK - Chf = 1 = Pc$

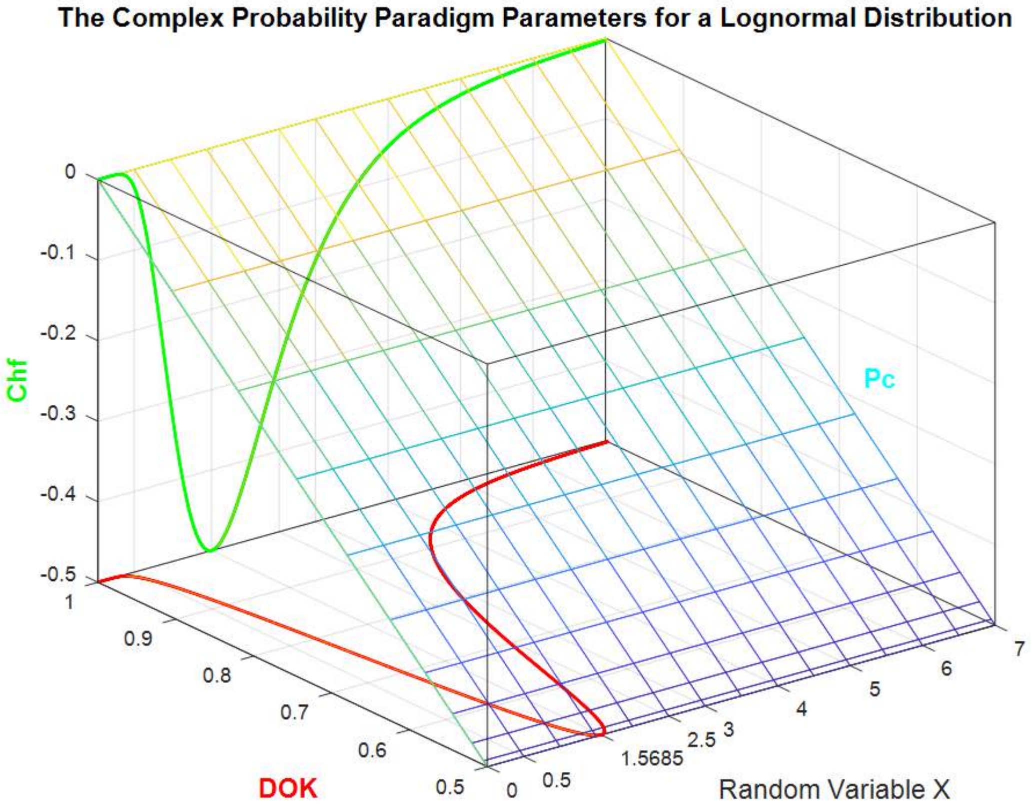


Figure 4: *DOK*, *Chf*, and *Pc* for a Lognormal probability distribution in 3*D* with $Pc^2 = DOK - Chf = 1 = Pc$

The graph below illustrates the linear relation between both *DOK* and *Chf*. (Figure 5)

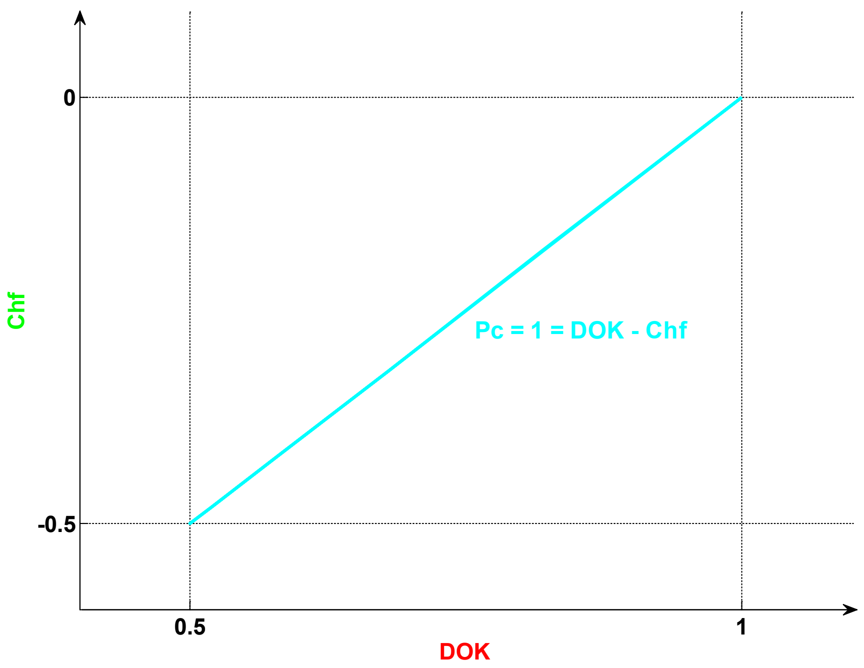


Figure 5: Graph of $Pc^2 = DOK - Chf = 1 = Pc$ for any probability distribution

Additionally, the absolute value of the chaotic factor will be required in our current work which will evaluate for us the magnitude of all the random and chaotic influences on the studied stochastic system that is materialized by the random simulation cycles time t and a probability density function, and which leads to an increasing system chaos in the set $\mathcal{R}$ and sometimes to a premature system failure. We will denote this new term accordingly Magnitude of the Chaotic factor or *MChf* [15-28]. Therefore, we can deduce what follows:

$$MChf = |Chf| = |2iP_rP_m| = -2iP_rP_m = 2P_r(1-P_r) \geq 0, \quad \forall P_r : 0 \leq P_r \leq 1, \tag{5}$$

And

$$\begin{aligned} Pc^2 &= DOK - Chf \\ &= DOK + |Chf|, \quad \text{since } -0.5 \leq Chf \leq 0 \\ &= DOK + MChf = 1, \end{aligned}$$

$$\Leftrightarrow 0 \leq MChf \leq 0.5 \text{ where } 0.5 \leq DOK \leq 1.$$

The graph below (Figure 6) illustrates the linear relation between both *DOK* and *MChf.* Additionally, Figures 7-13 illustrate the graphs of *Chf, MChf*, *DOK*, and *Pc* as functions of the real probability P_r and of the random variable X for any probability distribution and for a Lognormal distribution. It is significant to mention here that we could have considered deliberately any random distribution besides the Lognormal probability distribution like the discrete Binomial or Poisson random distributions or the continuous standard Gaussian normal distribution, etc. Although the graphs would have looked different whether in 2D or in 3D but the mathematical interpretations and consequences would have been similar for any imaginable and possible random distribution. This hypothesis is confirmed in my fourteen previous published papers by the mean of many illustrations encompassing both continuous and discrete probability distributions [15-28].

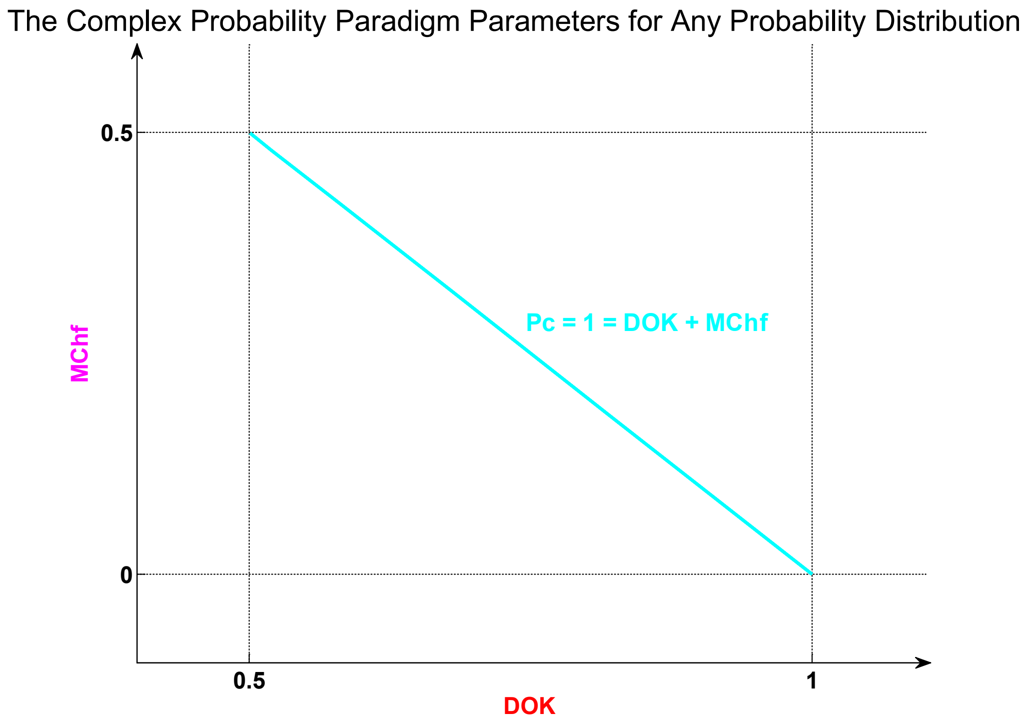


Figure 6: Graph of $Pc^2 = DOK + MChf = 1 = Pc$ for any probability distribution

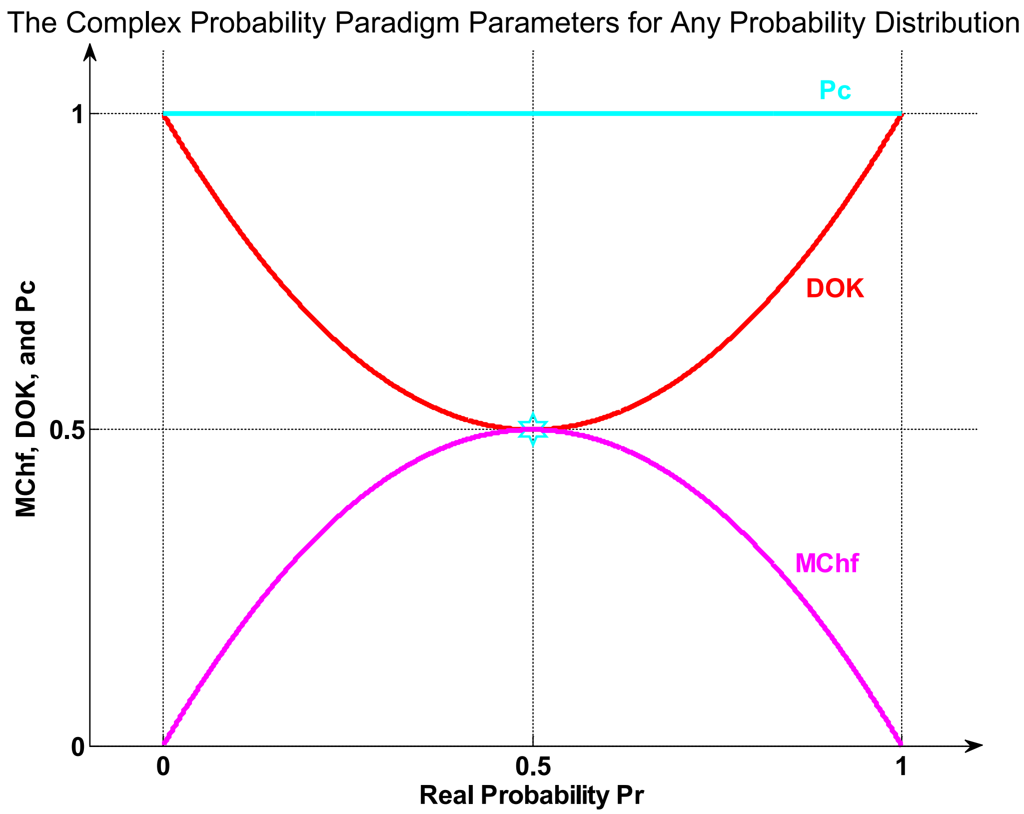


Figure 7: *MChf*, *DOK*, and *Pc* for any probability distribution in 2*D*

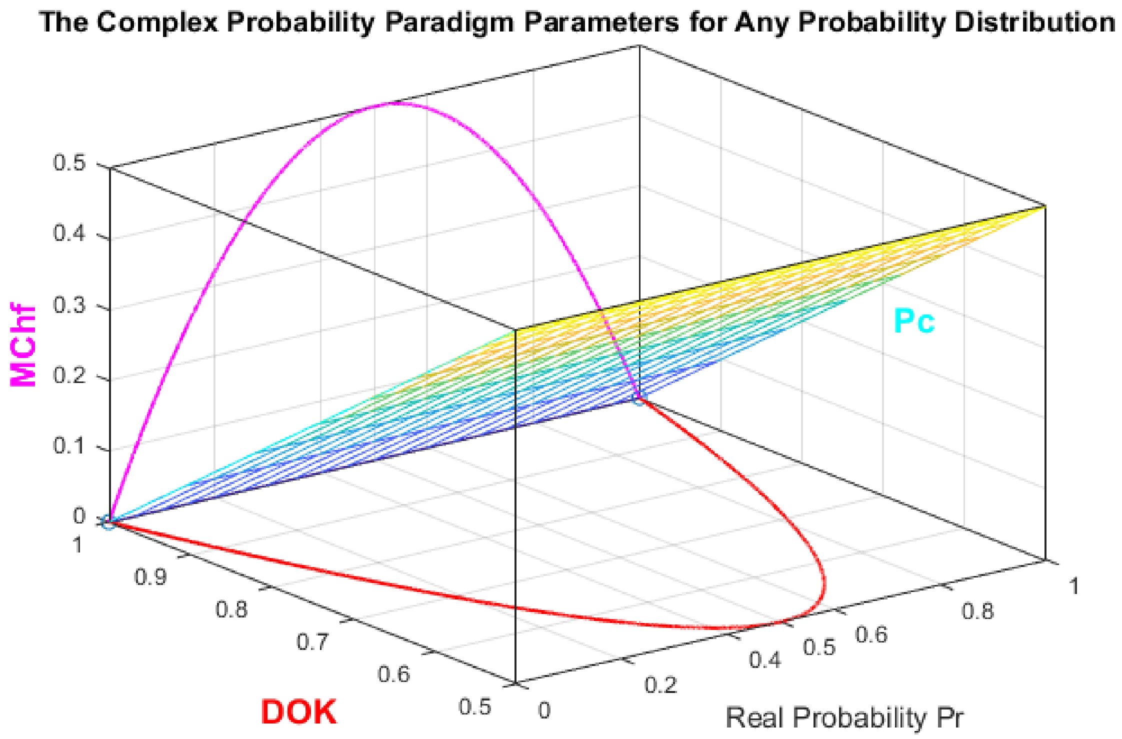


Figure 8: DOK, $MChf$, and Pc for any probability distribution in 3*D* with

$$Pc^2 = DOK + MChf = 1 = Pc$$

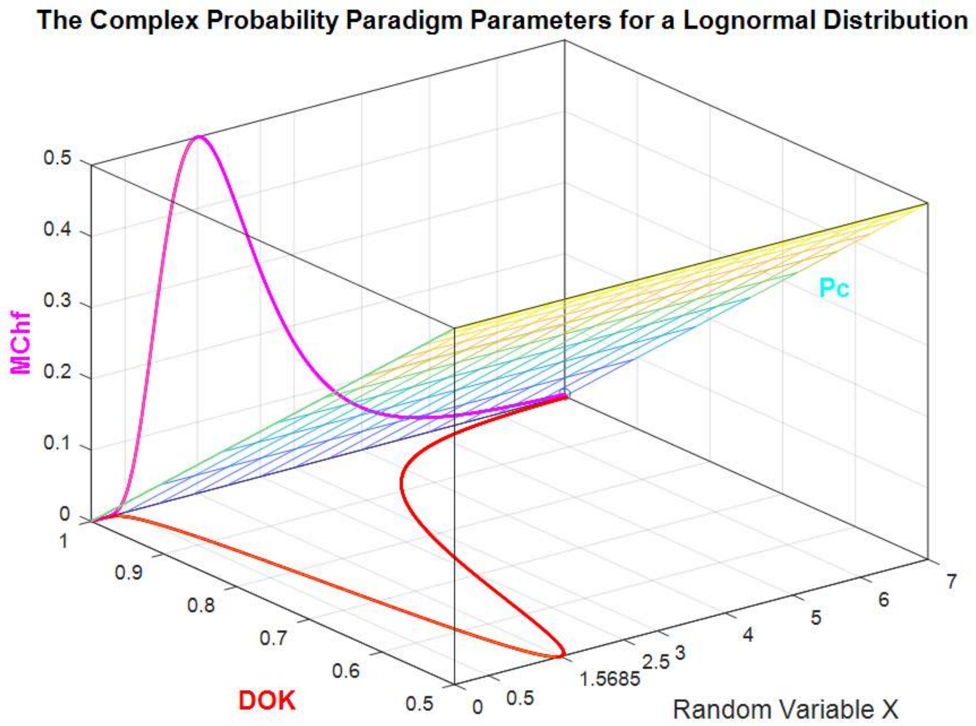


Figure 9: DOK, $MChf$, and Pc for a Lognormal probability distribution in $3D$ with $Pc^2 = DOK + MChf = 1 = Pc$

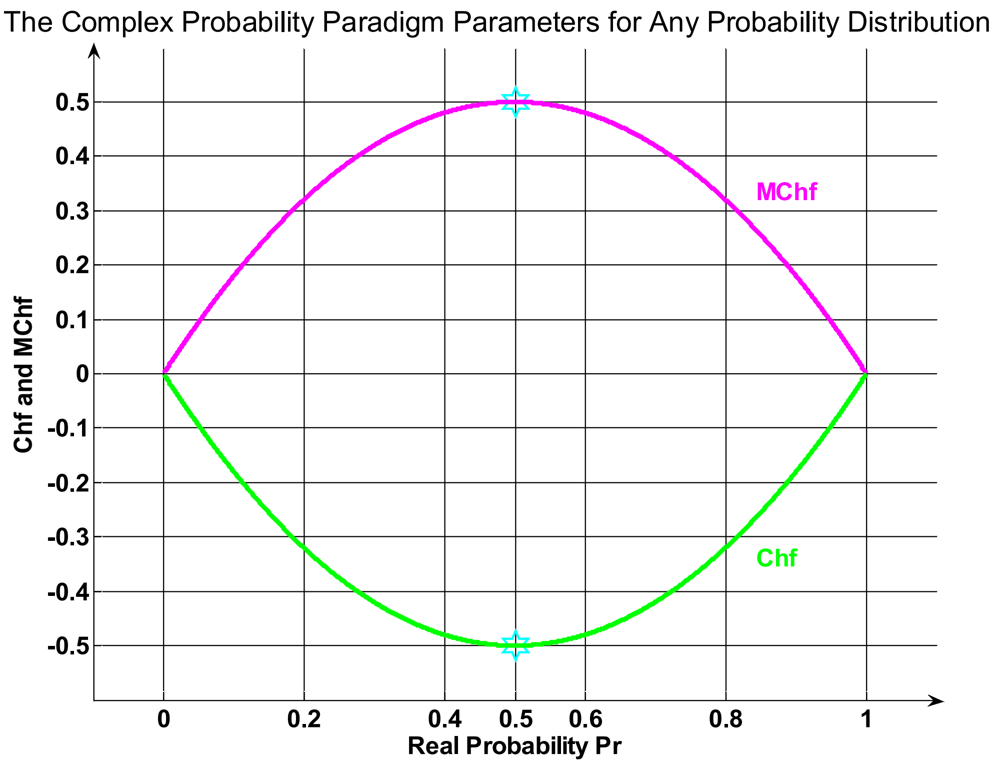


Figure 10: Chf and $MChf$ for any probability distribution in $2D$

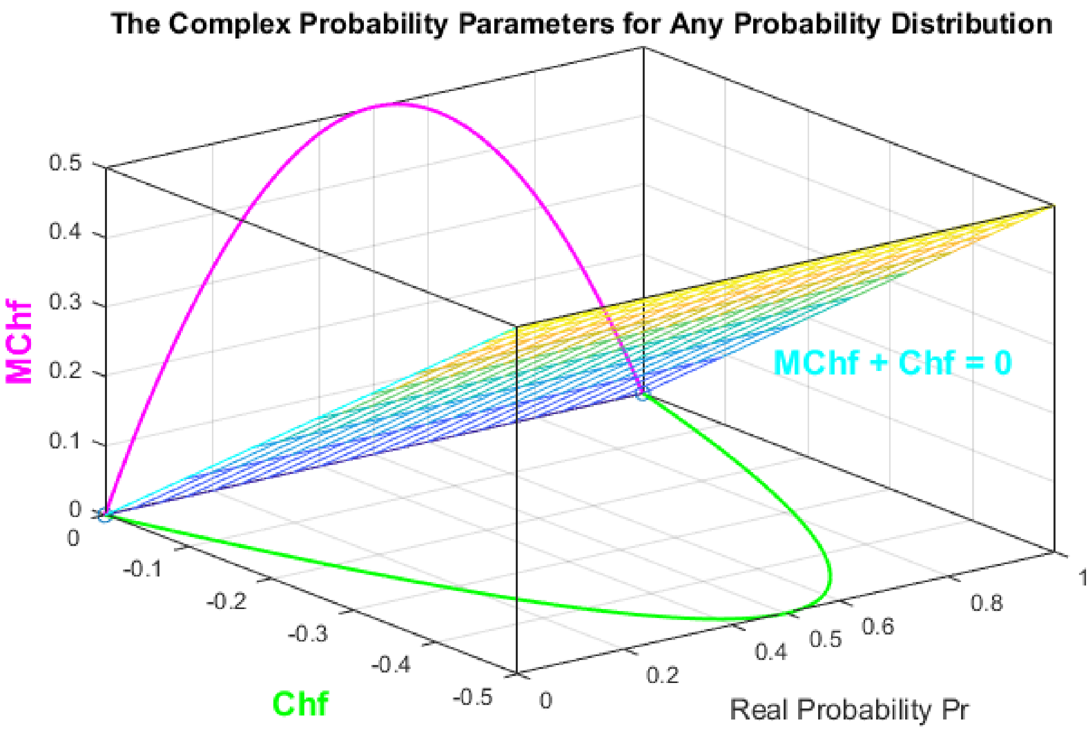


Figure 11: *Chf* and *MChf* for any probability distribution in 3*D* with *MChf* + *Chf* = 0

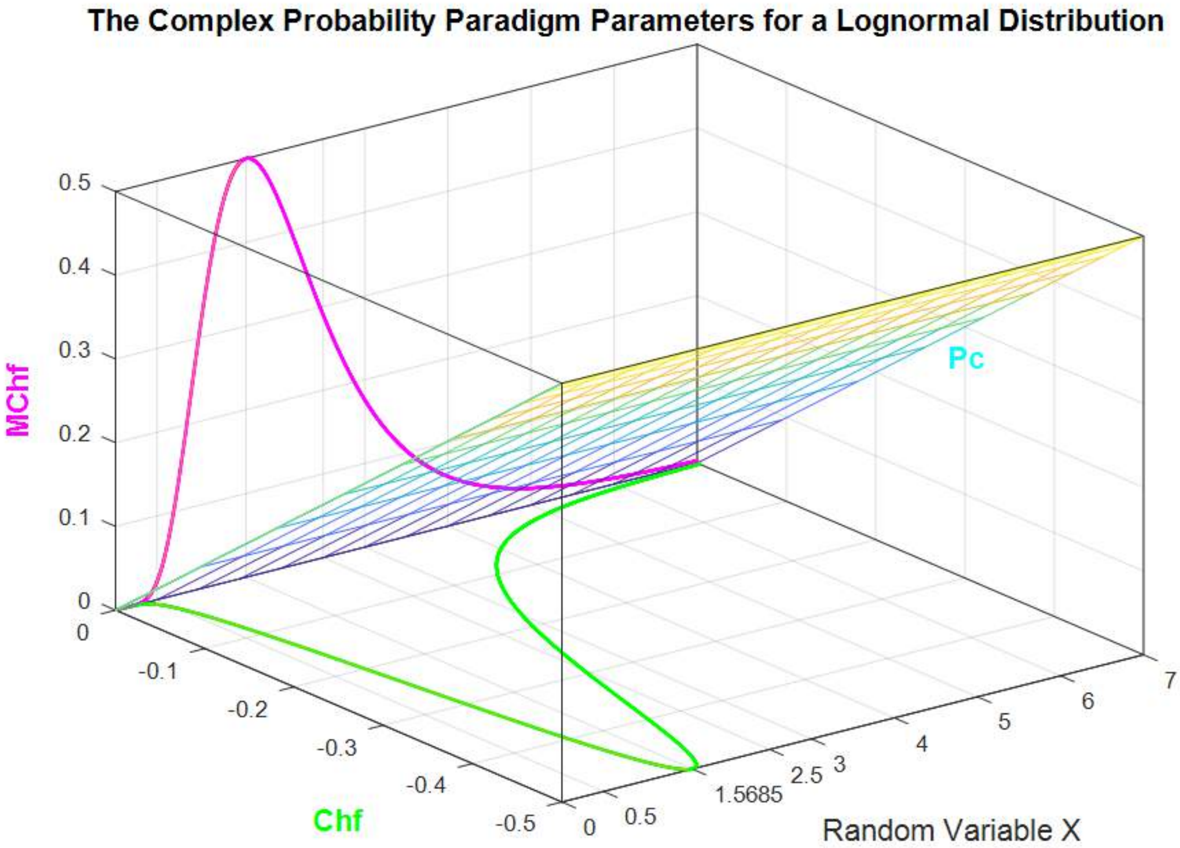


Figure 12: *Chf* and *MChf* for a Lognormal probability distribution in 3*D* with *MChf* + *Chf* = 0

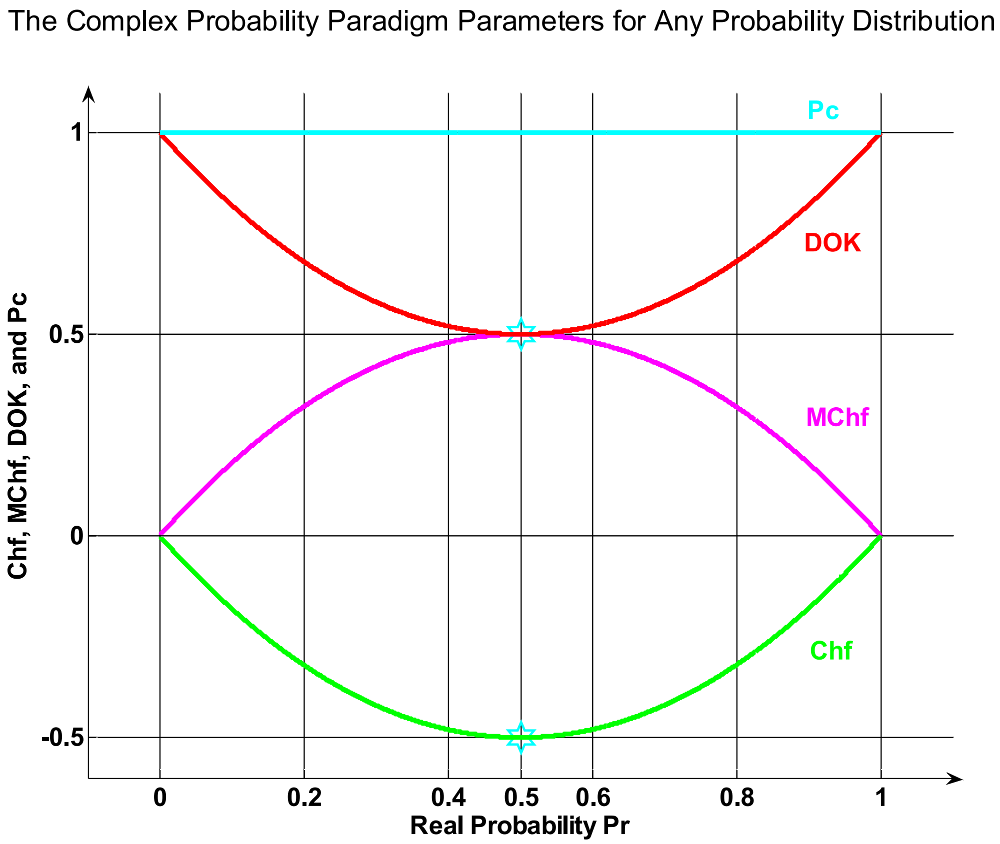


Figure 13: *Chf*, *MChf*, *DOK*, and *Pc* for any probability distribution in 2*D*

To recapitulate and to conclude, we state that in the real probability universe $\mathcal{R}$ our degree of our certain knowledge is undesirably imperfect and hence unsatisfactory, thus we extend our analysis to the set of complex numbers $\mathcal{C}$ which incorporates the contributions of both the set of real probabilities which is $\mathcal{R}$ and the complementary set of imaginary probabilities which is $\mathcal{M}$. Afterward, this will yield an absolute and perfect degree of knowledge in the probability universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ because $Pc = 1$ constantly. As a matter of fact, the work in the complex universe $\mathcal{C}$ gives way to a sure prediction of any stochastic experiment, because in $\mathcal{C}$ we remove and subtract from the computed degree of our knowledge the measured chaotic factor. This will generate a probability in the universe $\mathcal{C}$ equal to 1 ($Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc$). Many illustrations taking into consideration numerous continuous and discrete probability distributions in my fourteen previous research papers confirm this hypothesis and innovative paradigm [15-28].

The Extended Kolmogorov Axioms (*EKA* for short) or the Complex Probability Paradigm (*CPP* for short) can be shown and summarized in the next illustration (Figure 14):

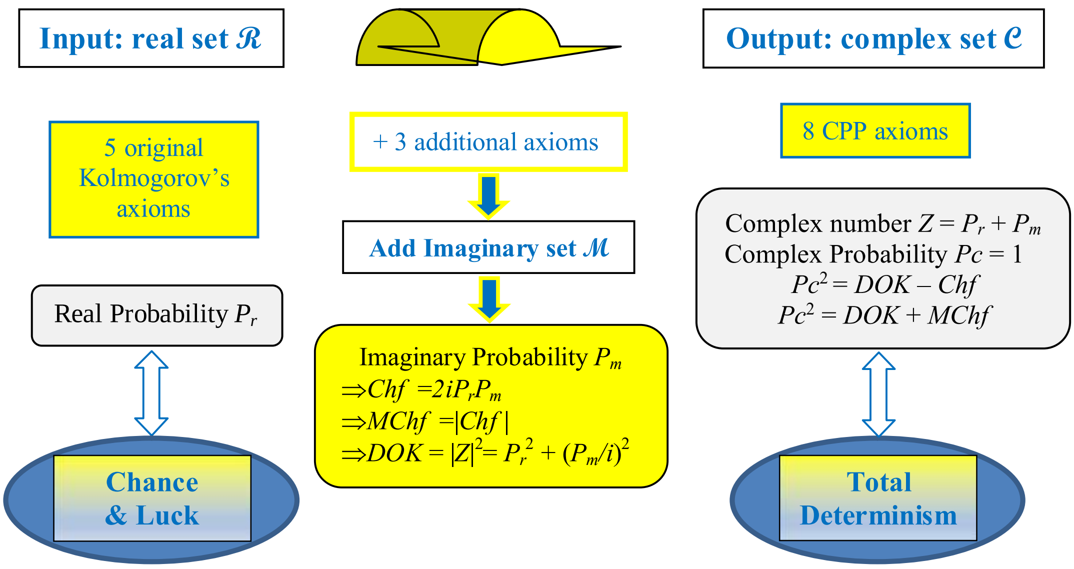


Figure 14: The *EKA* Paradigm or the Complex Probability Paradigm (*CPP*)

IV. Review of Reliability Theory [73-76]

The reliability is the probabilistic evaluation of a limit state of performance on a domain of basic variables. As a matter of fact, it is obtained by the computation of the probability of failure toward a limit state or criterion.

4.1 Methodology

1) Identify the basic parameters intervening in the limit state
2) Identify the limit states that govern the lifetime of the structure
3) Deduce their probability density functions
4) Compute the failure probability that expresses the risk when the limit states are not satisfied.

There exist two types of methods which are: The first method is the Monte Carlo simulation and the second method is the approximate method FORM (First-Order Reliability Method). The Monte Carlo simulation method is based on a large number of simulations and we must use N simulations when we want to evaluate a probability of order of $10^{-(N+4)}$.

The approximate method FORM is an iterative procedure that permits us to compute an index of reliability which is denoted by β. The index β is the distance between the origin and the limit state function $G(t)$ in a standard space. Once we have computed β we can deduce the failure cumulative probability which is: $P_r = \Phi(-\beta)$.

In FORM approximation the real (usually nonlinear) limit state is replaced by its tangent plane at a specific point called the Most Probable Failure Point (MPFP). This point is the closest point on $G(t)$ to the origin.

The limit state $G(t)$ divides the space into two regions:

- The first region where $G(t) > 0$ called safe region
- and the second region where $G(t) \le 0$ called failure region

4.2 Work Plan

In the general case, we choose N random variables correlated and of any density functions as well as a nonlinear limit state function. This method is based on the following iterative algorithm:

1) Converting basic random variables to standard normal variables $\mathcal{N}(0,1)$
2) Converting the limit state from the original space to the standard normal space
3) Searching of the MPFP point by replacing the limit state surface by its tangent hyper-plane at the same point.
4) Computing the index β and consequently the failure cumulative distribution function P_r.

4.3 Description of the Algorithm

The transformation from the basic state to the normalized state is implicit in the algorithm. The detailed steps of the algorithm are the following (Figure 15):

Let the limit state equation be: $g(z)$
where $z = z_1, z_2, z_3, \ldots, z_n$ is the random vector of the limit state; therefore:

1) Initialize the coordinates of the MPFP. The mean value of each variable is a good choice.

$$z^1 = \mu_{z_1}, \mu_{z_2}, \ldots, \mu_{z_n}$$

2) Calculate the following parameters: (m is the number of the iteration)

The value of the limit state at the MPFP:

$$g_0^m = g\left(z_1^m, \ldots, z_2^m\right)$$

The gradient at the MPFP is assumed to be:

$$g_i^m = \frac{\partial g}{\partial z_i}\left(z_1^m, \ldots, z_2^m\right)$$

The equivalent normal standard deviation and mean value of non-normal variables:

$$\sigma_i^m = \frac{\varphi\left(\Phi^{-1}\left(F_{z_i}\left(z_i^m\right)\right)\right)}{f_{z_i}\left(z_i^m\right)}$$

$$\mu_i^m = z_i^m - \sigma_i^m \Phi^{-1}\left(F_{z_i}\left(z_i^m\right)\right)$$

3) Calculate the intermediate parameters:

$$z^m = \sum_{i=1}^{n} g_i^m z_i^m$$

$$\mu_z^m = \sum_{i=1}^{n} g_i^m \mu_i^m$$

$$\sigma_z^m = \sqrt{\sum_{i=1}^{n} \left(g_i^m\right)^2 \left(\sigma_i^m\right)^2}$$

4) Calculate:

The directive cosine:

$$\alpha_i = -\frac{g_i^m \sigma_i^m}{\sigma_z^m}$$

The reliability index:

$$\beta^m = -\frac{z^m - g_0^m - \mu_z^m}{\sigma_z^m}$$

The new coordinates of the MPFP:

$$z_i^m = \mu_i^m + \alpha_i^m \beta^m \sigma_i^m$$

5) Verify the convergence criterion:

$$\left\| z^{m+1} - z^m \right\| \leq \text{tolerance and} \left| \beta^{m+1} - \beta^m \right| \leq \text{tolerance}$$

6) Repeat the steps from 2 till 5 until convergence.

7) Calculate the failure cumulative distribution function (CDF):

$$F = P_r = \Phi(-\beta)$$

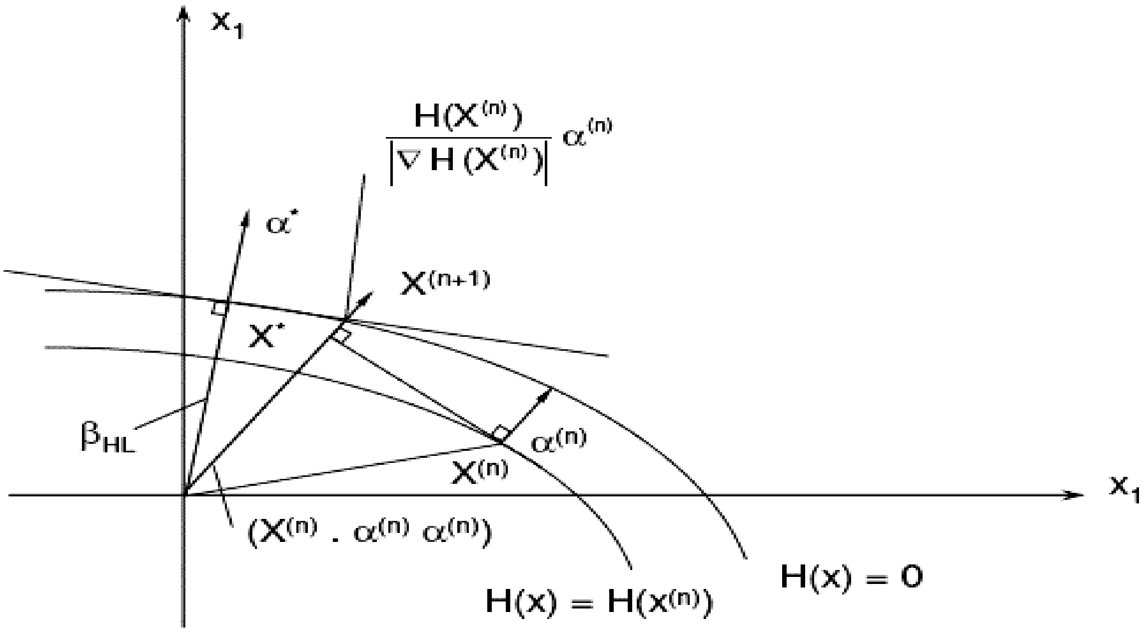


Figure 15: The First-Order Reliability Method illustration.

V. Application of the Complex Probability Paradigm to Prognostic Based on Reliability [15-28] [77-87]

5.1 The Basic Parameters of the New Model

In engineering systems, the prediction of the remaining useful lifetime is related deeply to many aspects and factors that generally have a chaotic and random behavior which decreases the degree of our knowledge of the entire system.

As we have shown, we can deduce from *CPP* that if we add to an event probability in the real set $\mathcal{R}$ the imaginary counterpart $\mathcal{M}$ (like the lifetime variables) then we can predict the exact probability of the remaining lifetime with certainty in $\mathcal{C}$ (since *Pc* = 1 permanently). We can apply this idea to prognostic analysis through the degradation evolution of a system. As a matter of fact, prognostic analysis consists in the prediction of the remaining useful lifetime of a system at any instant t_k and during the system functioning.

Let us consider a degradation trajectory $D(t)$ of a system where a specific instant t_k is studied. The instant t_k means here the time or age that can be measured also by the cycle number *N*. (Figure 16)

Referring to Figure 17, the previous statement means that at the system age t_k, the prognostic study must give the prediction of the failure instant t_C. Therefore, the *RUL* predicted here at the instant t_k is the following interval: $$RUL(t_k) = t_C - t_k \quad (6)$$

In fact, at the beginning ($t_k = 0$) (point J), the system failure probability $P_r = 0$ and the chaotic factor in our prediction is zero (*Chf* = 0) since chaos has not started its deteriorating and harmful effect on the system yet. The system is intact and in raw state; therefore, $RUL(0) = t_C - 0 = t_C$.

If $t_k = t_C$ (point L) then the $RUL(t_C) = t_C - t_C = 0$, the system failure probability is one ($P_r = 1$), and the chaotic factor in our prediction is zero (*Chf* = 0) since chaos has finished its deteriorating and harmful effect on the worn-out system and failure has certainly occurred.

If not (i.e. $0 < t_k < t_C$) (point K), the probability of the occurrence of this instant and the prediction probability of *RUL* are both less than one (not certain) due to non-zero chaotic factors since chaos has begun its damaging influence ($-0.5 \le Chf < 0$). Consequently, the system failure probability is: $0 < P_r < 1$ for the same reason. The degree of our knowledge *DOK* is accordingly less than 1. Thus, by applying here the *CPP* model, we can determine the system *RUL* and degradation with certainty in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ where *Pc* = 1 always.

Furthermore, we need in our current study the absolute value of the chaotic factor that will give us the magnitude of the chaotic and random effects on the studied system. Hence, we can deduce that at any instant $t_k : 0 \le t_k \le t_C$, the *MChf* influencing and acting on the system is the following:

$$MChf(t_k) = |Chf(t_k)| \ge 0 \text{ and}$$
$$\begin{aligned} Pc^2(t_k) &= DOK(t_k) - Chf(t_k) \\ &= DOK(t_k) + |Chf(t_k)|, \quad \text{since } -0.5 \le Chf(t_k) \le 0 \\ &= DOK(t_k) + MChf(t_k) = 1, \quad \forall t_k: \ 0 \le t_k \le t_C \end{aligned}$$
$$\Leftrightarrow 0 \le MChf(t_k) \le 0.5 \text{ where } 0.5 \le DOK(t_k) \le 1$$

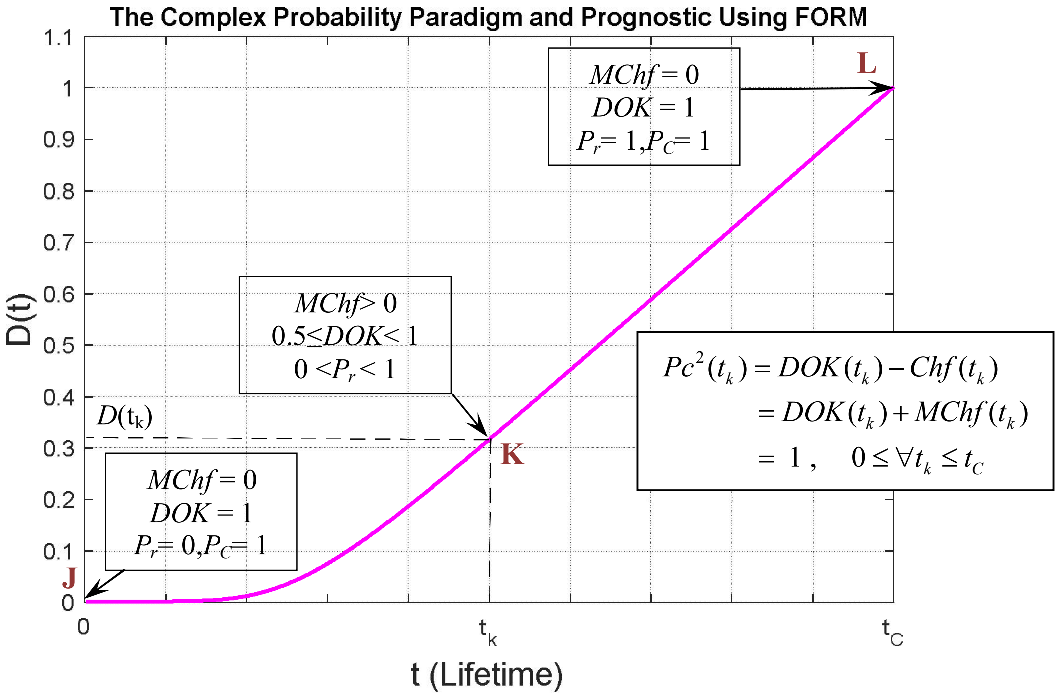


Figure 16: *CPP* and the prognostic of degradation

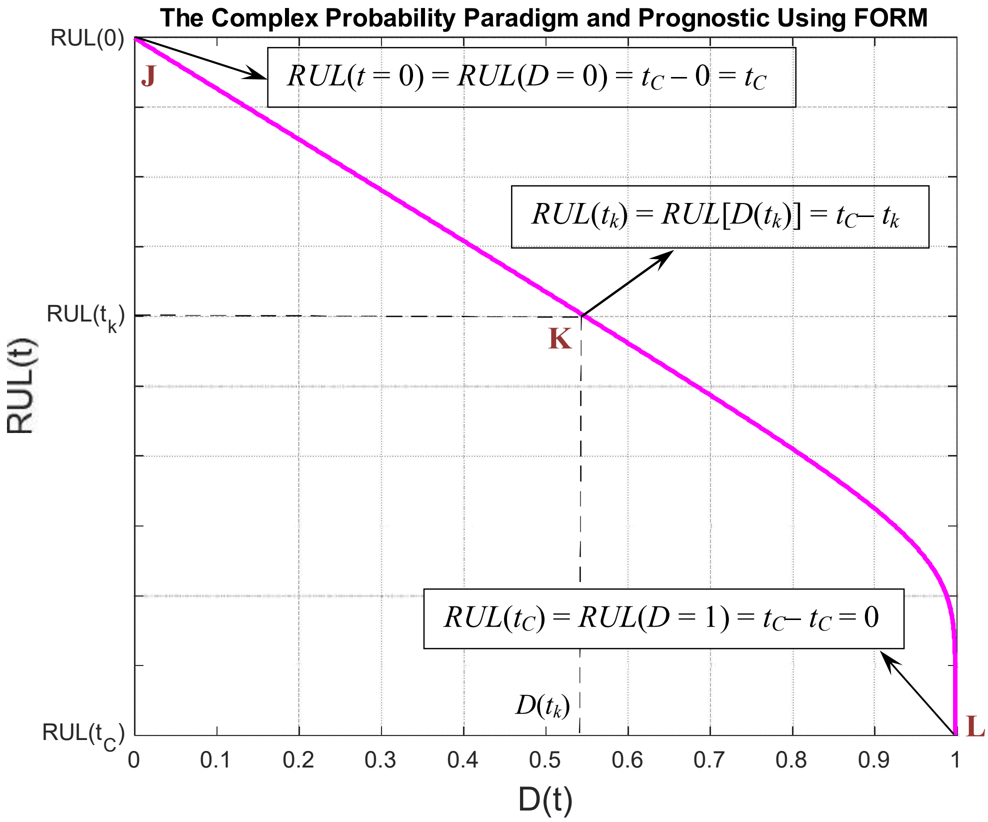


Figure 17: RUL prognostic model

Moreover, we can define two complementary events E and $\bar{E}$ with their respective probabilities:

$$P_{rob}(E) = p \text{ and } P_{rob}(\bar{E}) = q = 1 - p$$

Then $P_{rob}(E)$ in terms of the instant t_k is given by:

$$P_{rob}(E) = P_r(t_k) = P_{rob}\left(t \le t_k\right) = F(t_k) \quad (7)$$

where F is the cumulative probability distribution function (CDF) of the random variable t.

Since $P_{rob}(E) + P_{rob}(\bar{E}) = 1$, therefore,

$$P_{rob}(\bar{E}) = 1 - P_{rob}(E) = 1 - P_r(t_k) = 1 - P_{rob}(t \le t_k) = 1 - F(t_k) = P_{rob}(t > t_k) = P_m(t_k)/i \quad (8)$$

Let us define the two particular instants:

$t_k = 0$ assumed as the initial time of functioning (raw state) corresponding to $D = D_0 = 0$, and

t_C = the failure instant (wear-out state) corresponding to the degradation D=1.

The boundary conditions are:

For $t_k = 0$ then $D = D_0$ (initial damage that may be zero or not)

and $F(t_k) = P_{rob}(t \le 0) = 0$

For $t_k = t_C$ then D=1 and $F(t_k) = F(t_C) = P_{rob}(t \le t_C) = 1$.

Also $F(t_k)$ is a nondecreasing function that varies between 0 and 1. In fact, $F(t_k)$ is a cumulative CDF function (Figure 18).

In addition, since $RUL(t_k) = t_C - t_k$ and since time t_k is always increasing $(0 \le t_k \le t_C)$ then $RUL(t_k)$ is a nonincreasing remaining useful lifetime function (Figure 17).

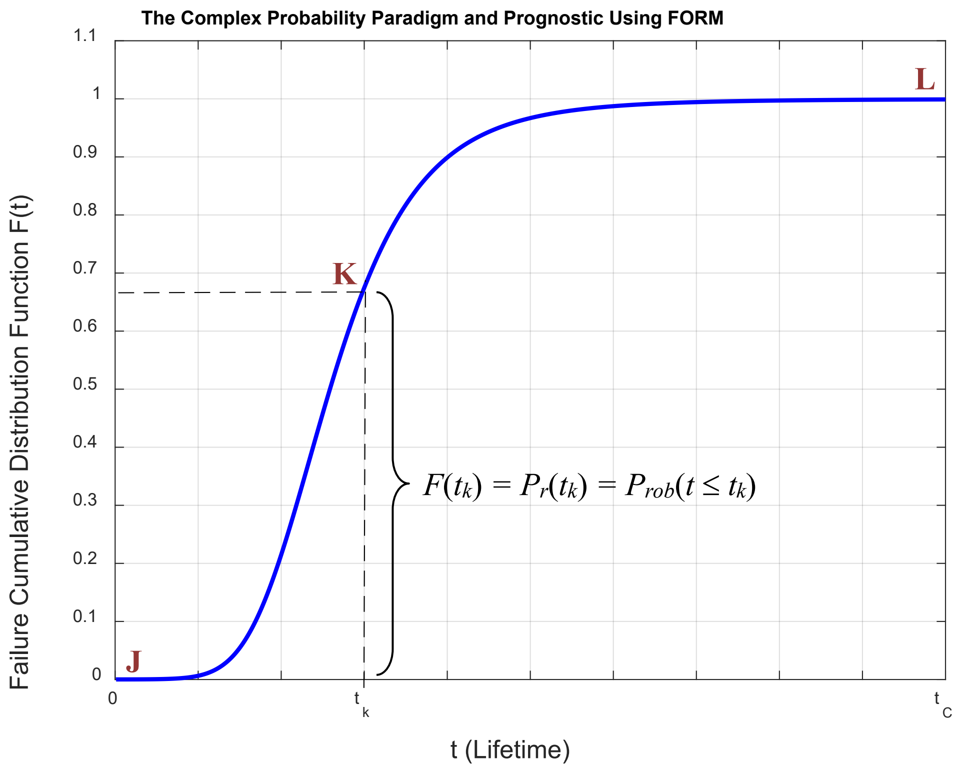


Figure 18: General system failure CDF

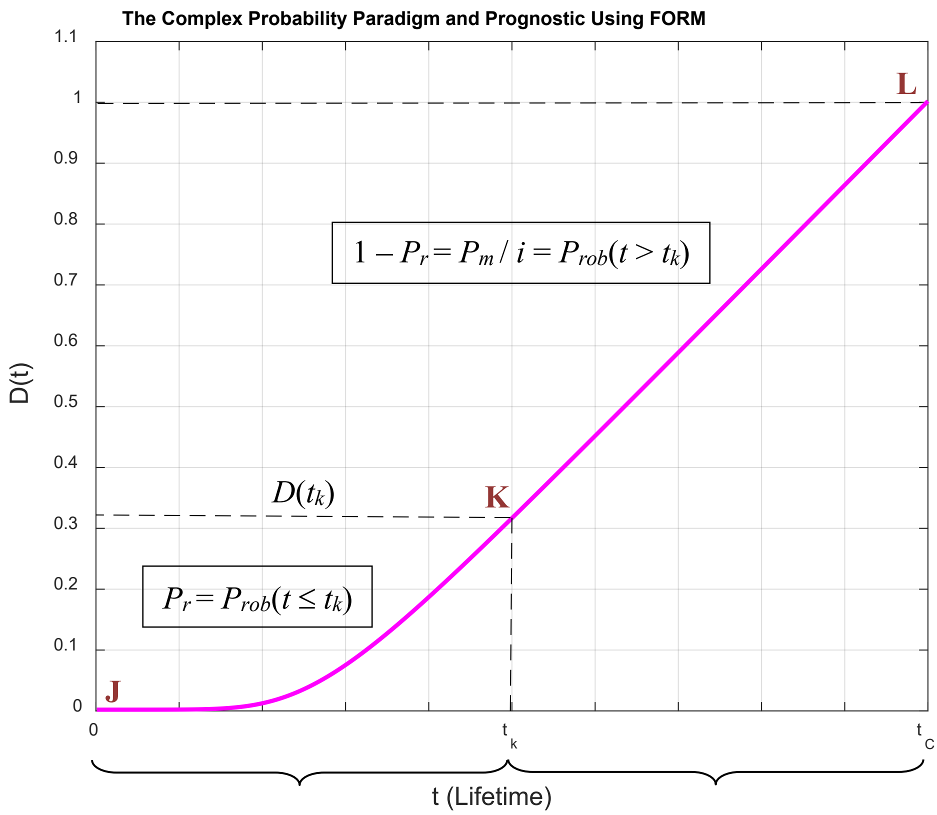


Figure 19: Degradation prognostic model

Referring to Figures16 to 19, we can infer the following:

The complementary imaginary probability to $P_r(t_k)$ in $\mathcal{M}$ is $P_m(t_k)$ and it is equal to:

$$P_m(t_k) = i[1 - P_r(t_k)] \quad (9)$$

The complex probability number or vector is:

$$Z(t_k) = P_r(t_k) + P_m(t_k) = P_r(t_k) + i[1 - P_r(t_k)] \quad (10)$$

The Degree of Our Knowledge DOK is the square of the norm of $Z(t_k)$ and it is equal to:

$$\begin{aligned} |Z(t_k)|^2 = DOK(t_k) &= [P_r(t_k)]^2 + [P_m(t_k)/i]^2 = [P_r(t_k)]^2 + [i\{1 - P_r(t_k)\}/i]^2 \\ &= 1 - 2P_r(t_k)[1 - P_r(t_k)] \\ &= 1 - 2P_r(t_k) + 2[P_r(t_k)]^2 \end{aligned} \quad (11)$$

The Chaotic Factor is:

$$\begin{aligned} Chf(t_k) &= 2iP_r(t_k)P_m(t_k) = 2iP_r(t_k)i[1 - P_r(t_k)] = -2P_r(t_k)[1 - P_r(t_k)] \quad \text{since} \quad i^2 = -1 \\ &= -2P_r(t_k) + 2[P_r(t_k)]^2 \end{aligned} \quad (12)$$

$Chf(t_k)$ is null when $P_r(t_k) = P_r(0) = 0$ (point J) or when $P_r(t_k) = P_r(t_C) = 1$ (point L).
The Magnitude of the Chaotic Factor is:

$$\begin{aligned} MChf(t_k) = |Chf(t_k)| &= -2iP_r(t_k)P_m(t_k) = -2iP_r(t_k)i[1 - P_r(t_k)] \\ &= 2P_r(t_k)[1 - P_r(t_k)] \quad \text{since} \quad i^2 = -1 \\ &= 2P_r(t_k) - 2[P_r(t_k)]^2 \end{aligned} \quad (13)$$

$MChf(t_k)$ is null when $P_r(t_k) = P_r(0) = 0$ (point J) or when $P_r(t_k) = P_r(t_C) = 1$ (point L),

At any instant $t_k : 0 \le t_k \le t_C$ (J ≤ K ≤ L), the probability expressed in the complex set $\mathcal{C}$ is:

$$\begin{aligned} Pc^2(t_k) &= DOK(t_k) - Chf(t_k) = \{1 - 2P_r(t_k) + 2[P_r(t_k)]^2\} - \{-2P_r(t_k) + 2[P_r(t_k)]^2\} \\ &= DOK(t_k) + MChf(t_k) = \{1 - 2P_r(t_k) + 2[P_r(t_k)]^2\} + \{2P_r(t_k) - 2[P_r(t_k)]^2\} \\ &= 1 \end{aligned} \quad (14)$$

$\Leftrightarrow Pc(t_k) = 1$ always

And

$$Pc(t_k) = P_r(t_k) + P_m(t_k)/i = P_r(t_k) + [1 - P_r(t_k)] = 1 \text{ always.} \quad (15)$$

Hence, the prediction of $RUL(t_k)$ andof the system degradation $D(t_k)$ in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is permanently and totally certain and perfect.

5.2 The New Prognostic Model

Let us present in this section the basic assumption of the new prognostic model. We consider firstly the cumulative probability distribution function $F(t)$ of the random time variable t which was calculated by FORM as being equal to:

$$F(t_k) = P_{rob}(t_0 \le t \le t_k) = \sum_{t=t_0}^{t=t_k} P_{rob}(t) = P_r(t_k) = \Phi(-\beta) \tag{16}$$

We note that we are dealing here with discrete random functions depending on the discrete random time t of simulations.

Then, we assume secondly that the real system failure probability $P_r(t)/\psi$ at the instant $t = t_k$ is equal to:

$$\begin{aligned} P_r(t_k) = F(t_k) &= \psi \times [P_{rob}(t \le t_k) - P_{rob}(t \le t_{k-1})] \\ &= \psi \times \left[\sum_{t=t_0}^{t=t_k} P_{rob}(t) - \sum_{t=t_0}^{t=t_{k-1}} P_{rob}(t) \right] \\ &= \psi \times \sum_{t=t_{k-1}}^{t=t_k} P_{rob}(t) = \psi \times P_{rob}(t_{k-1} \le t \le t_k) \\ &= \psi \times [D(t_k) - D(t_{k-1})] \end{aligned} \tag{17}$$

$=\psi$ times the jump in $D(t)$ from $t = t_{k-1}$ to $t = t_k$ (Figures 20 & 21),

where,

$t = [0, 1, 2, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_C]$ = the time of simulation cycles, and

$t_0 = 0$ = the initial time of cycles at the simulation beginning. It corresponds to a degradation $D = D(t_0) = D_0$ which is generally considered to be nearly equal to 0.

$t_1 = 1$ = the first simulation cycle time;

... ...

t_k = the k^{th} simulation cycle time;

... ...

t_C = the time of simulation cycles till system failure = the critical number of simulation time. It corresponds to $D = D(t_C) = D_C = 1$.

ψ = the simulation magnifying factor that depends on the simulation profile. ψ is equal to 724.3113.

Consequently, the recursive relation for degradation as a function of the failure probability $P_r(t)/\psi$ is the following:

$$\begin{aligned} D(t_k) &= D(t_{k-1}) + P_r(t_k)/\psi \\ \Rightarrow D(t_k) &= D(t_{k-1}) + F(t_k)/\psi \\ \Rightarrow D(t_k) &= D(t_{k-1}) + \Phi(-\beta)/\psi \end{aligned} \tag{18}$$

This basic assumption is plausible since:

1- Both D and F are cumulative functions starting from 0 and ending with 1.
2- Both are nondecreasing functions.
3- Both functions are without measure units: D is an indicator quantifying degradation and system damage as well as F which is an indicator quantifying chance and randomness.

Thus, initially we have:

$$P_r(t_k = t_0 = 0) = F(t_0) = 0$$

Moreover,

$$P_r(t_k) = \psi \times f(t_k) \Rightarrow P_r(t_k)/\psi = \Phi(-\beta)/\psi = f(t_k), \quad (19)$$

Where $1/\psi$ is a normalizing constant that is utilized to transform $P_r(t_k) = \Phi(-\beta)$ function to a probability density function with a total probability equal to one. $1/\psi$ depends on the simulation mode and conditions.Subsequently, we deduce that $f(t_k)$ is the usual probability density function (*PDF*) for any simulation mode. Knowing that, from classical probability theory, we have permanently:

$$\sum_{t_k=t_0}^{t_k=t_C} f(t_k) = \sum_{t_k=t_0}^{t_k=t_C} P_r(t_k)/\psi = 1$$

This result is reasonable since $P_r(t_k)/\psi$ is here a probability density function. (Figures 20& 21)

Therefore, we can deduce that:

$$\sum_{t_k=t_0}^{t_k=t_C} P_r(t_k) = \sum_{t_k=t_0}^{t_k=t_C} F(t_k)$$

$$= \psi \times \sum_{t=t_0}^{t=t_C} P_{rob}(t) = \psi \times P_{rob}(t_0 \le t \le t_C)$$

$$= \psi \times \left[D(t = t_C) - D(t = t_0)\right]$$

$$= \psi \times \left[D_C - D_0\right] = \psi \times [1 - 0] = \psi,$$

since $D(t_C) = 1$ and $D(t_0) = 0$ and $F(t_0)$ is taken as $= 0$

$$= \psi \times \sum_{t_k=t_0}^{t_k=t_C} f(t_k) = \psi \times 1 = \sum_{t_k=t_0}^{t_k=t_C} \Phi(-\beta) = \psi$$

$$\Rightarrow \sum_{t_k=t_0}^{t_k=t_C} P_r(t_k)/\psi = \sum_{t_k=t_0}^{t_k=t_C} F(t_k)/\psi = \sum_{t_k=t_0}^{t_k=t_C} \Phi(-\beta)/\psi = \psi/\psi = 1. \quad (20)$$

We can observe that $D(t)$ is a discrete random function where the amount of the jump in the degradation discrete curve is $P_r(t)/\psi$; therefore, $P_r(t)/\psi$ is a function of degradation and damage evolution (Figures 20 & 21). And we can realize from the previous calculations that $P_r(t)/\psi$ is a probability density function. Consequently, we can understand now that $P_r(t)/\psi$ measures the probability of the system failure or degradation. Accordingly, what we have done here is that we have linked probability theory to degradation measure.

Notice that:

$0 \le P_r(t_k)/\psi \le 1$, $0 \le F(t_k) \le 1$, and $(D_0 = 0) \le D(t_k) \le (D_C = 1)$, for every t_k: $0 \le t_k \le t_C$

and

If $t_k \to 0 \Rightarrow D \to D_0 = 0 \Rightarrow F \to 0 \Rightarrow P_r(t_k) \to 0$

if $t_k \to t_C \Rightarrow D \to D_C = 1 \Rightarrow F \to 1 \Rightarrow P_r(t_k) \to 1$.

This, since the degradation is flat near 0 and starts increasing and becoming more acute with time t; hence, at t_C, D is the greatest and is equal to 1. (Figures 21 & 22)

Furthermore, we have:

$RUL(t_k) = t_C - t_k$ and it corresponds to a degradation $D(t_k)$,

And

$RUL(t_{k-1}) = t_C - t_{k-1}$ and it corresponds to a degradation $D(t_{k-1})$.

This implies that (Figure 23):

$$\begin{aligned} P_r(t_k) &= \psi \times [D(t_k) - D(t_{k-1})] \\ &= \psi \times \{D[t_C - RUL(t_k)] - D[t_C - RUL(t_{k-1})]\} \end{aligned} \tag{21}$$

5.3 Analysis and Extreme Random and Stochastic Environments

Even though the prognostic laws are sometimes deterministic and well-known in general [88-100] but there are chaotic and stochastic aspects (such as in engineering: geometry dimensions, humidity, water action, material nature, atmospheric pressure, applied load location, corrosion, soil pressure and friction, temperature, etc…). Additionally, many variables in the expression of degradation which are believed to be deterministic may as well adopt a stochastic behavior, such as in engineering and in pipelines and suspension prognostic: the magnitude of the applied pressure (due to the various conditions of pressure profile) and the length of the initial crack (potentially occurring during the process of manufacturing). All those stochastic aspects, embodied in the prognostic models by their average values, influence the system and make its function of degradation diverge from its computed path foretold by these deterministic rules. An updated follow-up of the behavior of degradation with cycle number or time, and which is under the influence of chaotic and non-chaotic aspects, is done by $P_r(t)/\psi$ due to its definition that evaluates the jumps in D. In fact, chaos alters and affects all the environment and system parameters included in the expression of degradation. Consequently, chaos total effect on the system contributes to shape the curve of degradation D and is embodied and counted in the system failure probability $P_r(t)/\psi$. Actually, $P_r(t)/\psi$ quantifies the resultant of all the nondeterministic (stochastic) and deterministic (non-stochastic) factors and parameters which are included in the equation of D, which influence the system, and which determine the consequent final curve of degradation. Accordingly, an accentuated effect of chaos on the system can lead to a smaller (or bigger) jump in the trajectory of degradation and thus to a smaller (or bigger) probability of failure $P_r(t)/\psi$. If for example, due to extreme random influences and deterministic causes, D jumps directly from $D_0 = 0$ to 1 then RUL goes straight from t_C to 0 and subsequently $P_r(t)/\psi$ jumps instantly from 0 to 1:

$$P_r(t_k)/\psi = D(t_k) - D(t_{k-1}) = D(t_C) - D(0) = 1 - 0 = \sum_{t=0}^{t=t_C} P_{rob}(t) = \sum_{t=0}^{t=t_C} \Phi(-\beta)/\psi = \psi/\psi = 1$$

where t goes directly from 0 to t_C.

In the extreme ideal situation, if the system never deteriorates (no pressure or stresses) and with zero chaotic effects and random influences, then the resultant of all the deterministic and nondeterministic aspects is null (like in the system isolated and idle state). Accordingly, the system remains indefinitely at $D_0 = 0$ and RUL stays equal to t_C . So consequently, the jump in D is always 0. Therefore, ideally, the probability of failure persists 0:

$$\begin{aligned} P_r(t_k)/\psi &= [D(t_k) - D(t_{k-1})] \\ &= [D_0 - D_0] \\ &= 0 \end{aligned}$$

where $D(t_0) = D(t_1) = \ldots = D(t_{k-1}) = D(t_k) = D(t_{k+1}) = \ldots\ldots\ldots = D_0 = 0,$
for $k = 0,1,2,3,\ldots\infty$

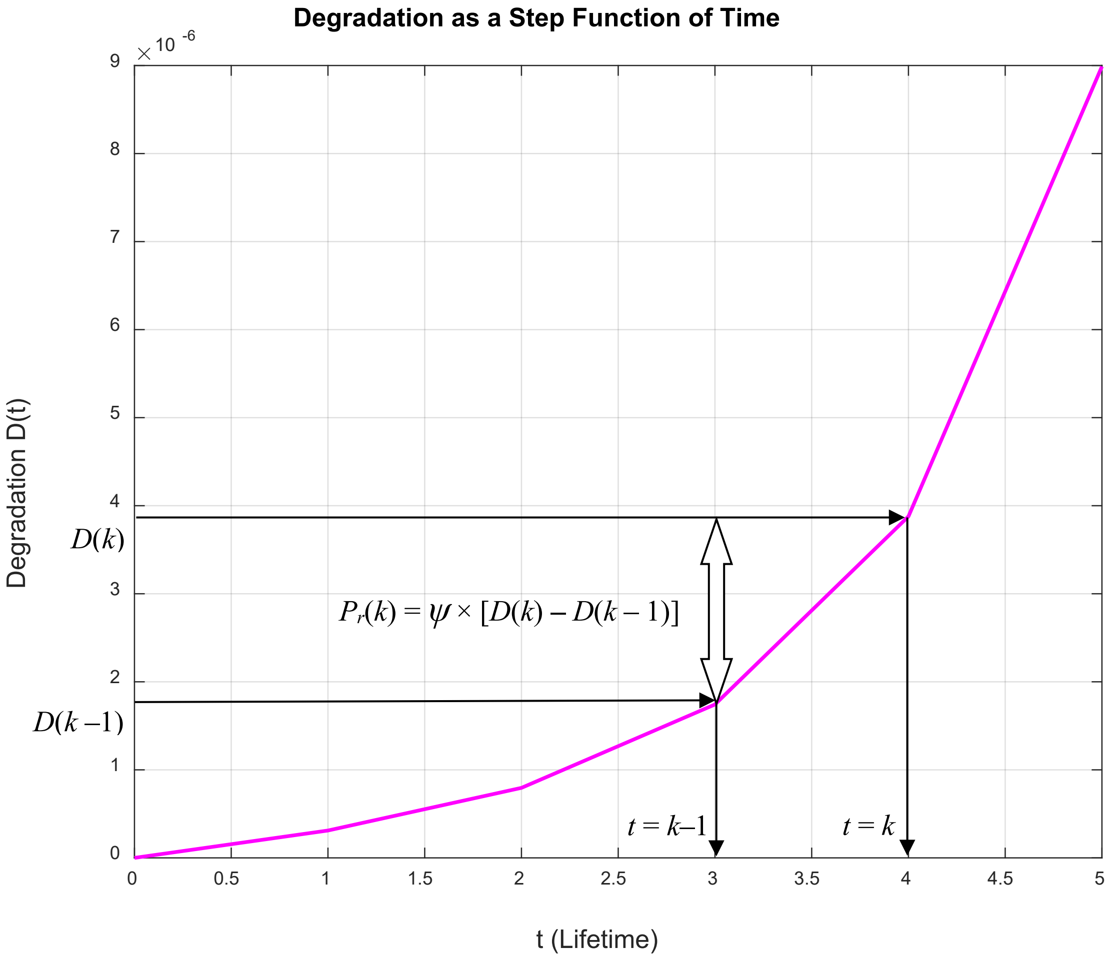


Figure 20: P_r, degradation, and the CDF step function

Figure 20 shows the real failure probability $P_r(t)$ as a function of the random system degradation step CDF in terms of the simulation cycles time t.

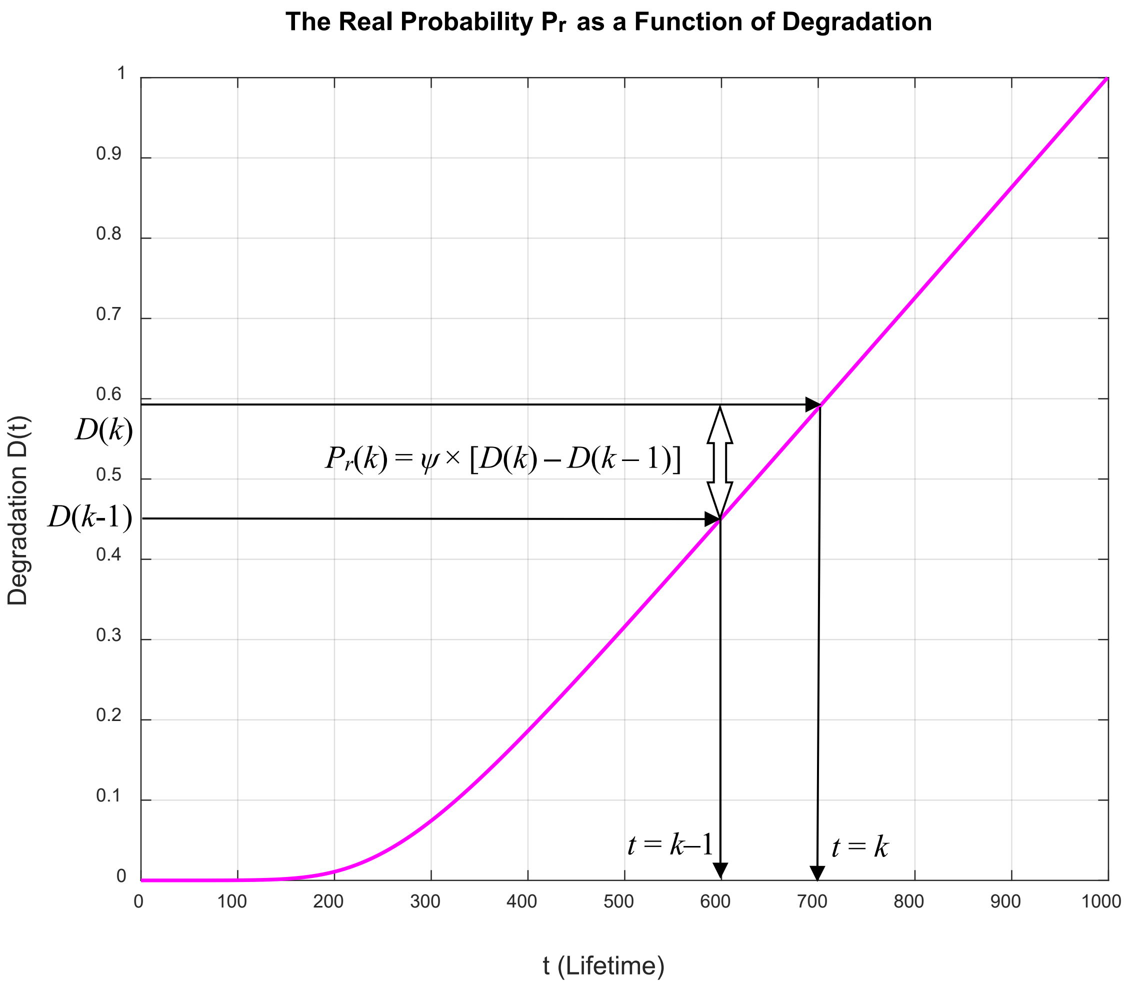


Figure 21: P_r as a function of Degradation $D(t)$

Figure 21 shows the real failure probability $P_r(t)$ as a function of the random system degradation in terms of the simulation cycles time t.

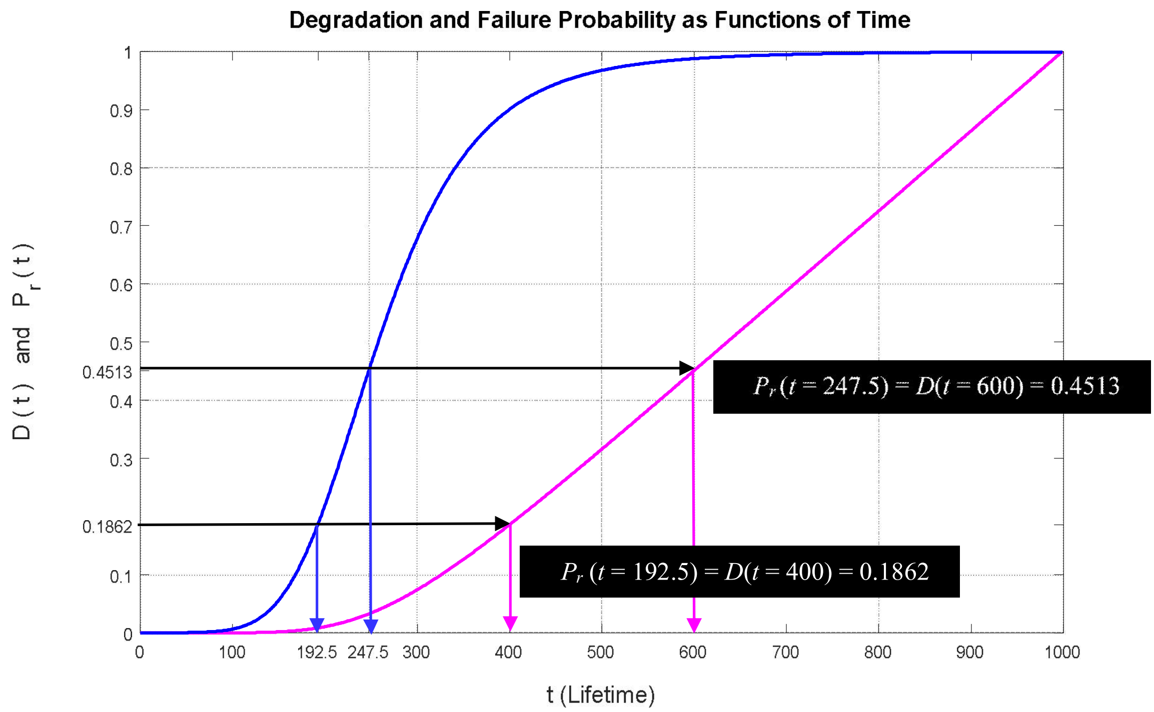


Figure 22: Degradation and P_r

Figure 22 shows the real failure probability $P_r(t)$ and the random system degradation $D(t)$ as functions of the number of the simulation cycles time t.

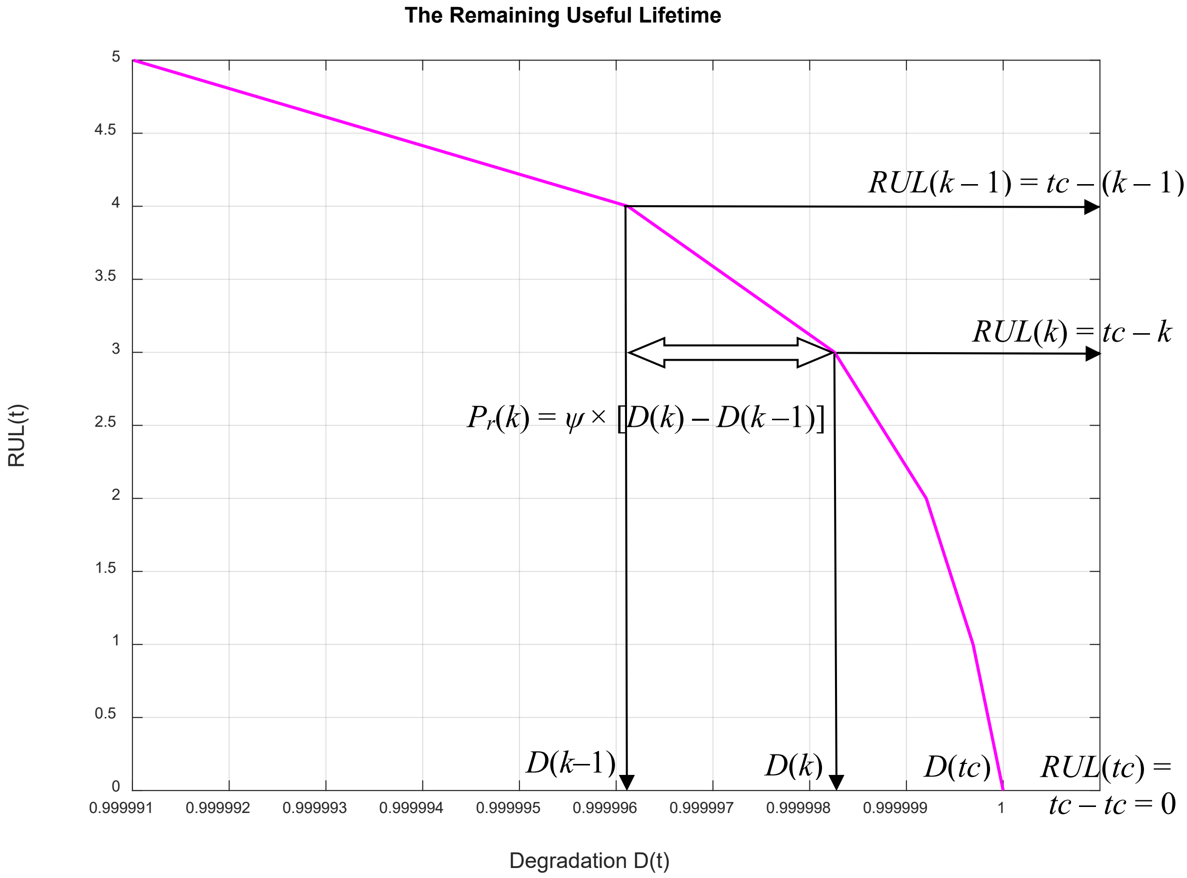


Figure 23: P_r , D, and RUL

Figure 23 shows the real failure probability $P_r(t)$ as a function of the random system degradation $D(t)$ and the random system $RUL(t)$ in terms of the simulation cycles time t.

VI. Application of FORM to Prognostic [15-28] [73-76]

In this part, we study the *CPP* in the context of reliability by defining a limit state G that describes the lifetime margin of the system.

We have:

$$G(t_k) = t_C - t_k = RUL(t_k) \tag{22}$$

where $G(t_k)$ is the limit state of lifetime.

t_C: is the fixed lifetime of the system which follows a normal distribution $\mathcal{N}(60, 1)$
t_k : is an arbitrary instant that varies from 0 to t_C and which follows a normal distribution $\mathcal{N}\left(t_k, 0.1\times t_k\right)$

When $G(t_k)$ is zero or negative then we have a case of $t_k \geq t_C$ that means that we have a system failure that cannot live until the instant t_k. In the other case where $t_k < t_C$, the system can live above the instant t_k and we have a case of success.

The reliability index $\beta = -\Phi^{-1}[P_r(t_k)]$ where $P_r(t_k)$ is the cumulative distribution probability function and Φ is the normal cumulative distribution function. Hence, Φ^{-1} is the inverse of Φ and $P_r(t_k) = \Phi(-\beta)$ (23)

The failure cumulative distribution function computed by the FORM procedure is:

$$P_r(t_k) = P_{rob}\{G(t_k) \leq 0\} = P_{rob}\{t_k \geq t_C\} = \Phi(-\beta) \quad (24)$$

It corresponds in system prognostic to: $P_{rob}\{RUL(t_k) \leq 0\}$

Therefore, the survival cumulative distribution function computed by the FORM procedure is:

$$P_{rob}\{G(t_k) > 0\} = P_{rob}\{t_k < t_C\} = 1 - P_r(t_k) = P_m(t_k)/i = 1 - \Phi(-\beta) \quad (25)$$

It corresponds in system prognostic to: $P_{rob}\{RUL(t_k) > 0\}$

In *CPP*, the real part of probability is taken here $P_r(t_k)$. As we make the instant t_k vary between 0 and t_C, then $P_r(t_k) = F(t_k)$ varies between 0 and 1 as shown in Figure 24. Moreover, Figure 25 illustrates the system failure PDF which is $P_r(t_k)/\psi = \Phi(-\beta)/\psi$.

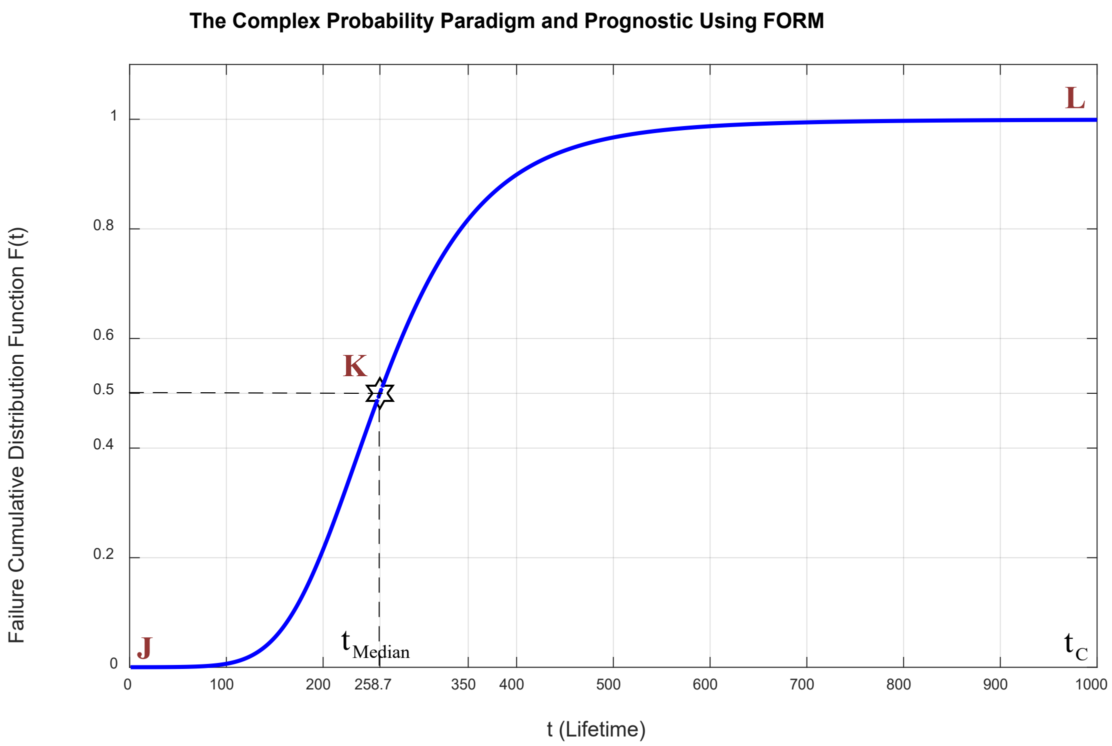


Figure 24: System failure CDF for the current simulation

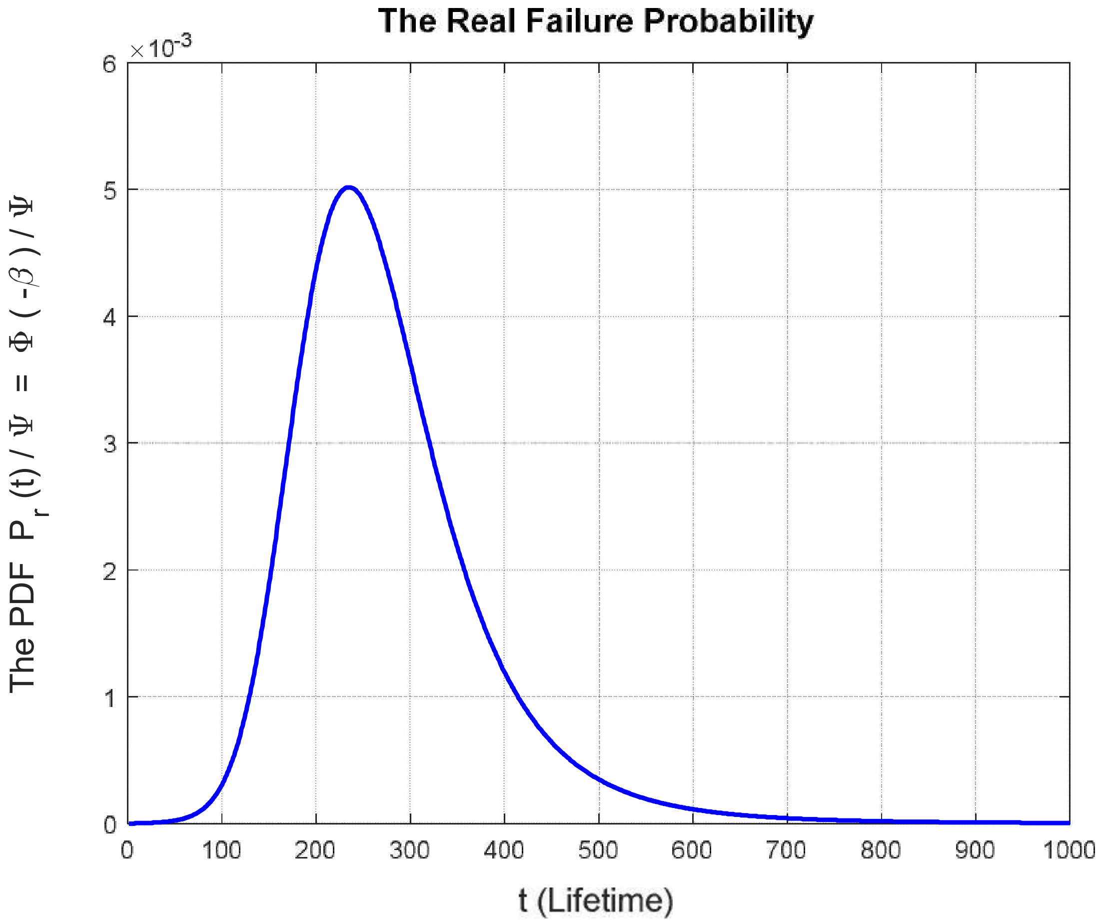


Figure 25: System failure PDF for the current simulation

We Know that we take t_k and t_C as two normal random variables where the value of t_C corresponds to 1000 cycles (t_C = critical value). After a reliability calculation using a Matlab version 2020 program, we deduce a value of $P_r(t_k)$ for each value of instant t_k. Figure 26 illustrates all the new prognostic model functions and proves all the mathematical derivations. We have computed and ploted for this set of $P_r(t_k)$ all the *CPP* parameters and components and which are: $Chf(t_k)$, $MChf(t_k)$, $DOK(t_k)$, $Pc(t_k)$, $P_m(t_k)/i$, $D(t_k)$, and $P_{rob}[RUL(t_k)]$.

We note from the figure that the *DOK* is maximum (*DOK* = 1) when absolute value of *Chf* which is *MChf* is minimum (*MChf* = 0) (points J & L), that means when the magnitude of the chaotic factor (*MChf*) diminishes our certain knowledge (*DOK*) grows. Subsequently, *MChf* begins to grow during the functioning due to the environment and intrinsic circumstances thus leading to a diminution in *DOK* until they both reach 0.5 at $t_k = t_{Median} = 258.7$ (point K). The real cumulative failure probability P_r and the real cumulative complementary survival probability P_m/i will meet with *DOK* and *MChf* also at the point (258.7, 0.5) (point K). The point K' is the point corresponding to K and which is (637, 0.5). K' is the point where the degradation $D(t_k)$ and $P_{rob}[RUL(t_k)]$ intersect. With the growth of t_k, the *Chf* and *MChf* return to zero and the *DOK* returns to 1 where we attain total damage ($D = 1$) and hence the total certain failure of the system ($P_r = 1$) (point L). At this last point the failure here is definite, $P_r(t_C) = 1$ and $RUL(t_C) = t_C - t_C = 0$ with $Pc(t_C) = 1$, so the logical consequence of the value $DOK = 1$ ensues.

We note that the point K corresponding to $t_{Median} \neq \overline{t} \neq t_{Mode}$ which is the median of the distribution is not at the middle of the simulation since the probability of failuredistribution evaluated by FORM is not symmetric. Therefore, the corresponding graphs are skewed to the right or positively skewed.

Furthermore, at each instant t_k, we can predict with certainty the remaining useful lifetime $RUL(t_k)$ in the complex probability set $\mathcal{C}$ with Pc preserved as equal to one through an unceasing compensation between DOK and Chf. This compensation is from instant $t_k = 0$ where $D(t_k) = 0$ until the instant of failure t_C where $D(t_C) = 1$.

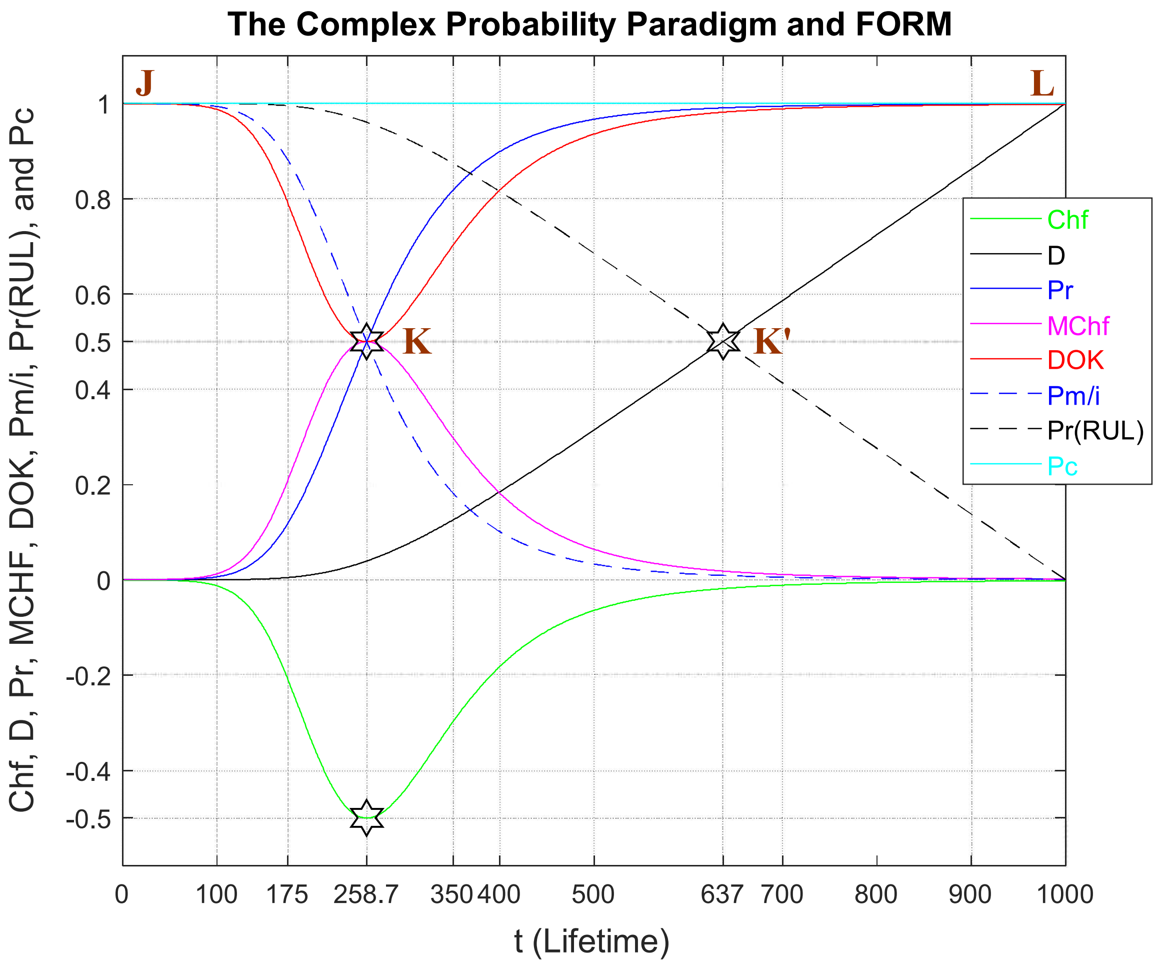


Figure 26: *CPP* and FORM applied to prognostic.

6.1 The Complex Probability Cubes

In the first cube (Figure 27), the simulation of *Chf*and *DOK*as functions of each other and of the simulation cyclestime t can be seen. The line in cyan is the projection of $Pc^2(t) = DOK(t) - Chf(t) = 1 = Pc(t)$ on the plane $t = 0$ cycles. This line starts at the point J ($DOK = 1$, $Chf = 0$) when $t = 0$ cycles, reaches the point ($DOK = 0.5$, $Chf= -0.5$) when $t = t_{Median}$=258.7 cycles, and returns at the end to J ($DOK = 1$, $Chf= 0$) when $t = t_C$ = 1000cycles. The other curves are the graphs of $Chf(t)$(pink, blue,green) and$DOK(t)$(red)in different planes. Notice that they all have a minimum at the point K ($DOK = 0.5$, $Chf= -0.5$, $t = t_{Median}$=258.7cycles). The point L corresponds to ($DOK = 1$, $Chf= 0$, $t = t_C$ = 1000 cycles). The three points J, K, L are similar to those in the previous figures.

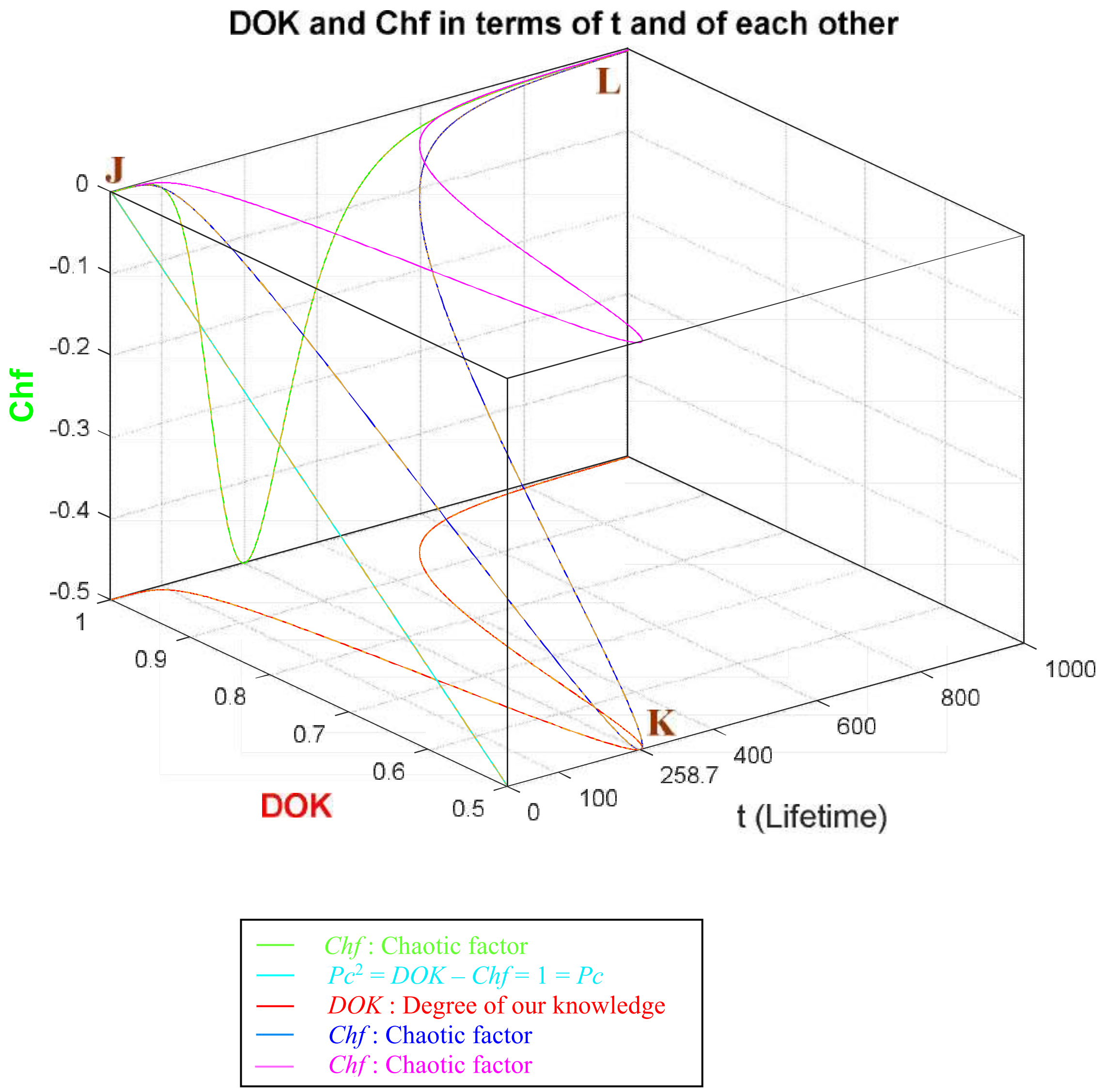


Figure 27: *DOK* and *Chf* in terms of *t* and of each other

In the second cube (Figure 28), we can notice the simulation of the failure probability $P_r(t)$ and its complementary real probability $P_m/i(t)$ in terms of the simulation cycles time t. The line in cyan is the projection of $Pc^2(t) = P_r(t) + P_m/i(t) = 1 = Pc(t)$ on the plane $t = 0$ cycles. This line starts at the point ($P_r = 0$, $P_m/i= 1$) and ends at the point ($P_r = 1$, $P_m/i = 0$). The red curve represents $P_r(t)$ in the plane $P_r(t) = P_m/i(t)$. This curve starts at the point J ($P_r = 0$, $P_m/i= 1$, $t = 0$ cycles), reaches the point K ($P_r = 0.5$, $P_m/i= 0.5$, $t = t_{Median} =258.7$cycles), and gets at the end to L ($P_r = 1$, $P_m/i= 0$, $t = t_C = 1000$ cycles). The blue curve represents $P_m/i(t)$ in the plane $P_r(t) + P_m/i(t) = 1$. Notice the importance of the point K which is the intersection of the red and blue curves at $t = t_{Median} =258.7$ cycles and when $P_r(t) = P_m/i(t) = 0.5$.The three points J, K, L are similar to those in the previous figures.

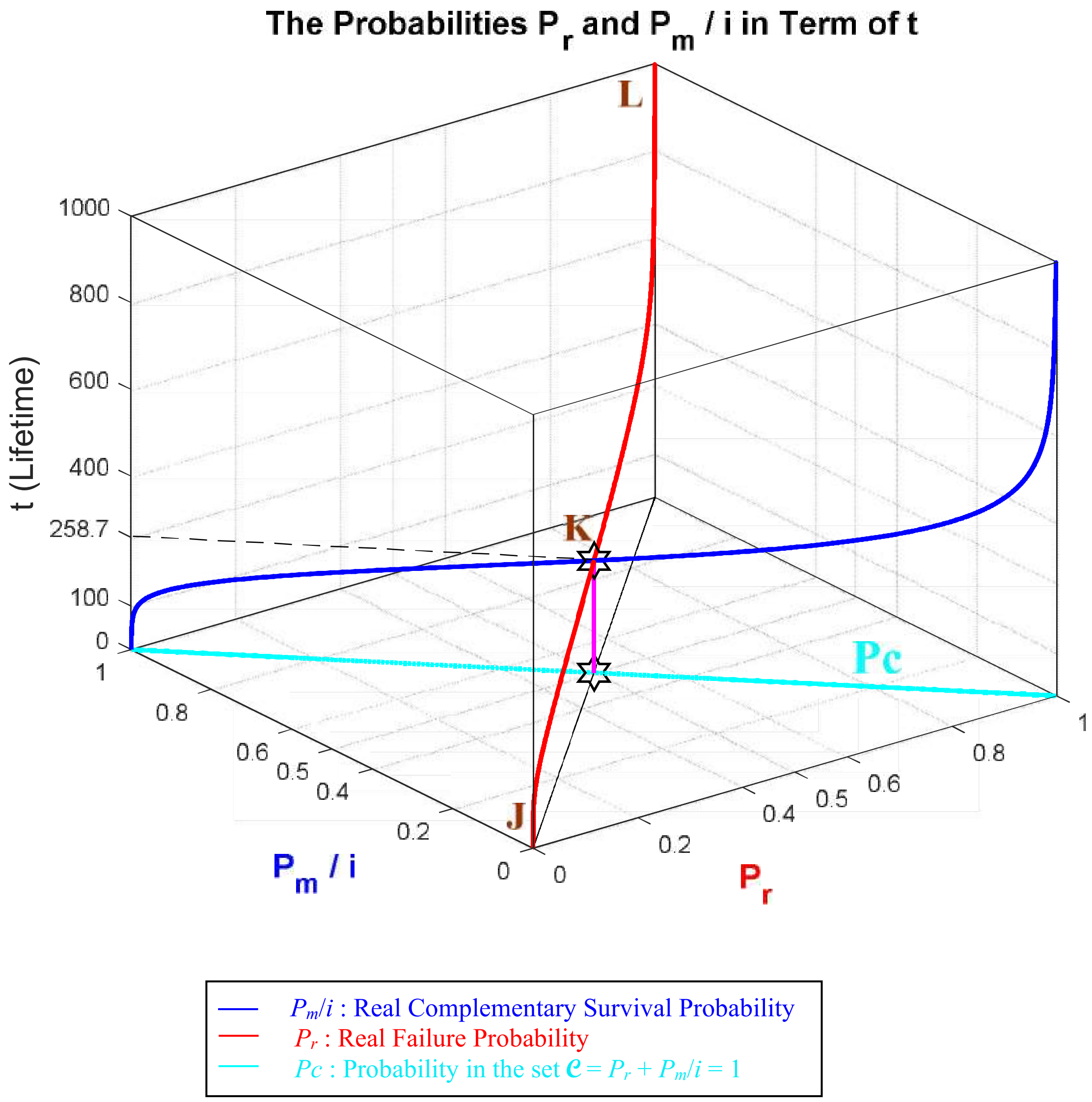
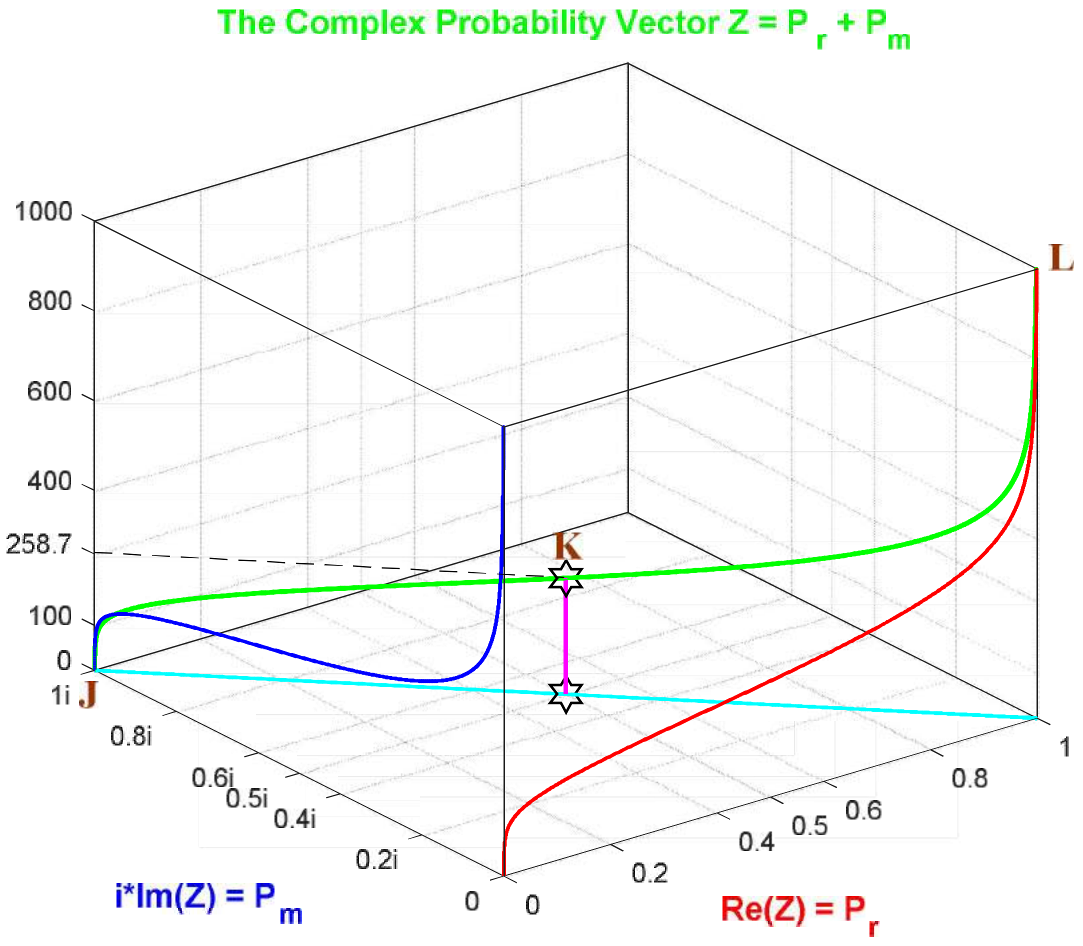


Figure 28: P_r and P_m/iin terms of t and of each other

In the third cube (Figure 29), we can notice the simulation of the complex random vector $Z(t)$ in $\mathcal{C}$ as a function of the real failure probability $P_r(t) = \mathrm{Re}(Z)$ in $\mathcal{R}$ and of its complementary imaginary probability $P_m(t) = i\times\mathrm{Im}(Z)$ in $\mathcal{M}$, and this in terms of the simulation cycles time. The red curve represents $P_r(t)$ in the plane $P_m(t) = 0$ and the blue curve represents $P_m(t)$ in the plane $P_r(t) = 0$. The green curve represents the complex probability vector $Z(t) = P_r(t) + P_m(t) = \mathrm{Re}(Z) + i\times\mathrm{Im}(Z)$ in the plane $P_r(t) = iP_m(t) + 1$. The curve of $Z(t)$ starts at the point J ($P_r = 0$, $P_m = i$, $t = 0$ cycles) and ends at the point L ($P_r = 1$, $P_m = 0$, $t = t_C = 1000$ cycles). The line in cyan is $P_r(0) = iP_m(0) + 1$ and it is the projection of the $Z(t)$ curve on the complex probability plane whose equation is $t = 0$ cycles. This projected line starts at the point J ($P_r = 0$, $P_m = i$, $t = 0$ cycles) and ends at the point ($P_r = 1$, $P_m = 0$, $t = 0$ cycles). Notice the importance of the point K corresponding to $t = t_{Median} = 258.7$ cycles and when $P_r = 0.5$ and $P_m = 0.5i$. The three points J, K, L are similar to those in the previous figures.

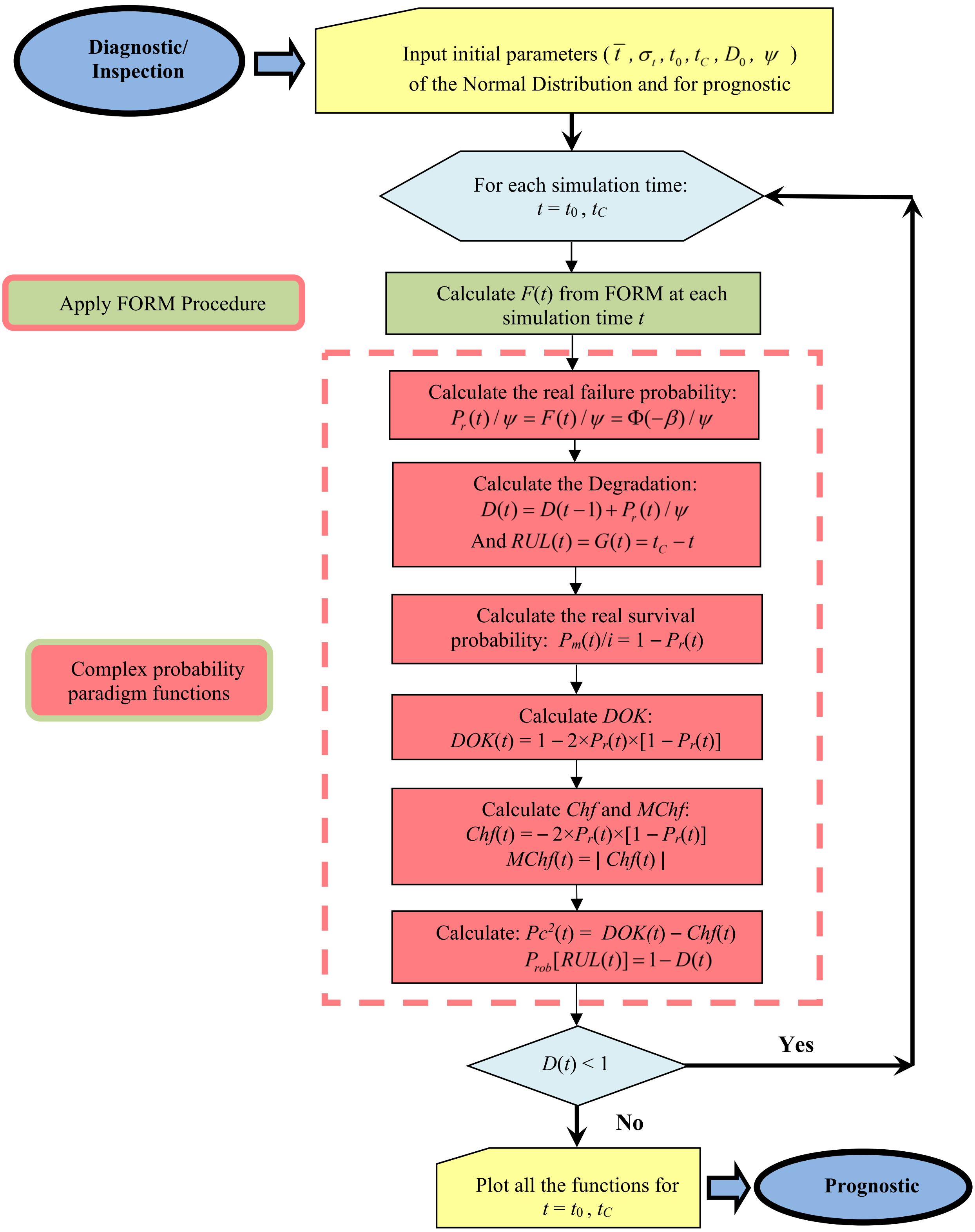


——	P_r : Real Failure Probability in the set $\mathcal{R}$ = Re(Z)
——	P_m : Complementary Imaginary Survival Probability in the set $\mathcal{M}$ = i×Im(Z)
——	$Z = P_r + P_m$: The Complex Probability Vector in the set $\mathcal{C}$

Figure 29: The Complex Probability Vector Z in terms of t

6.2 Flowchart of CPP Applied to Prognostic

The following flowchart summarizes all the explained procedures of the proposed complex probability prognostic model:



VII. Application of the New Model to Young Modulus [15-28] [73-76]

We apply now the novel prognostic model to the very well-known Young modulus. Let E be the Young modulus in a material bar domain (Figure 30) and we suppose that it follows a Normal Gaussian probability distribution.

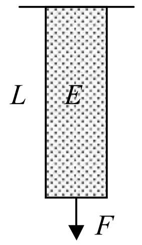


Figure 30: The Young modulus E in a material domain

The novel prognostic model expressions for Young modulus are the following:

The CDF (cumulative probability distribution function) $F(E)$ of the random variable $E$and which was calculated by FORM is equal to:

$$F(E_k) = P_{rob}(E_0 \le E \le E_k) = \sum_{E=E_0}^{E=E_k} P_{rob}(E) = P_r(E_k) = \Phi(-\beta) \tag{26}$$

And

$$\begin{aligned} P_r(E_k) = F(E_k) &= \psi \times [P_{rob}(E \le E_k) - P_{rob}(E \le E_{k-1})] \\ &= \psi \times \left[\sum_{E=E_0}^{E=E_k} P_{rob}(E) \; - \sum_{E=E_0}^{E=E_{k-1}} P_{rob}(E) \right] \\ &= \psi \times \sum_{E=E_{k-1}}^{E=E_k} P_{rob}(E) = \psi \times P_{rob}(E_{k-1} \le E \le E_k) \\ &= \psi \times [D(E_k) - D(E_{k-1})] \qquad (27) \\ &= \psi \text{ times the jump in } D(E) \text{ from } E = E_{k-1} \text{ to } E = E_k. \text{ (Figures 31 \& 32)} \end{aligned}$$

ψ = the simulation magnifying factor that depends on the simulation profile. ψ is equal to 1449.4 in the case of Young modulus prognostic.

Consequently, the recursive relation for degradation as a function of the failure probability $P_r(E)/\psi$ is the following:

$$D(E_k) = D(E_{k-1}) + P_r(E_k)/\psi \tag{28}$$

$$\Rightarrow D(E_k) = D(E_{k-1}) + F(E_k)/\psi \tag{29}$$

$$\Rightarrow D(E_k) = D(E_{k-1}) + \Phi(-\beta)/\psi \tag{30}$$

Thus, initially we have:

$$P_r(E_k = E_0 = 0) = F(E_0) = 0$$

Moreover,

$$P_r(E_k) = \psi \times f\ (E_k) \Rightarrow P_r(E_k)/\psi = \Phi(-\beta)/\psi = f\ (E_k)$$

Where $1/\psi$ is a normalizing constant that is utilized to transform $P_r(E_k) = \Phi(-\beta)$ function to a probability density function with a total probability equal to one. $1/\psi$ is a function of the simulation mode and conditions. Subsequently, we deduce that $f\ (E_k)$ is the usual probability density function (*PDF*) for any simulation mode. Knowing that, from classical probability theory, we have continuously:

$$\sum_{E_k=E_0}^{E_k=E_C} f\ (E_k) = \sum_{E_k=E_0}^{E_k=E_C} P_r(E_k)/\psi = 1$$

This result is reasonable since $P_r(E_k)/\psi$ is here a probability density function.

Therefore, we can deduce that:

$$\sum_{E_k=E_0}^{E_k=E_C} P_r(E_k)/\psi = \sum_{E_k=E_0}^{E_k=E_C} F(E_k)/\psi = \sum_{E_k=E_0}^{E_k=E_C} \Phi(-\beta)/\psi = 1. \quad (31)$$

(Figures 31 & 32)

Notice that:
$0 \le P_r(E_k)/\psi \le 1$, $0 \le F(E_k) \le 1$, and $(D_0 = 0) \le D(E_k) \le (D_C = 1)$, for every E_k: $0 \le E_k \le E_C$
and

If $E_k \to 0 \Rightarrow D \to D_0 = 0 \Rightarrow F \to 0 \Rightarrow P_r(E_k) \to 0$
if $E_k \to E_C \Rightarrow D \to D_C = 1 \Rightarrow F \to 1 \Rightarrow P_r(E_k) \to 1$.

This, since the degradation is flat near 0 and starts increasing and becoming more acute with Young modulus E; hence, at E_C, D is the greatest and is equal to 1. (Figures 32 & 33)

Furthermore, we have (Figure 34):

$RUL(E_k) = E_C - E_k$ and it corresponds to a degradation $D(E_k)$,
And

$RUL(E_{k-1}) = E_C - E_{k-1}$ and it corresponds to a degradation $D(E_{k-1})$.

This implies that:

$$\begin{aligned} P_r(E_k) &= \psi \times [D(E_k) - D(E_{k-1})] \\ &= \psi \times \{D[E_C - RUL(E_k)] - D[E_C - RUL(E_{k-1})]\} \end{aligned} \quad (32)$$

Figures 31 to 34 illustrate the application of the new prognostic model to Young modulus.

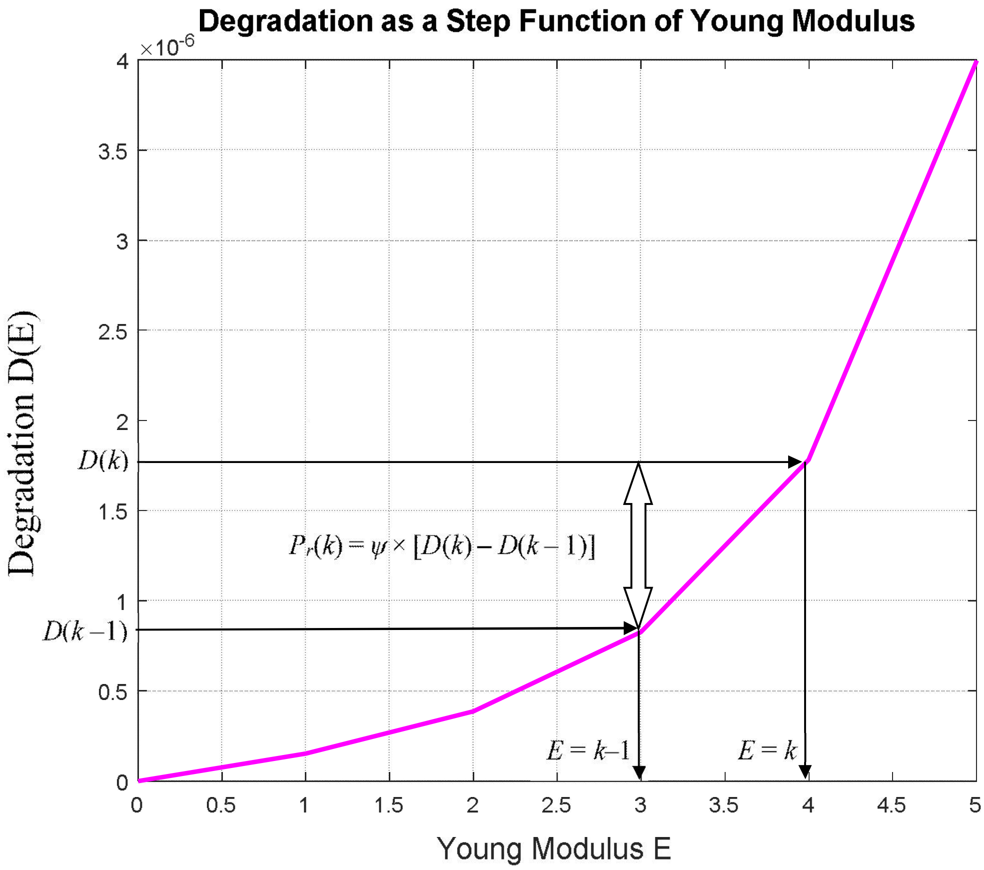


Figure 31: P_r, degradation, and the *CDF* step function

Figure 31 shows the real failure probability $P_r(E)$ as a function of the random system degradation step *CDF* in terms of the simulation Young modulus E.

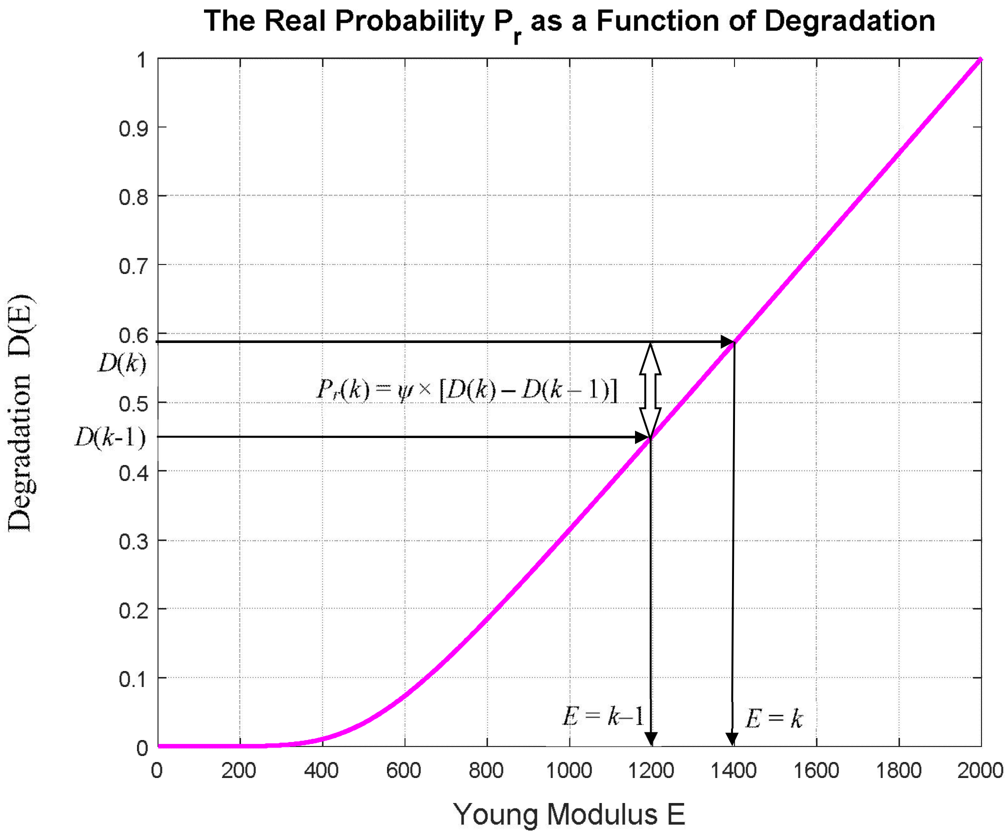


Figure 32: P_r as a function of Degradation $D(E)$

Figure 32 shows the real failure probability $P_r(E)$ as a function of the random system degradation in terms of the simulation Young modulus E.

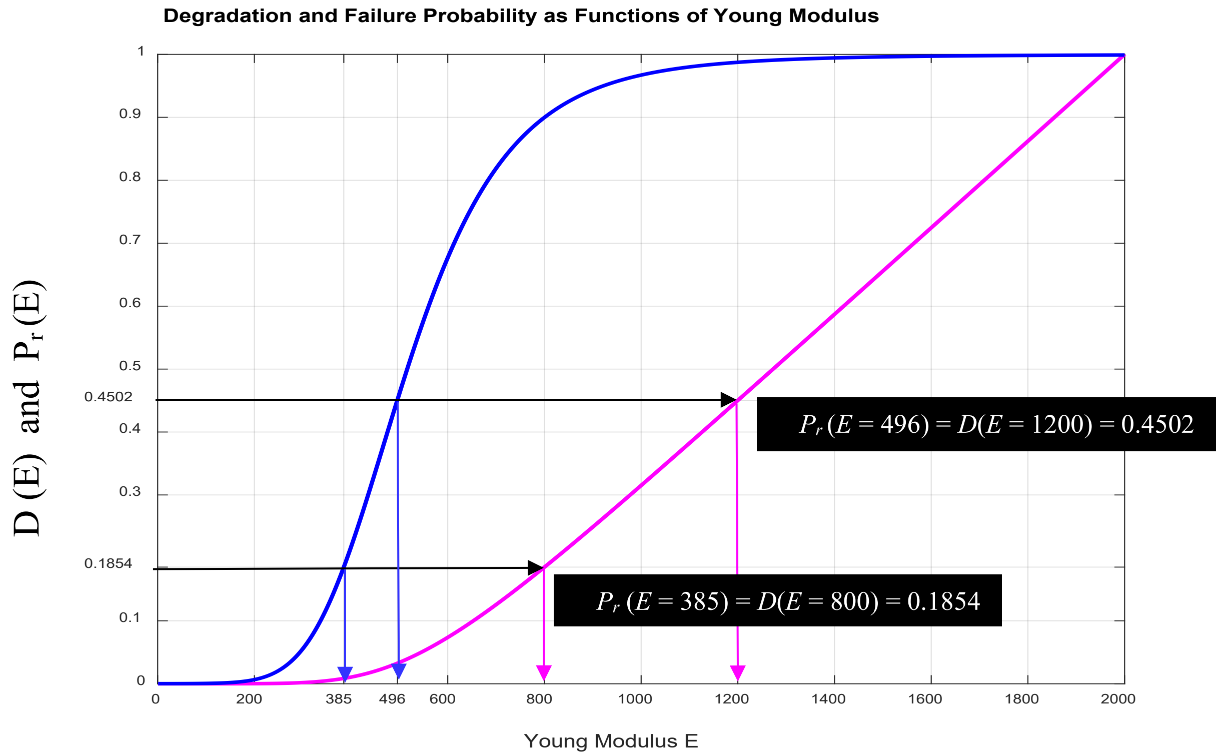


Figure 33: Degradation and P_r

Figure 33 shows the real failure probability $P_r(E)$ and the random system degradation $D(E)$ as functions of the simulationYoung modulus E.

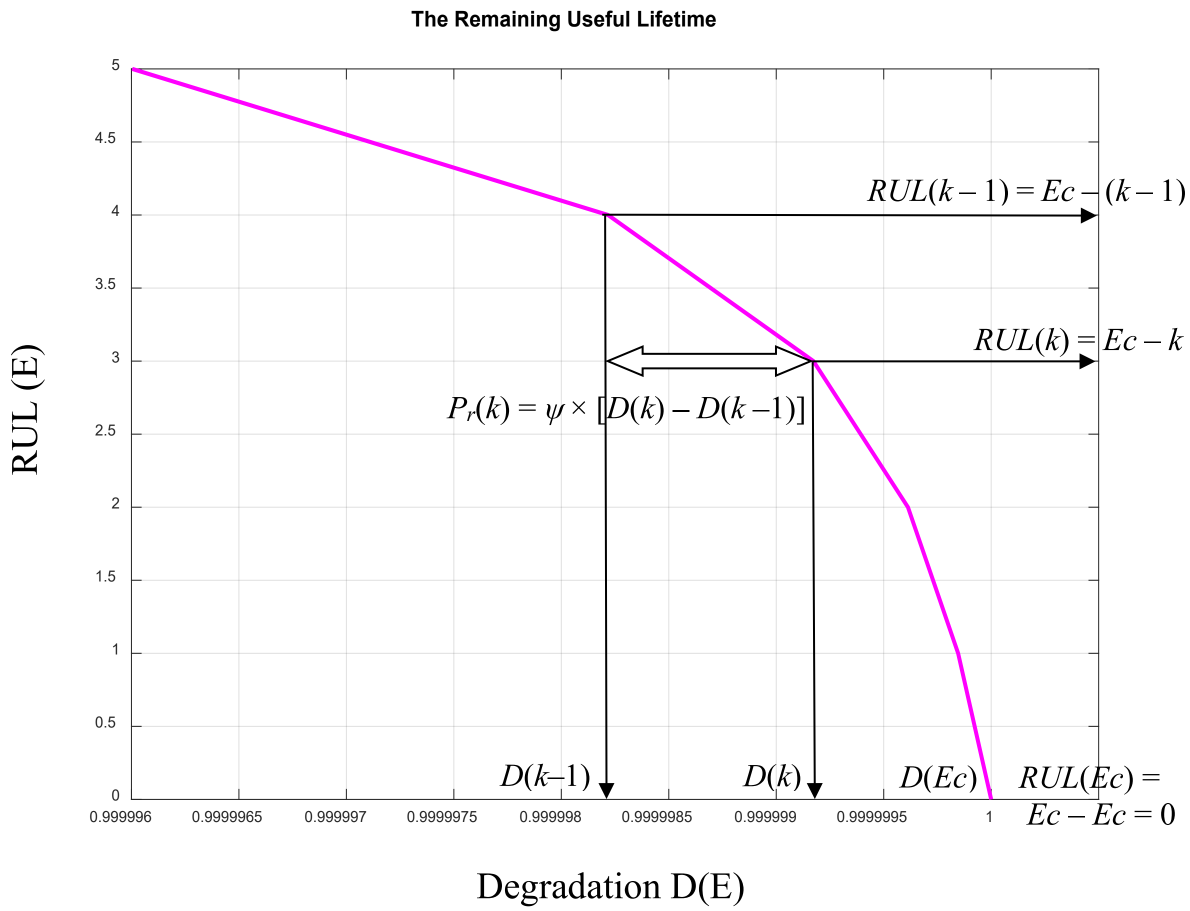


Figure 34: P_r, D, and RUL

Figure 34 shows the real failure probability $P_r(E)$ as a function of the random system degradation $D(E)$ and the random system $RUL(E)$ in terms of the simulation Young modulus E.

Moreover, we have from FORM:

$$G(E_k) = E_C - E_k = RUL(E_k) \tag{33}$$

where $G(E_k)$ is the limit state of lifetime.

E_C: is the fixed lifetime of the system which follows a normal probability distribution $\mathcal{N}(516, 1)$
E_k: is an arbitrary instant that varies from 0 to E_C and which follows a normal probability distribution $\mathcal{N}(E_k, 0.051 \times E_k)$

When $G(E_k)$ is zero or negative then we have a case of $E_k \geq E_C$ that means that we have a system failure that cannot live until the instant E_k. In the other case where $E_k < E_C$, the system can live above the instant E_k and we have a case of success.

The reliability index $\beta = -\Phi^{-1}\left[P_r(E_k)\right]$ where $P_r(E_k)$ is the cumulative distribution probability function and Φ is the normal cumulative distribution function. Hence, Φ^{-1} is the inverse of Φ and

$$P_r(E_k) = \Phi(-\beta) \tag{34}$$

The failure cumulative distribution functioncomputed by the FORM procedure is:

$$P_r(E_k) = P_{rob}\{G(E_k) \leq 0\} = P_{rob}\{E_k \geq E_C\} = \Phi(-\beta) \tag{35}$$

It corresponds in system prognostic to: $P_{rob}\{RUL(E_k) \leq 0\}$

Therefore, the survival cumulative distribution function computed by the FORM procedure is:

$$P_{rob}\{G(E_k) > 0\} = P_{rob}\{E_k < E_C\} = 1 - P_r(E_k) = P_m(E_k)/i = 1 - \Phi(-\beta) \tag{36}$$

It corresponds in system prognostic to: $P_{rob}\{RUL(E_k) > 0\}$

In *CPP*, the real part of probability is taken here $P_r(E_k)$. As we make the instant $E_k$vary between 0 and E_C, then $P_r(E_k) = F(E_k)$ varies between 0 and 1 as shown in Figures 35 and 36. Moreover, Figure 37 illustrates the system failure PDF which is $P_r(E_k)/\psi = \Phi(-\beta)/\psi$.

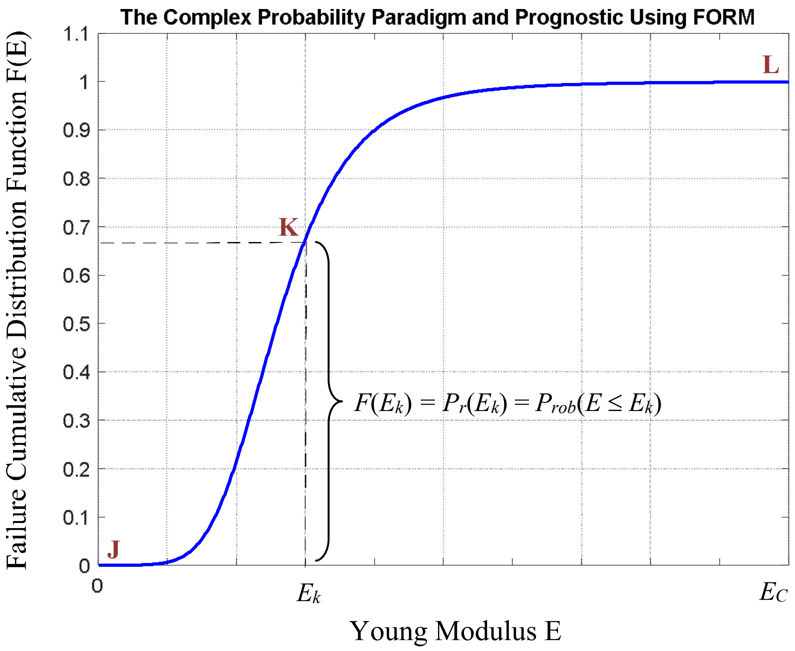


Figure 35: General system failure CDF

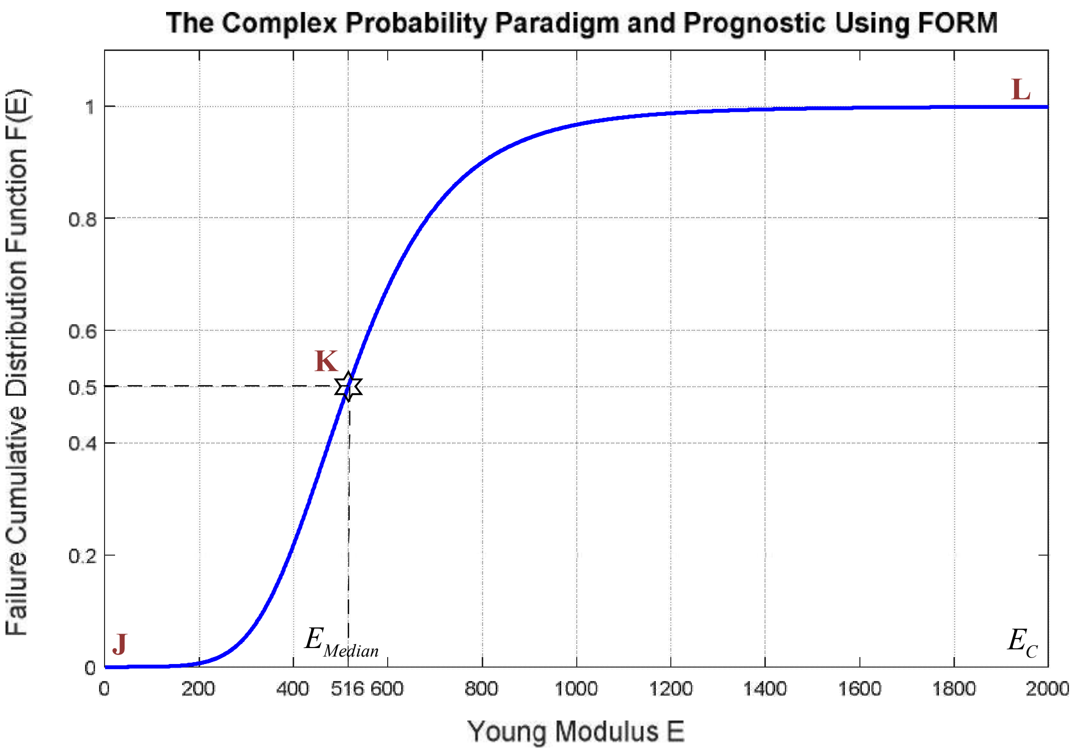


Figure 36: System failure CDF for the current simulation

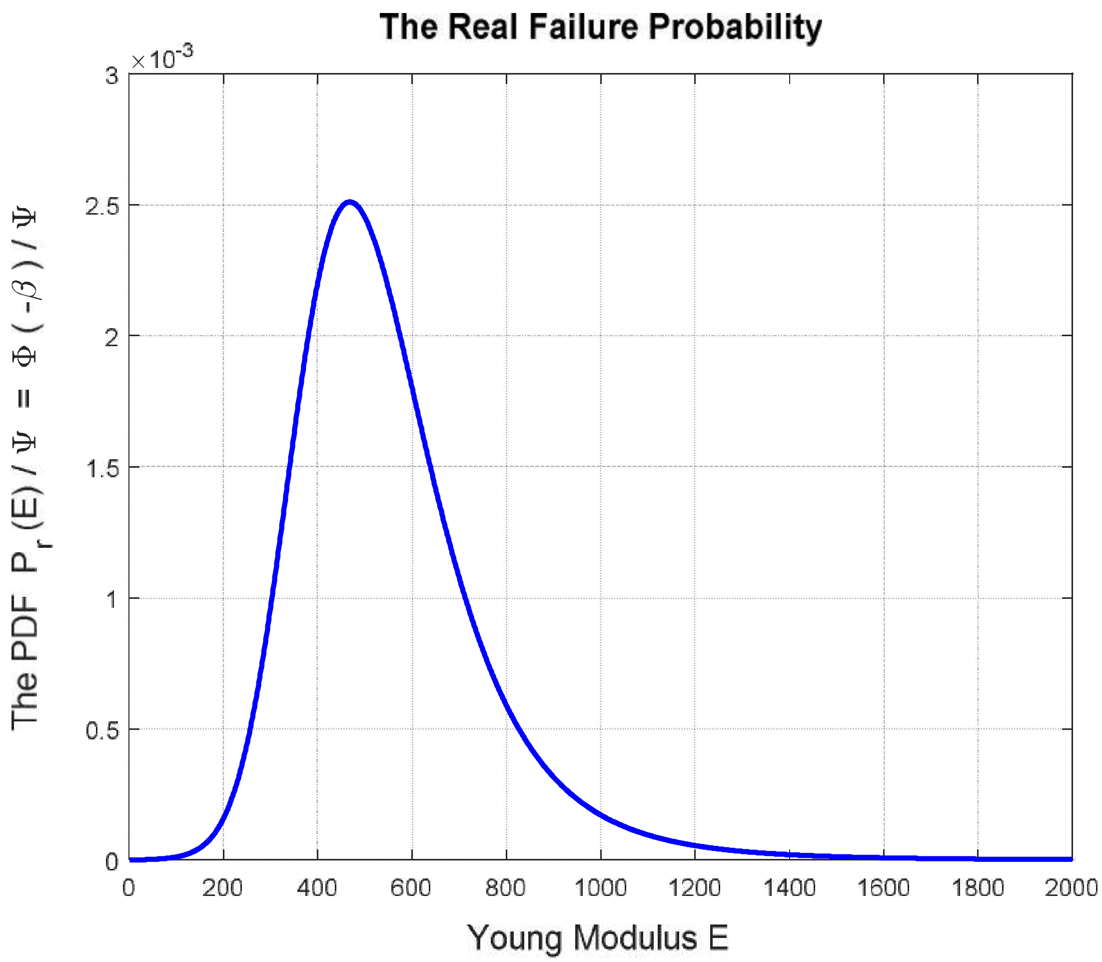


Figure 37: System failure PDF for the current simulation

7.1 Mathematical Analysis: A Numerical Example

Let $\overline{E}$ be the mean value of Young modulus E in the simulations and let σ_E be the standard deviation of E then the coefficient of variation in the simulations is:

$$c_v = \frac{\sigma_E}{\overline{E}} = 0.051 .$$

We can infer from the simulations that the median of the distribution is $E = E_{Median} = 516 \text{ Ksi}$ and is denoted accordingly E_{Md}. We know from classical statistical theory that it is the value that divides the distributioninto two equal parts. From the simulations we have the following results:

$$P_{rob}[-\infty < E \le (E_{Md} = 516 \text{ Ksi})] = 0.5 \text{, and } P_{rob}[(E_{Md} = 516 \text{ Ksi}) \le E < +\infty] = 0.5 ,$$

$$\text{And } P_{rob}[-\infty < E \le 663 \text{ Ksi}] = 0.7748 \text{, and } P_{rob}[663 \text{ Ksi} \le E < +\infty] = 0.2252 ,$$

As well

$$P_{rob}[E \le 0 \text{ Ksi}] = 0 .$$

Moreover,

$dF = f_E(E)\,dE = \frac{1}{\sqrt{2\pi}\sigma_E}\exp\left[-\frac{1}{2}\left(\frac{E-\bar{E}}{\sigma_E}\right)^2\right]dE$, which is the normal distribution corresponding to Young modulus E with mean equal to $\bar{E}$ and standard deviation equal to σ_E.

$\Rightarrow dF = f_E(u)\,du = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-u^2}{2}\right)du$, where $u = \frac{E-\bar{E}}{\sigma_E}$, which is the standard normal distribution corresponding to Young modulus E with $\bar{u} = 0$ and $\sigma_u = 1$.

Therefore,

$$\Phi(u_E) = \int_{-\infty}^{u_E} \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-u^2}{2}\right)du = P_{rob}[u \le u_E].$$

Note that:

$$\int_{-\infty}^{+\infty} dF = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_E}\exp\left[-\frac{1}{2}\left(\frac{E-\bar{E}}{\sigma_E}\right)^2\right]dE = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-u^2}{2}\right)du = 1$$

Now,in the real probability domain $\mathcal{R}$ we have:

$$\begin{aligned} P_{rob}[-\infty < E \le (E = 663)] &= F(E = 663) \\ &= P_r(E = 663) = \int_{-\infty}^{E=663} \frac{1}{\sqrt{2\pi}\times\sigma_E}\exp\left[-\frac{1}{2}\left(\frac{E-\bar{E}}{\sigma_E}\right)^2\right]dE \\ &= 0.7748 \end{aligned}$$

The correspondingcomplementary probability in the imaginary domain $\mathcal{M}$ is:

$$\begin{aligned} P_m(E = 663) &= i\times[1 - P_r(E = 663)] = i\times P_{rob}[E > 663] = i\times[1 - F(E = 663)] \\ &= i\times \int_{E=663}^{+\infty} \frac{1}{\sqrt{2\pi}\times\sigma_E}\exp\left[-\frac{1}{2}\left(\frac{E-\bar{E}}{\sigma_E}\right)^2\right]dE \\ &= i\times 0.2252 \end{aligned}$$

If we compute the norm of the complex number or vector $Z = P_r + P_m$ we have:

$$|Z|^2 = P_r^2 + (P_m / i)^2 = P_r^2 + (1 - P_r)^2 = 1 + 2P_r(P_r - 1) = 1 - 2P_r(1 - P_r);$$

This implies that:

$$1=|Z|^2+2P_r(1-P_r)=|Z|^2-2i^2\times P_r(1-P_r)=|Z|^2-2i\times P_r\times[i(1-P_r)]=|Z|^2-2iP_rP_m$$
$$=P_r^2+(P_m/i)^2-2iP_rP_m=P_r^2+(P_m/i)^2+2P_r(P_m/i)=(P_r+P_m/i)^2=Pc^2\Leftrightarrow Pc=1$$

since $i^2=-1\Leftrightarrow -i=\dfrac{1}{i}$

We note that:

$Z_E=P_r(E)+P_m(E)=\int_{-\infty}^{E}f_E(u)\,du+i\int_{E}^{+\infty}f_E(u)\,du$, written for short $\int_{-\infty}^{E}+i\int_{E}^{+\infty}$

and

$$Z_{E_{Md}}=P_r(E_{Md}=516)+P_m(E_{Md}=516)=\int_{-\infty}^{E_{Md}=516}f_E(u)\,du+i\times\int_{E_{Md}=516}^{+\infty}f_E(u)\,du=0.5+i\times 0.5$$

and

$$Z_{E=663}=P_r(E=663)+P_m(E=663)=\int_{-\infty}^{E=663}f_E(u)\,du+i\times\int_{E=663}^{+\infty}f_E(u)\,du=0.7748+i\times 0.2252.$$

We have also:

$$Pc_E^{\ 2}=[P_r(E)+P_m(E)/i]^2=\left(\int_{-\infty}^{E}+\frac{i\int_{E}^{+\infty}}{i}\right)^2=\left(\int_{-\infty}^{E}+\int_{E}^{+\infty}\right)^2=\left(\int_{-\infty}^{+\infty}\right)^2=1^2=1$$

And the chaotic factor is:

$$Chf_E=2i\times P_r(E)\times P_m(E)=2i\times\int_{-\infty}^{E}\times i\times\int_{E}^{+\infty}=-2\times\int_{-\infty}^{E}\times\left(1-\int_{-\infty}^{E}\right),\text{ where:}$$

$$Chf_E=0\ \text{if}\begin{cases}E\to-\infty, & \text{hence } P_r=0\\ E\to+\infty, & \text{hence } P_r=1\end{cases}$$

Moreover, the Magnitude of the chaotic factor is:

$$MChf_E=|Chf_E|=|2i\times P_r(E)\times P_m(E)|=\left|2i\times\int_{-\infty}^{E}\times i\times\int_{E}^{+\infty}\right|=\left|-2\times\int_{-\infty}^{E}\times\left(1-\int_{-\infty}^{E}\right)\right|$$
$$=2\times\int_{-\infty}^{E}\times\left(1-\int_{-\infty}^{E}\right)$$

where:

$$MChf_E=0\ \text{if}\begin{cases}E\to-\infty, & \text{hence } P_r=0\\ E\to+\infty, & \text{hence } P_r=1\end{cases}$$

Additionally,

$$DOK_E = |Z_E|^2 = P_r(E)^2 + [P_m(E)/i]^2 = \left[\int_{-\infty}^{E}\right]^2 + \left[\int_{E}^{+\infty}\right]^2 = \left[\int_{-\infty}^{E}\right]^2 + \left[1 - \int_{-\infty}^{E}\right]^2, \text{ where:}$$

$$DOK_E = 1 \text{ if } \begin{cases} E \to -\infty, & \text{hence } P_r = 0 \\ E \to +\infty, & \text{hence } P_r = 1 \end{cases}$$

Consequently, we state that:

$$Pc_E^{\;2} = |Z_E|^2 - 2i.P_r(E).P_m(E) = \text{Degree of our knowledge} - \text{Chaotic factor} = 1.$$

And if $Chf_E = 0 \Rightarrow |Z_E|^2 = DOK = 1$, in other words, if the chaotic factor is zero then the degree of our knowledge is 1 = 100%.

In addition, we say that:

$$Pc_E^{\;2} = \text{Degree of our knowledge} + \text{Magnitude of the Chaotic factor} = 1.$$

And if $MChf_E = 0 \Rightarrow |Z_E|^2 = DOK = 1$, in other words, if the magnitude of the chaotic factor is zero then the degree of our knowledge is 1 = 100%.

Numerically, we write:

$$|Z_{E_{Md}=516}|^2 = DOK_{E_{Md}=516} = (0.5)^2 + (0.5)^2 = 0.25 + 0.25 = 0.5$$

$$\Rightarrow |Z_{E_{Md}=516}| = 0.707106781 \Rightarrow Chf_{E_{Md}=516} \neq 0,$$

hence, $Chf_{E_{Md}=516} = 0.5 - 1 = -0.5$,

and $MChf_{E_{Md}=516} = |Chf_{E_{Md}=516}| = |-0.5| = 0.5$.

Additionally,

$$|Z_{E=663}|^2 = (0.7748)^2 + (0.2252)^2 = 0.60031504 + 0.05071504 = 0.65103008$$

$$\Rightarrow |Z_{E=663}| = 0.80686435 \Rightarrow Chf_{E=663} \neq 0, \quad \text{Notice that: } \frac{1}{2} \le |Z_E|^2 = DOK \le 1$$

Hence, $Chf_{E=663} = 0.65103008 - 1 = -0.34896992$, Notice that: $-\frac{1}{2} \le Chf_E \le 0$

and $MChf_{E=663} = |Chf_{E=663}| = |-0.34896992| = 0.34896992$, Notice that: $0 \le MChf_E \le \frac{1}{2}$

Accordingly, we can say that:

The degree of our knowledge $DOK_{E=663} = |Z_{E=663}|^2 = 0.65103008$, the chaotic factor $Chf_{E=663} = -0.34896992$, and the magnitude of the chaotic factor $MChf_{E=663} = 0.34896992$.

What is interesting here is thus we have quantified both the degree of our knowledge and the chaotic factor of the stochastic event as well as the corresponding magnitude of the chaotic factor.

Notice that:

$DOK_{E=663} - Chf_{E=663} = 0.65103008 - (-0.34896992) = 0.65103008 + 0.34896992 = 1 = Pc_{E=663}$
and $DOK_{E=663} + MChf_{E=663} = 0.65103008 + 0.34896992 = 1 = Pc_{E=663}$

Also
$DOK_{E_{Md}=516} - Chf_{E_{Md}=516} = 0.5 - (-0.5) = 0.5 + 0.5 = 1 = Pc_{E_{Md}=516}$
and $DOK_{E_{Md}=516} + MChf_{E_{Md}=516} = 0.5 + 0.5 = 1 = Pc_{E_{Md}=516}$

Conversely, if we assume that:

$$Chf_E = 0 \Rightarrow MChf_E = 0 \Rightarrow |Z_E|^2 = DOK_E = 1 \Rightarrow P_r(E)^2 + [P_m(E)/i]^2 = 1$$

$$\Rightarrow 2P_r(E)[1 - P_r(E)] = 0 \Rightarrow \begin{cases} P_r(E) = 0 \\ \text{or} \\ P_r(E) = 1 \end{cases} \Rightarrow \begin{cases} E \to -\infty \\ \text{or} \\ E \to +\infty \end{cases}$$

And if $Chf_E = -\frac{1}{2} \Rightarrow MChf_E = \frac{1}{2} \Rightarrow |Z_E|^2 = DOK_E = \frac{1}{2} \Rightarrow E = E_{Md} = 516\text{ Ksi}$,
And

if $Chf_E = -0.34896992 \Rightarrow MChf_E = 0.34896992 \Rightarrow |Z_E|^2 = DOK_E = 0.65103008$
$\Rightarrow E = 663$ Ksi.

Now if E increases to become $= 1000$ Ksi then both $|Z_E|^2 = DOK$ and Chf_E increase, and $MChf_E$ decreases.

Therefore, we can infer that:

$$\lim_{E\to\pm\infty}(Chf_E) = 0,\ \lim_{E\to\pm\infty}(MChf_E) = 0, \text{ and } \lim_{E\to\pm\infty}\left(|Z_E|^2 = DOK_E\right) = 1$$

where we have always:

$$Pc_E^{\ 2} = |Z_E|^2 - Chf_E = DOK_E - Chf_E = |Z_E|^2 + MChf_E = DOK_E + MChf_E = 1,$$

for every value of E in the real set of numbers.

Figure 38 illustrates all the novel prognostic model functions when applied to Young modulus and proves all the mathematical derivations. We have computed and ploted for this set of $P_r(E_k)$ all the *CPP* parameters and components and which are: $Chf(E_k)$, $MChf(E_k)$, $DOK(E_k)$, $Pc(E_k)$, $P_m(E_k)/i$, $D(E_k)$, and $P_{rob}[RUL(E_k)]$.

We note from the figure that the *DOK* is maximum (*DOK* = 1) when absolute value of *Chf* which is *MChf* is minimum (*MChf* = 0) (points J & L), that means when the magnitude of the chaotic factor (*MChf*) diminishes our certain knowledge (*DOK*) grows. Afterward, *MChf* starts to grow during the functioning due to the environment and intrinsic conditions thus leading to a diminution in *DOK* until they both reach 0.5 at $E_k = E_{Md} = 516$ (point K). The real cumulative failure probability P_r and the real cumulative complementary survival probability P_m/i will meet with *DOK* and *MChf* also at the point (516, 0.5) (point K). The point K' is the point corresponding to K and which is (1273, 0.5). K' is the point where the degradation $D(E_k)$ and $P_{rob}[RUL(E_k)]$ meet. With the increase of E_k, the *Chf* and *MChf* return to zero and the *DOK* returns to 1 where we reach total damage ($D = 1$) and hence the total certain failure of the system ($P_r = 1$) (point L). At this last point the failure here is definite, $P_r(E_C = 2000) = 1$ and $RUL(E_C = 2000) = E_C - E_C = 0$ with $Pc(E_C = 2000) = 1$, so the logical consequence of the value *DOK* = 1 ensues.

We note that the point K corresponding to $E_{Md} \neq \overline{E} \neq E_{Mode}$ is not at the middle of the simulation since the probability of failure distribution evaluated by FORM is not symmetric. Therefore, the corresponding graphs are skewed to the right or positively skewed.

Furthermore, at each instant E_k, we can predict the remaining useful lifetime $RUL(E_k)$ with certainty in the complex probability set $\mathcal{C}$ with *Pc* preserved as equal to one through an unceasing compensation between *DOK* and *Chf*. This compensation is from instant $E_k = 0$ where $D(E_k) = 0$ until the failure instant E_C where $D(E_C) = 1$.

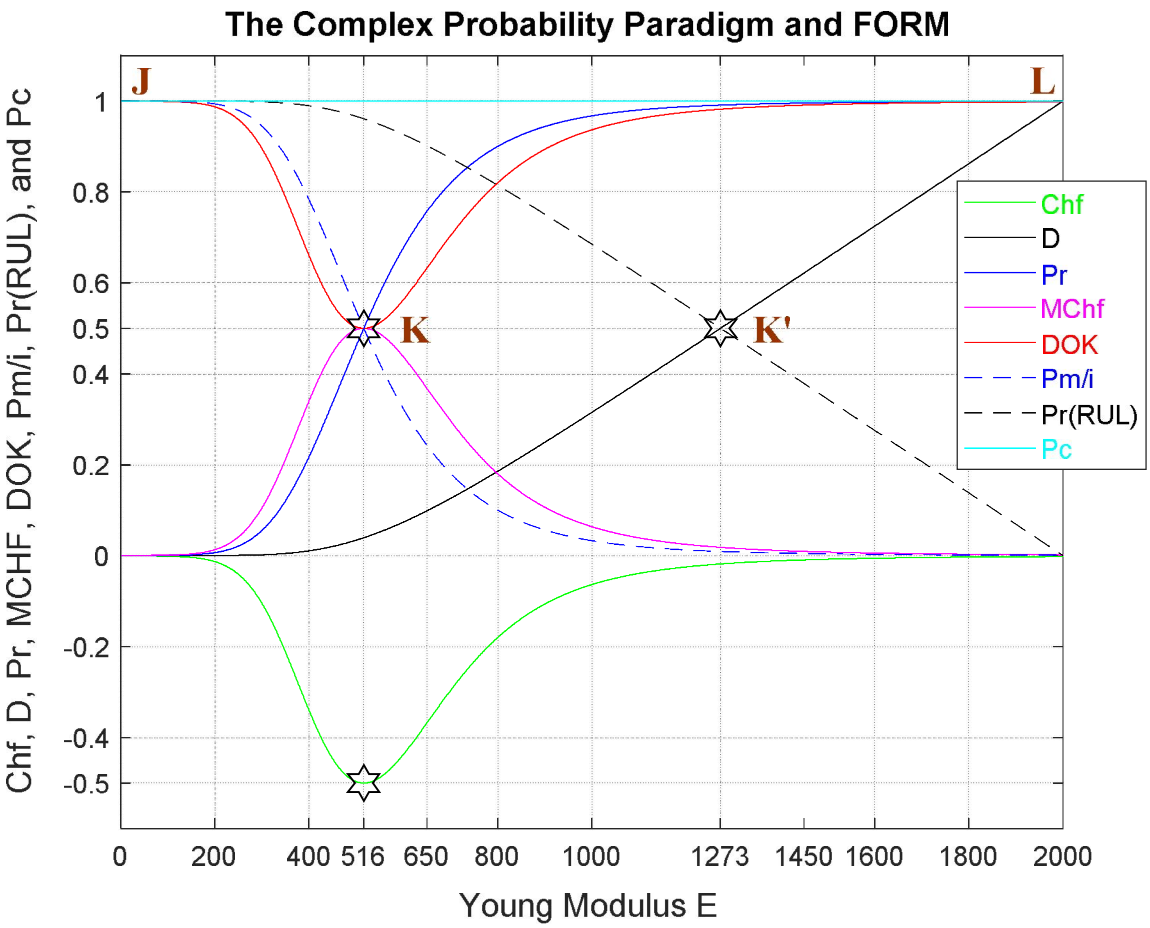


Figure 38: *CPP* and FORM applied to prognostic and to Young modulus.

7.2 The Complex Probability Cubes

In the first cube (Figure 39), the simulation of *Chf* and *DOK* as functions of each other and of the simulation of Young modulus E can be seen. The line in cyan is the projection of $Pc^2(E) = DOK(E) - Chf(E) = 1 = Pc(E)$ on the plane $E = 0$ Ksi. This line starts at the point J ($DOK = 1$, $Chf = 0$) when $E = 0$ Ksi, reaches the point ($DOK = 0.5$, $Chf = -0.5$) when $E = E_{Md} = 516$ Ksi, and returns at the end to J ($DOK = 1$, $Chf = 0$) when $E = E_C = 2000$ Ksi. The other curves are the graphs of $Chf(E)$ (pink, blue, green) and $DOK(E)$ (red) in different planes. Notice that they all have a minimum at the point K ($DOK = 0.5$, $Chf = -0.5$, $E = E_{Md} = 516$ Ksi). The point L corresponds to ($DOK = 1$, $Chf = 0$, $E = E_C = 2000$ Ksi). The three points J, K, L are similar to those in the previous figures.

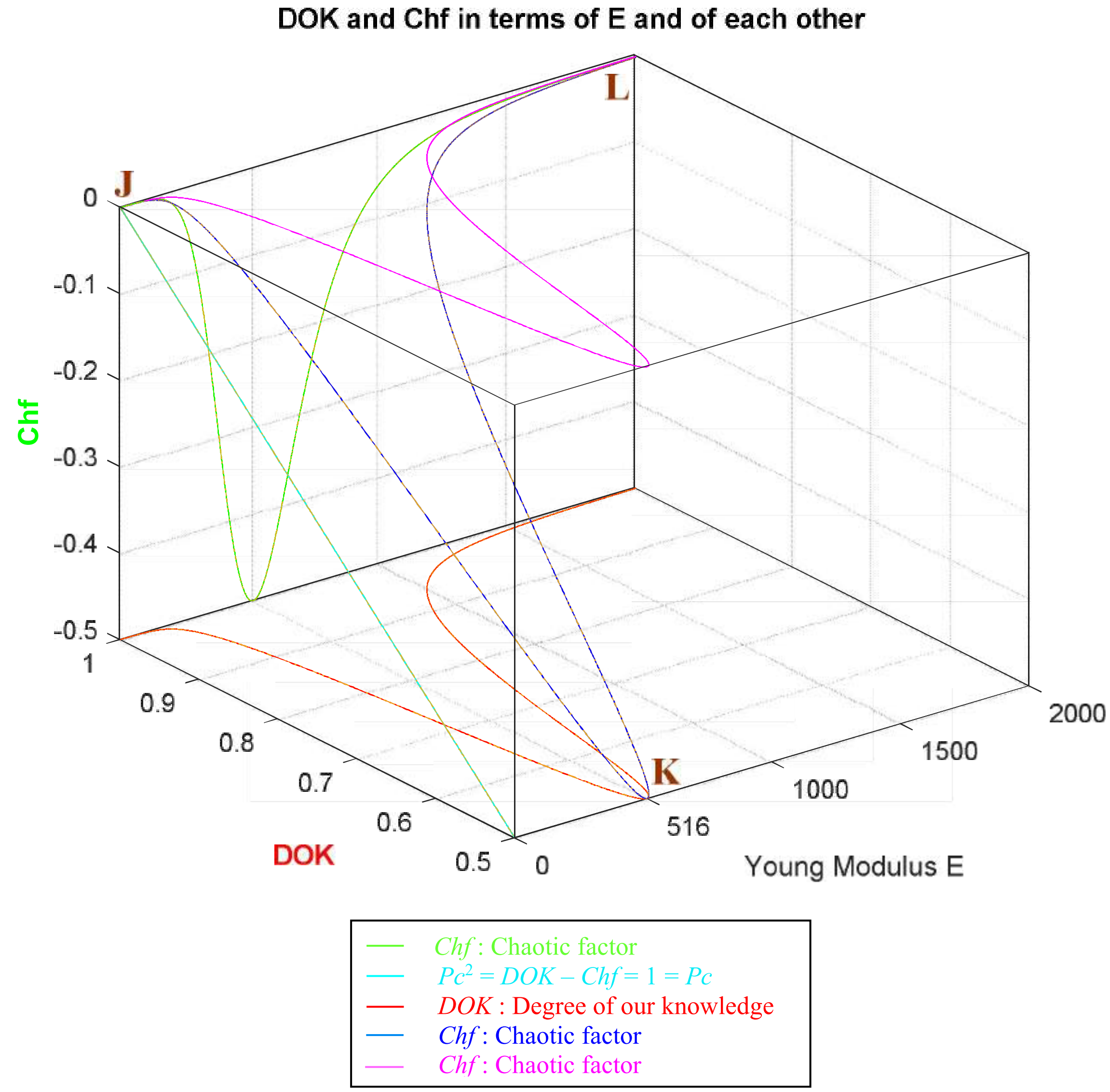


Figure 39: *DOK* and *Chf* in terms of *E* and of each other

In the second cube (Figure 40), we can notice the simulation of the failure probability $P_r(E)$ and its complementary real probability $P_m/i(E)$ in terms of the simulation of Young modulus$E$. The line in cyan is the projection of $Pc^2(E) = P_r(E) + P_m/i(E) = 1 = Pc(E)$ on the plane $E = 0$Ksi. This line starts at the point ($P_r = 0$, $P_m/i= 1$) and ends at the point ($P_r = 1$, $P_m/i = 0$). The red curve represents $P_r(E)$ in the plane $P_r(E) = P_m/i(E)$. This curve starts at the point J ($P_r = 0$, $P_m/i= 1$, $E = 0$Ksi), reaches the point K ($P_r = 0.5$, $P_m/i= 0.5$, $E= E_{Md}$ =516 Ksi), and gets at the end to L ($P_r = 1$, $P_m/i= 0$, $E = E_C = 2000$ Ksi). The blue curve represents $P_m/i(E)$ in the plane $P_r(E) + P_m/i(E) = 1$. Notice now the importance of the point K which is the intersection of the red and blue curves at $E = E_{Md}$ =516 Ksi and when $P_r(E) = P_m/i(E) = 0.5$.The three points J, K, L are similar to those in the previous figures.

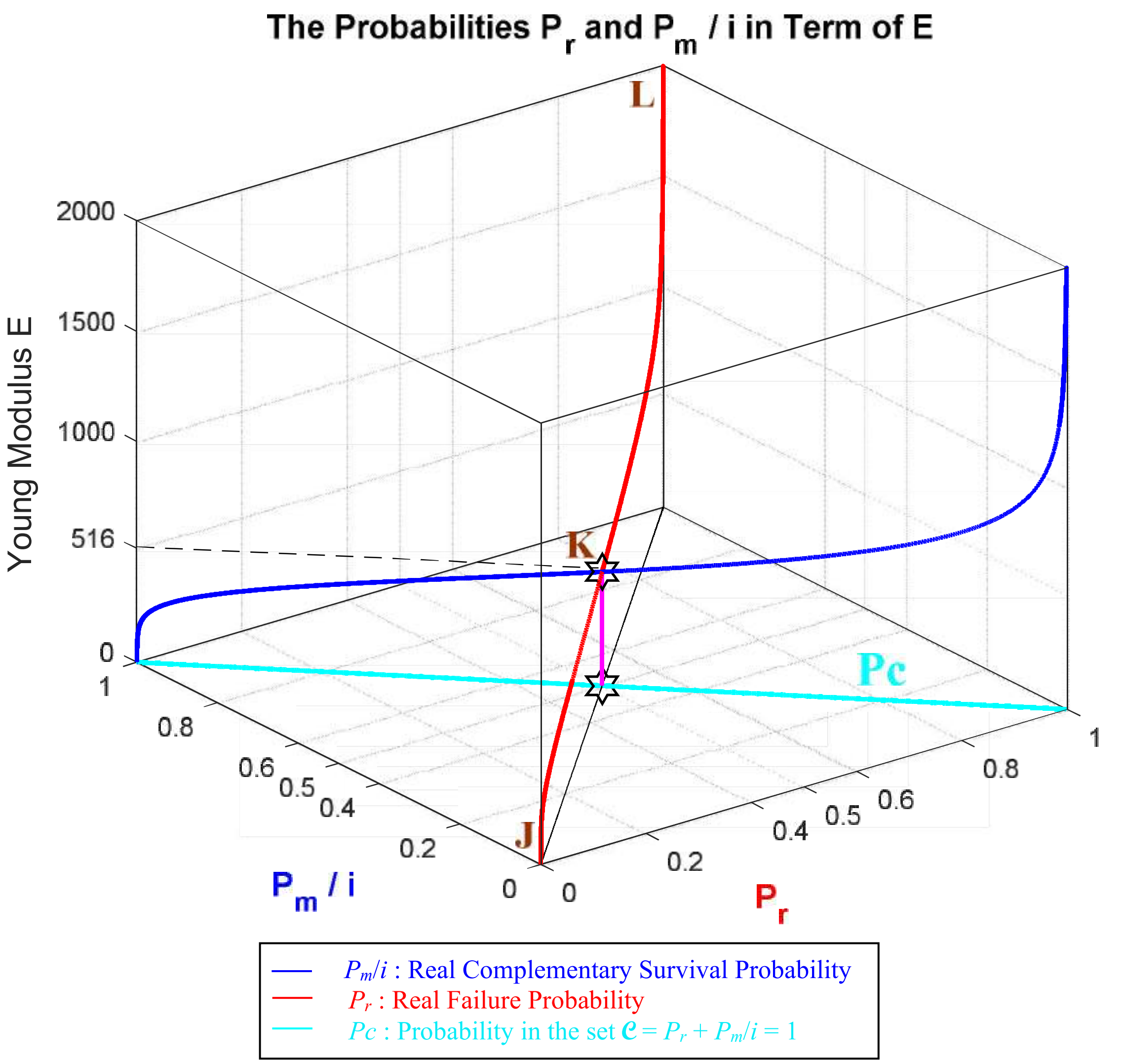


Figure 40: P_r and P_m/iin terms of E and of each other

In the third cube (Figure 41), we can notice the simulation of the complex random vector $Z(E)$ in $\mathcal{C}$ as a function of the real failure probability $P_r(E) = \text{Re}(Z)$ in $\mathcal{R}$ and of its complementary imaginary probability $P_m(E) = i\times\text{Im}(Z)$ in $\mathcal{M}$, and this in terms of the simulation of Young modulus E. The red curve represents $P_r(E)$ in the plane $P_m(E) = 0$ and the blue curve represents $P_m(E)$ in the plane $P_r(E) = 0$. The green curve represents the complex probability vector $Z(E) = P_r(E) + P_m(E) = \text{Re}(Z) + i\times\text{Im}(Z)$ in the plane $P_r(E) =iP_m(E) + 1$. The curve of $Z(E)$ starts at the point J ($P_r = 0$, $P_m= i$, $E = 0$ Ksi) and ends at the point L ($P_r = 1$, $P_m = 0$, $E = E_C = 2000$ Ksi). The line in cyan is $P_r(0) =iP_m(0) + 1$ and it is the projection of the $Z(E)$ curve on the complex probability plane whose equation is $E = 0$ Ksi. This projected line starts at the point J ($P_r = 0$,$P_m= i$, $E = 0$ Ksi) and ends at the point ($P_r = 1$, $P_m = 0$, $E = 0$ Ksi). Notice the importance of the point K corresponding to $E = E_{Md}$ =516Ksi and when $P_r = 0.5$ and $P_m = 0.5i$. The three points J, K, L are similar to those in the previous figures.

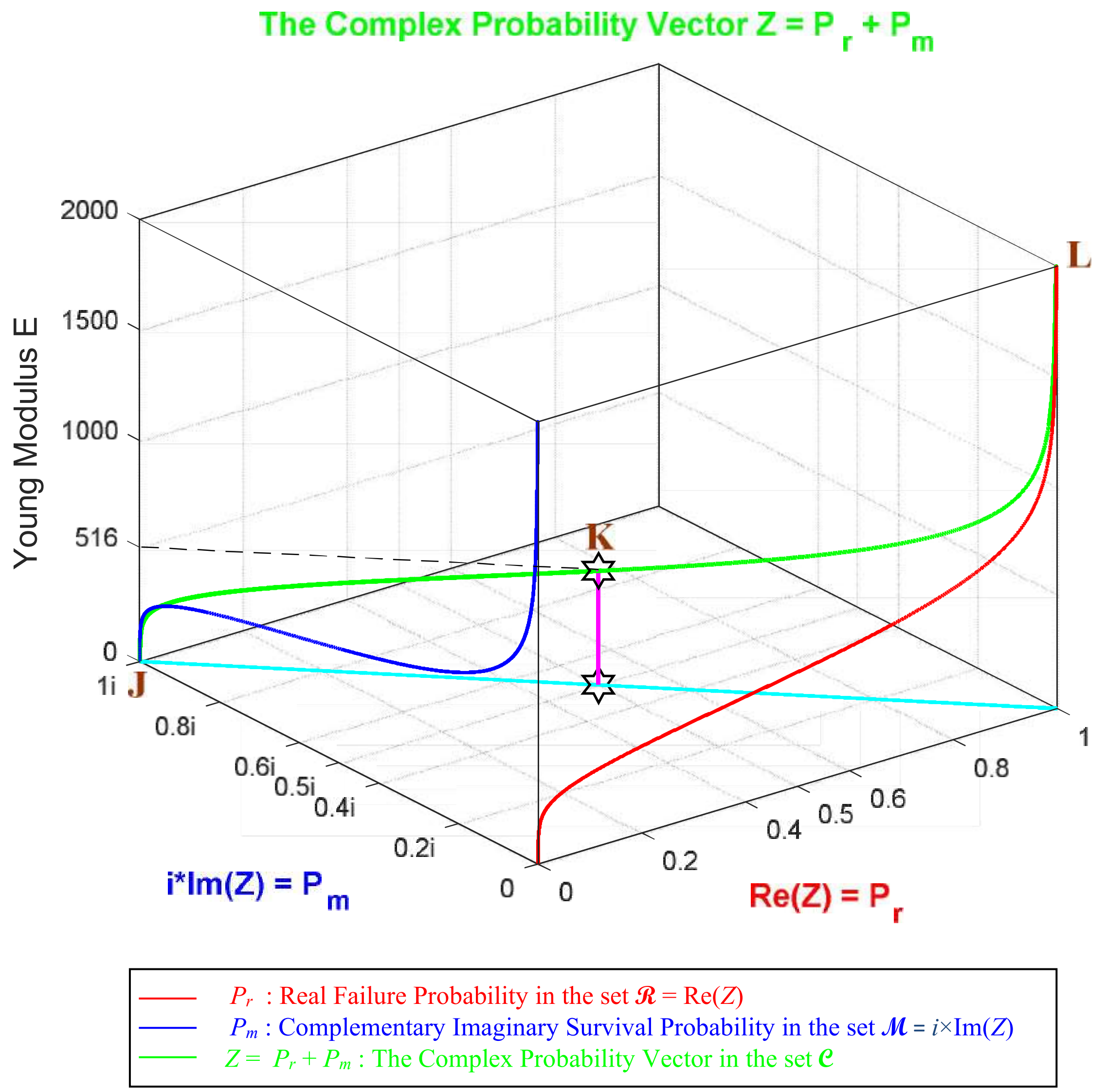
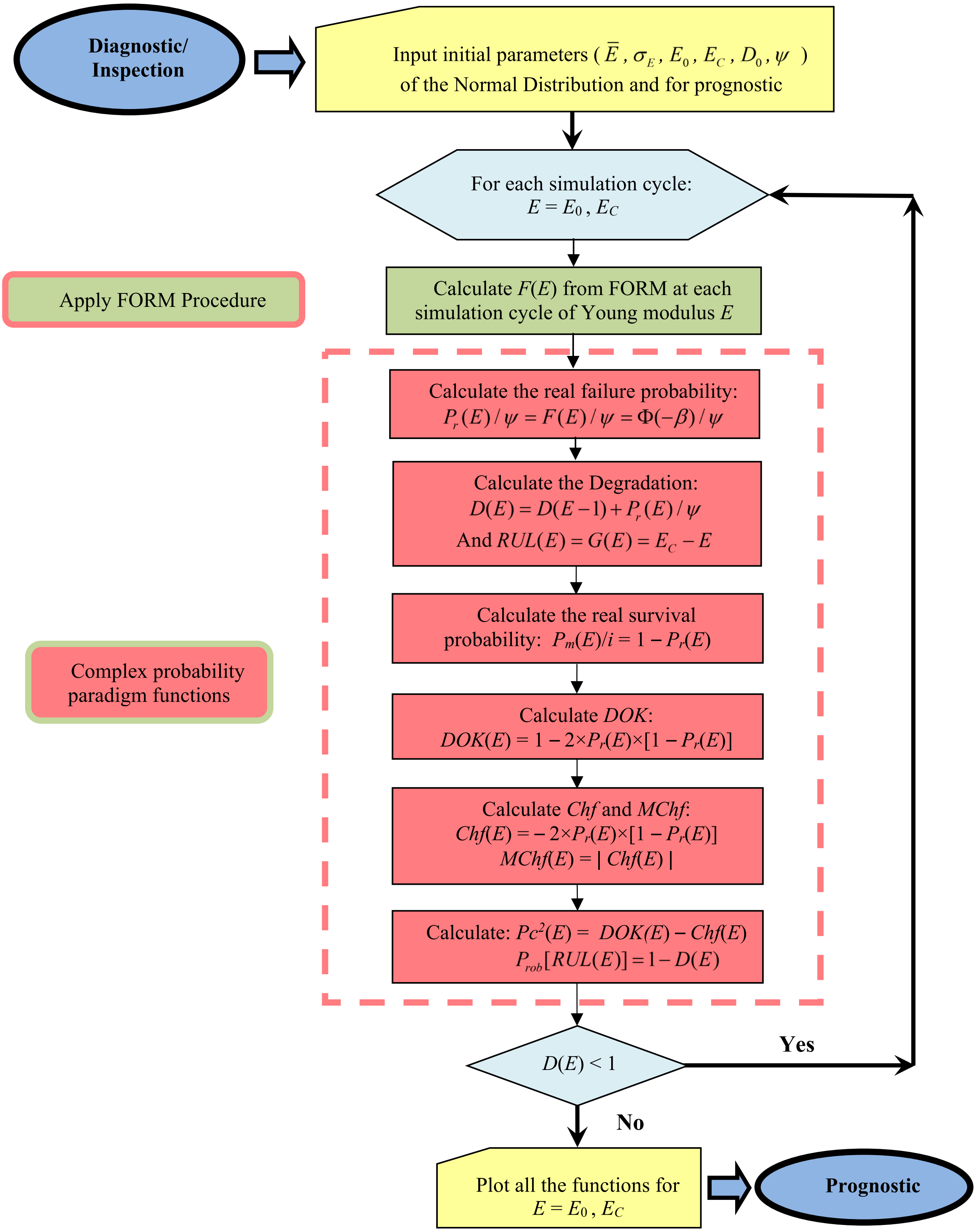


Figure 41: The Complex Probability Vector Z in terms of E

7.3 Flowchart of CPP Applied to Young Modulus Prognostic

The following flowchart summarizes all the explained procedures of the proposed complex probability prognostic model:



VIII. Final Analysis: Explanation and the General Prognostic Equations

In this section, all the obtained data and achieved simulations will be interpreted, a final analysis will be done, and the novel general prognostic equations will be presented. A detailed discussion of all the previous figures will be executed to understand the results.

Firstly, probability theory embodied by the *CDF* $F(t)$ calculated and made available by FORM was connected to prognostic materialized by the degradation $D(t)$ by supposing that $F(t_k) = \psi \times [D(t_k) - D(t_{k-1})] = \Phi(-\beta)$ and the good reason for this postulation was presented. Subsequently, the deterministic $D(t)$ quantified from deterministic prognostic converts to a nondeterministic function. Consequently, the deterministic and discrete variable of simulation cycles time t converts to a stochastic and discrete variable. Thus, the resultant of all the factors influencing the system which was deterministic converts to a stochastic resultant because $D(t)$ quantifies now the stochastic degradation of the system in terms of the random cycles time t. Accordingly, all the parameters exact values of the $D(t)$ expression become now mean values of the stochastic factors influencing the system and are embodied by *PDF*s as functions of the stochastic variable of simulation cycles time t. As a matter of fact, this is the real-world case where randomness is omnipresent in one way or another. What we consider and judge as a deterministic phenomenon is nothing in reality but a simplification and an approximation of an actual chaotic and stochastic phenomenon and experiment due to the impact of a huge number of nondeterministic and deterministic forces and factors (a good illustration is a lottery machine).

Subsequently, an updated follow-up of the stochastic degradation behavior with time or cycle number, and which is subject to chaotic and non-chaotic effects, is done by the quantity $P_r(t_k)/\psi_j$ due to its definition that embodies the jumps in the stochastic degradation $D(t)$. Henceforth,

$$P_r(t_k) = \psi \times [D(t_k) - D(t_{k-1})] = \Phi(-\beta),$$

Referring to the theory of classical probability, this converts $P_r(t_k)/\psi = \Phi(-\beta)/\psi$ to become the system failure probability at $t = t_k$, with $0 \le P_r(t_k)/\psi \le 1$ and $\sum_{t=t_0}^{t=t_C} P_r(t)/\psi = \sum_{t=t_0}^{t=t_C} \Phi(-\beta)/\psi =$ [sum of all the jumps in D from t_0 to t_C] $= D_C = 1$, similar to any probability density function (*PDF*).

Additionally, in the simulations a constant and extremely minor increases in t have been considered and which yield extremely minor increases in D and consequently in $P_r(t_k)/\psi$. So, we have multiplied those extremely minor jumps in D by a simulation magnifying factor that we called ψ. Note that $1/\psi$ is a normalizing constant that is utilized to transform $P_r(t_k)$ function to a probability density function with a total probability equal to one. $1/\psi$ is a function of the simulation mode and conditions and it depends on the parameters in the degradation equation and in the FORM algorithm. We have from the simulations that ψ = 724.3113 and ψ = 1449.4 for Young modulus. So we get: if t tends to $t_0 = 0$ then $P_r(t_k)$ tends to 0, and if t tends to t_C then $P_r(t_k)$ tends to 1, so $0 \le P_r(t_k) \le 1$ and $\sum_{t=t_0}^{t=t_C} P_r(t) = \sum_{t=t_0}^{t=t_C} \Phi(-\beta) = \psi \times (D_C - D_0) = \psi \times (1-0) = \psi$ as $P_r(t_k) = \Phi(-\beta)$ is a *CDF* since $P_r(t_k)$ is cumulative, it is ψ times the probability of failure at $t =$

t_k. Hence, in the simulations, $P_r(t_k)$ is the cumulative probability that the system failure occurs at $t = t_k$ and is used accordingly to compute all the *CPP* parameters.

Therefore, $F(t_k) = P_{rob}(0 \le t \le t_k) = P_{rob}(t = 0$ or $t = 1$ or $t = 2$ or … or $t = t_k)$ = sum of all probabilities of failure between 0 and t_k = probability that failure will happen somewhere between 0 and t_k. So, if $t_k = 0$ then $P_{rob}(t \le 0) = 0 =$ probability that failure will happen at $t = 0$ and before. If $t_k = t_C$ then $P_{rob}(0 \le t \le t_C) = 1 =$ sum of all probabilities of failure between 0 and t_C = probability that failure will happen somewhere between 0 and t_C. If $t_k > t_C$ then $P_{rob}(t > t_C) = 1 =$ probability that failure will happen beyond t_C. We can observe that the probability of failure grows with the growth of the cycles time t_k until at the end it becomes equal to 1 when $t_k \ge t_C$.
Thus, If $t_0 = 0$ and $D(t_0) = 0$ therefore:

$$P_{rob}(0 \le t \le t_k) = \sum_{t=0}^{t=t_k} P_{rob}(t) = \sum_{t=0}^{t=t_k} P_r(t)/\psi = \sum_{t=0}^{t=t_k} \Phi(-\beta)/\psi$$

This implies that: $P_{rob}(0 \le t \le t_C) = \sum_{t=0}^{t=t_C} P_{rob}(t) = \sum_{t=0}^{t=t_C} P_r(t)/\psi = \sum_{t=0}^{t=t_C} \Phi(-\beta)/\psi = \psi/\psi = 1$ and

$$P_{rob}(t \le 0) = \sum_{t=0}^{t=0} P_{rob}(t) = \sum_{t=0}^{t=0} P_r(t)/\psi = P_r(0)/\psi = 0/\psi = 0$$

If $t_0 \ne 0$ and $D(t_0) \ne 0$ then the prognostic equation in the new model is:

$$P_{rob}(t_0 \le t \le t_k) = \sum_{t=t_0}^{t=t_k} P_{rob}(t) = \sum_{t=t_0}^{t=t_k} P_r(t)/\psi = \sum_{t=t_0}^{t=t_k} \Phi(-\beta)/\psi \tag{37}$$

with $P_r(t_0)/\psi = D_0$.

Moreover, since $P_r(t_k) = \psi[D(t_k) - D(t_{k-1})]$, this leads to the following recursive relation:

$$D(t_k) = D(t_{k-1}) + P_r(t_k)/\psi = D(t_{k-1}) + \Phi(-\beta)/\psi \;;\forall t_k : \; t_0 \le t_k \le t_C. \tag{38}$$

In the general prognostic case, if we possess the *PDF*of system failure then we can include it in the equations (37) and (38) above and computedegradation at any instant t_k and vice versa. Then, all the other model *CPP* functions (*Chf*, *MChf*, *DOK*, *Z*, P_r, P_m, P_m/i, *Pc*) will follow. This would be our new prognostic paradigm general equations:

$$P_{rob}(t_0 \le t \le t_k) = \sum_{t=t_0}^{t=t_k} P_{rob}(t) = \sum_{t=t_0}^{t=t_k} PDF_{failure}(t) \tag{39}$$

And the recursive relation:

$$D(t_k) = D(t_{k-1}) + PDF_{failure}(t_k) \tag{40}$$

with $PDF_{failure}(t_0) = D_0$.

It is crucial to indicate here that the $PDF_{failure}$ function of the system failure has all the mathematical characteristics and all the possible features of a probability density function whether it is a continuous or a discrete stochastic function and it can follow any imaginable probability distribution in condition only that it characterizes the failure function and the random degradation of the studied system whether it is a petrochemical pipe in the buried, unburied, or offshore case, or a vehicle suspension system in engineering, or any nondeterministic system under the effect of randomness and chaos. In fact, the function $PDF_{failure}$ inherits all the attributes and features of the failure system function and of the nondeterministic degradation.

Furthermore, by applying *CPP* to the system prognostic, and in all the simulations, we were successful in the original prognostic model to quantify in $\mathcal{R}$ (our real laboratory) both our chaos embodied by *Chf* and *MChf* and our certain knowledge embodied by *DOK*. These three parameters of *CPP* are evaluated and caused by the resultant of all the non-random (deterministic) and random (nondeterministic) aspects influencing the system. Knowing that, in the novel paradigm, the factors resultant effect on *RUL* and *D* is materialized by the jumps in their curves and is accordingly expressed and concretized in $\mathcal{R}$ by P_r and in $\mathcal{M}$ by P_m. As it was defined in *CPP*, $\mathcal{M}$ is an imaginary probability extension of the real probability set $\mathcal{R}$ and the complex probability set $\mathcal{C}$ is the sum of both probability sets; thus, $\mathcal{C} = \mathcal{R} + \mathcal{M}$. Because $P_m = i(1 - P_r)$ therefore it is the complementary probability of P_r in $\mathcal{M}$. Hence, if P_r is identified as the failure probability of the system in $\mathcal{R}$ at the simulation cycles time $t = t_k$, then P_m is identified as the corresponding probability in the set $\mathcal{M}$ that the system failure will not happen at the same simulation cycles time $t = t_k$. So, P_m is the associated probability in the set $\mathcal{M}$ of the system survival at $t = t_k$. It follows that, $P_m/i = 1 - P_r$ is the associated probability but in the set $\mathcal{R}$ of the system survival at the same simulation cycles time. Accordingly, we know that the sum in $\mathcal{R}$ of both complementary probabilities is surely 1 from classical probability theory. This sum is nothing but *Pc* which is equal to $P_r + P_m/i = P_r + (1 - P_r) = 1$ always. The sum in $\mathcal{C}$ of both complementary probabilities is the complex random number and vector *Z* which is equal to $P_r + P_m = P_r + i(1 - P_r)$. And as the complex probability cubes show and illustrate, we realize that *Z* is the sum in $\mathcal{C}$ of the real probability of failure and of the imaginary probability of survival in the complex probability plane that has the equation: $P_r(t) = iP_m(t) + 1$ for $\forall t: 0 \le t \le t_C$, $\forall P_r: 0 \le P_r \le 1$, $\forall P_m: 0 \le P_m \le i$. What is noteworthy is that the square of the norm of *Z* which is $|Z|^2$ is nothing but *DOK*, as it was proved in *CPP* and in the new model. Moreover, since $MChf = -2iP_rP_m = 2P_rP_m/i$, therefore it is twice the product in $\mathcal{R}$ of both the probability of failure and the probability of survival and it embodies the magnitude of chaos since it is always 0 or positive. All the simulations show and prove all these facts.

From all the above we can conclude that since $D(t)$ is stochastic, the factors resultant is random, the jumps in *D* are the simulations failure probabilities $P_r(t_k)$, then we are dealing with a random experiment, thus the natural appearance of *Chf*, *MChf*, *DOK*, *Z*, and hence *Pc*. So, we get in the simulations:

$$\begin{aligned} Chf(t_k) &= -2P_r(t_k)P_m(t_k)/i = -2\Phi(-\beta)[1-\Phi(-\beta)] \\ &= -2\{\psi[D(t_k)-D(t_{k-1})]\}\{1-\psi[D(t_k)-D(t_{k-1})]\} \end{aligned} \tag{41}$$

$$\begin{aligned} MChf(t_k) &= |Chf(t_k)| = 2\Phi(-\beta)[1-\Phi(-\beta)] \\ &= 2\{\psi[D(t_k)-D(t_{k-1})]\}\{1-\psi[D(t_k)-D(t_{k-1})]\} \end{aligned} \tag{42}$$

$$DOK(t_k) = 1 - 2P_r(t_k)P_m(t_k)/i = 1 - 2\Phi(-\beta)[1-\Phi(-\beta)]$$
$$= 1 - 2\{\psi[D(t_k) - D(t_{k-1})]\}\{1 - \psi[D(t_k) - D(t_{k-1})]\} \quad (43)$$

$$Z(t_k) = P_r(t_k) + P_m(t_k) = \Phi(-\beta) + i[1-\Phi(-\beta)] = \psi[D(t_k) - D(t_{k-1})] + i\{1 - \psi[D(t_k) - D(t_{k-1})]\} \quad (44)$$

$$Pc^2(t_k) = DOK(t_k) - Chf(t_k) = DOK(t_k) + MChf(t_k) = 1, \text{ for every } t_k\text{: } 0 \le t_k \le t_C \quad . \quad (45)$$

Furthermore, in the novel paradigm we have:

$$RUL(t_k) = t_C - t_k \, .$$

Note that, since t and D are random then RUL is also a random function of t. Thus, this will yield in the set $\mathcal{R}$:

$$P_{rob}[RUL(t_k)] = P_{rob} \text{ (the system will survive for } t_k < t \le t_C)$$
$$= 1 - P_{rob} \text{ (the system will fail for } t \le t_k)$$
$$= 1 - D(t_k) \quad (46)$$

Thenceforth, we get continuously: $P_{rob}[RUL(t_k)] + D(t_k) = 1$ everywhere.

This implies that: $P_{rob}[RUL(t_k = 0)] = 1 - D(t_k = 0) = 1 - D_0 = 1 - 0 = 1$

And $P_{rob}[RUL(t_k = t_C)] = 1 - D(t_k = t_C) = 1 - D_C = 1 - 1 = 0$

Henceforth, we attain a general and an original prognostic equation for RUL. In fact, if $t_0 \neq 0$ and $D(t_0) \neq 0$ therefore:

$$P_{rob}[RUL(t_k)] = P_{rob}(Survival : t_k < t \le t_C) = 1 - P_{rob}(Failure : t_0 \le t \le t_k)$$

$$= 1 - \sum_{t=t_0}^{t=t_k} P_r(t)/\psi = 1 - \sum_{t=t_0}^{t=t_k} \Phi(-\beta)/\psi; \quad \text{with } P_r(t_0)/\psi = D_0 \quad (47)$$
$$= 1 - D(t_k)$$

$$= \sum_{t=t_{k+1}}^{t=t_C} P_r(t)/\psi = \sum_{t=t_{k+1}}^{t=t_C} \Phi(-\beta)/\psi \quad (48)$$

$$= 1 - \sum_{t=t_0}^{t=t_k} PDF_{failure}(t); \quad \text{with } PDF_{failure}(t_0) = D_0 \quad (49)$$

$$= \sum_{t=t_{k+1}}^{t=t_C} PDF_{failure}(t) \quad (50)$$

for any mode of simulation profile.

Moreover, from equations (38), (39), and (40) and for any mode of simulation profile we have the following recursive relations:

$$P_{rob}[RUL(t_k)] = 1 - D(t_k) = 1 - \{D(t_{k-1}) + P_r(t_k)/\psi\} = 1 - \{D(t_{k-1}) + \Phi(-\beta)/\psi\} \quad (51)$$

$$= 1 - \{D(t_{k-1}) + PDF_{failure}(t_k)\} \quad (52)$$

$$= 1 - \{1 - P_{rob}[RUL(t_{k-1})] + P_r(t_k)/\psi\} \quad (53)$$

$$= 1 - \{1 - P_{rob}[RUL(t_{k-1})] + \Phi(-\beta)/\psi\} \quad (54)$$

$$= P_{rob}[RUL(t_{k-1})] - P_r(t_k)/\psi \quad (55)$$

$$= P_{rob}[RUL(t_{k-1})] - \Phi(-\beta)/\psi \quad (56)$$

$$= P_{rob}[RUL(t_{k-1})] - PDF_{failure}(t_k) \quad (57)$$

where $P_{rob}[RUL(t_{k-1})] = 1 - D(t_{k-1})$.

In the ideal situation, if all the factors are 100% deterministic then we have in $\mathcal{R}$: the probability of failure for $t_k < t_C$ is 0 and is 1 for $t_k \geq t_C$, accordingly the probability of system survival for $t_k < t_C$ is 1 and is 0 for $t_k \geq t_C$, since certain failure will happen only at $t_k = t_C$. So, degradation is determined surely everywhere in $\mathcal{R}$ and its random function is replaced by a deterministic curve. Therefore, chaos is null and hence $Chf = MChf = 0$ and $DOK = 1$ always for all $0 \leq t_k \leq t_C$. Thus, $P_{rob}[RUL(t_k < t_C)] = 1$ and $P_{rob}[RUL(t_k \geq t_C)] = 0$.

Furthermore, at each instant t in the original prognostic paradigm, the stochastic $RUL(t)$ and $D(t)$ are predicted with certitude in the complex probability set $\mathcal{C}$ with $Pc^2 = DOK - Chf = DOK + MChf$ preserved as equal to 1 through a permanent compensation between Chf and DOK. This compensation is from the instant $t = 0$ where $D(t) = D_0 = 0$ until the instant of failure t_C where $D(t_C) = 1$. Furthermore, we can realize that DOK does not comprise any uncertain knowledge (with a probability less than 100%), it is the measure of our certain knowledge (probability = 100%) about the expected event. We can understand that we have eliminated and subtracted in the equation above all the random factors and chaos (Chf) from our random experiment when computing Pc^2, thus no chaos exists in $\mathcal{C}$, it is only present (if it does) in $\mathcal{R}$; consequently, this has led to a 100% deterministic outcome and experiment in $\mathcal{C}$ since the probability Pc is constantly equal to one. This is one of the advantages of extending $\mathcal{R}$ to $\mathcal{M}$ and therefore of conducting random experiments in the set $\mathcal{C} = \mathcal{R} + \mathcal{M}$. Thus, in the original prognostic paradigm, our knowledge of all the indicators and parameters (RUL, P_{rob}, D, etc.) is totally predictable, always perfect, and constantly complete because $Pc = 1$ permanently, independently of any random factors or any simulation profile.

IX. Conclusion

In this paper I applied the theory of Extended Kolmogorov Axioms to Prognostic based on Reliability. I used for this purpose the very well-known First-Order Reliability Method or FORM analysis and procedure. Consequently, I established a tight link between the new theory degradation or the remaining useful lifetime and reliability. Hence, I developed the theory of "Complex Probability" beyond the scope of my previous fourteen papers on this topic.

As it was proved and illustrated, when the degradation index is 0 or 1 and correspondingly the *RUL* is t_C or 0 then the degree of our knowledge (*DOK*) is one and the chaotic factor (*Chf* and *MChf*) is 0 since the state of the system is totally known. During the process of degradation ($0 < D < 1$) we have: $0.5 \leq DOK < 1$, $-0.5 \leq Chf < 0$, and $0 < MChf \leq 0.5$. Notice that during the whole process of degradation we have $Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc$, that means that the phenomenon which seems to be random and stochastic in $\mathcal{R}$ is now deterministic and certain in $\mathcal{C} = \mathcal{R} + \mathcal{M}$, and this after adding to $\mathcal{R}$ the contributions of $\mathcal{M}$ and hence after subtracting the chaotic factor from the degree of our knowledge. Moreover, for each value of an instant t_k or E_k, I have determined their corresponding probability of survival or of the remaining useful lifetime $RUL(t_k) = t_C - t_k$ or $RUL(E_k) = E_C - E_k$. In other words, at each instant t_k or E_k, $RUL(t_k)$ or $RUL(E_k)$ are certainly predicted in the complex set $\mathcal{C}$ with Pc preserved as equal to one through an incessant compensation between *DOK* and *Chf*. This compensation is from instant $t_k = 0$ where $D(t_k) = 0$ until the failure instant t_C where $D(t_C) = 1$. And this compensation is also from $E_k = 0$ where $D(E_k) = 0$ until failure at E_C where $D(E_C) = 1$. Furthermore, using all these graphs illustrated throughout the whole paper, we can materialize and illustrate both the system chaos (*Chf* and *MChf*) and the system certain knowledge (*DOK* and *Pc*). Additionally, an application to Young modulus *E* was successfully done here and proves the success of the entire novel prognostic paradigm. This is certainly wonderful, very fruitful, and fascinating and shows once again the benefits of extending the axioms of Kolmogorov and hence the benefits and novelty of this innovative field in applied and in pure mathematics that can be called verily: "The Complex Probability Paradigm".

As a prospective and future challenges and research, we intend to more develop the novel conceived prognostic paradigm and to apply it to a diverse set of nondeterministic events like for other stochastic phenomena as in the classical theory of probability and in stochastic processes. Additionally, we will implement *CPP* to other topics in the field of prognostic in engineering and also to the problems of random walk which have huge consequences when applied to economics, to chemistry, to physics, to pure and applied mathematics.

Data Availability

The data used to support the findings of this study are available from the author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

References

[1] Verderaime, V. (1994). "Illustrated Structural Application of Universal First-Order Reliability Method", NASA Technical Paper 3501.

[2] Cizelj, L., Mavko, B., & Riesch-Oppermann, H. (1994). "Application of first and second order reliability methods in the safety assessment of cracked steam generator tubing", Nuclear Engineering and Design, 147.

[3] Huang, J., & D. V. Griffiths. (2011). "Observations on FORM in a simple geomechanics example." Structural Safety 33, no. 1, pp. 115-119.

[4] Dudzik, A., & U. Radoń. (2016). "The reliability assessment for steel industrial building." Advances in Mechanics: Theoretical, Computational and Interdisciplinary Issues, pp. 163-166.

[5] Choi, C., & H. Hee Yoo. (2012). "Uncertainty analysis of nonlinear systems employing the first-order reliability method." Journal of Mechanical Science and Technology 26, no. 1, pp. 39-44.

[6] Institute of Electrical and Electronics Engineers. (1990). IEEE Standard Computer Dictionary: A Compilation of IEEE Standard Computer Glossaries. New York, NY ISBN 1-55937-079-3.
[7] RCM II, Reliability Centered Maintenance, Second edition. (2008). pp. 250-260, the role of Actuarial analysis in Reliability.
[8] Why You Cannot Predict Electronic Product Reliability (PDF). (2012). ARS, Europe. Warsaw, Poland.
[9] O'Connor, P. D. T. (2002). Practical Reliability Engineering (Fourth Edition), John Wiley & Sons, New York. ISBN 978-0-4708-4462-5.
[10] Barnard, R. W. A. (2008). "What is wrong with Reliability Engineering?" (PDF). Lambda Consulting. Retrieved 30 October 2014.
[11] Saleh, J. H., & Marais, K. (February 2006). "Highlights from the Early (and pre-) History of Reliability Engineering", Reliability Engineering and System Safety, Volume 91, Issue 2, pp. 249–256.
[12] Juran, J., & Gryna, F. (1988). Quality Control Handbook, Fourth Edition, McGraw-Hill, New York, pp. 3-24.
[13] Reliability of military electronic equipment; report. Washington: United States Department of Defense. 4 June 1957. hdl:2027/mdp.39015013918332.
[14] Wong, K. (1981). "Unified Field (Failure) Theory-Demise of the Bathtub Curve", Proceedings of Annual RAMS, pp. 402-408.
[15] Abou Jaoude, A., El-Tawil, K., & Kadry, S. (2010). "Prediction in Complex Dimension Using Kolmogorov's Set of Axioms", Journal of Mathematics and Statistics, Science Publications, vol. 6(2), pp. 116-124.
[16] Abou Jaoude, A. (2013)."The Complex Statistics Paradigm and the Law of Large Numbers", Journal of Mathematics and Statistics, Science Publications, vol. 9(4), pp. 289-304.
[17] Abou Jaoude, A. (2013). "The Theory of Complex Probability and the First Order Reliability Method", Journal of Mathematics and Statistics, Science Publications, vol. 9(4), pp. 310-324.
[18] Abou Jaoude, A. (2014). "Complex Probability Theory and Prognostic", Journal of Mathematics and Statistics, Science Publications, vol. 10(1), pp. 1-24.
[19] Abou Jaoude, A. (2015). "The Complex Probability Paradigm and Analytic Linear Prognostic for Vehicle Suspension Systems", American Journal of Engineering and Applied Sciences, Science Publications, vol. 8(1), pp. 147-175.
[20] Abou Jaoude, A. (2015). "The Paradigm of Complex Probability and the Brownian Motion", Systems Science and Control Engineering, Taylor and Francis Publishers, vol. 3(1), pp. 478-503.
[21] Abou Jaoude, A. (2016). "The Paradigm of Complex Probability and Chebyshev's Inequality", Systems Science and Control Engineering, Taylor and Francis Publishers, vol. 4(1), pp. 99-137.
[22] Abou Jaoude, A. (2016). "The Paradigm of Complex Probability and Analytic Nonlinear Prognostic for Vehicle Suspension Systems", Systems Science and Control Engineering, Taylor and Francis Publishers, vol. 4(1), pp. 99-137.
[23] Abou Jaoude, A. (2017). "The Paradigm of Complex Probability and Analytic Linear Prognostic for Unburied Petrochemical Pipelines", Systems Science and Control Engineering, Taylor and Francis Publishers, vol. 5(1), pp. 178-214.
[24] Abou Jaoude, A. (2017). "The Paradigm of Complex Probability and Claude Shannon's Information Theory", Systems Science and Control Engineering, Taylor and Francis Publishers, vol. 5(1), pp. 380-425.
[25] Abou Jaoude, A. (2017). "The Paradigm of Complex Probability and Analytic Nonlinear Prognostic for Unburied Petrochemical Pipelines", Systems Science and Control Engineering, Taylor and Francis Publishers, vol. 5(1), pp. 495-534.

[26] Abou Jaoude, A. (2018). "The Paradigm of Complex Probability and Ludwig Boltzmann's Entropy", Systems Science and Control Engineering, Taylor and Francis Publishers, vol. 6(1), pp. 108-149.
[27] Abou Jaoude, A. (2019). "The Paradigm of Complex Probability and Monte Carlo Methods", Systems Science and Control Engineering, Taylor and Francis Publishers, vol. 7(1), pp. 407-451.
[28] Abou Jaoude, A. (2020). "Analytic Prognostic in the Linear Damage Case Applied to Buried Petrochemical Pipelines and the Complex Probability Paradigm", Fault Detection, Diagnosis and Prognosis, IntechOpen. DOI: 10.5772/intechopen.90157.
[29] Benton, W. (1966). *Probability*, *Encyclopedia Britannica*. vol. 18, pp. 570-574, Chicago, Encyclopedia Britannica Inc.
[30] Benton, W. (1966). *Mathematical Probability*, *Encyclopedia Britannica*. vol. 18, pp. 574-579, Chicago, Encyclopedia Britannica Inc.
[31] Feller, W. (1968). *An Introduction to Probability Theory and Its Applications*. 3rd Edition. New York, Wiley.
[32] Montgomery, D. C., & Runger, G. C. (2003). *Applied Statistics and Probability for Engineers*. 3rd Edition, New York, John Wiley & Sons, Inc.
[33] Walpole, R., Myers, R., Myers, S., & Ye, K. (2002). *Probability and Statistics for Engineers and Scientists*. 7th Edition, New Jersey, Prentice Hall.
[34] Freund, J. E. (1973). *Introduction to Probability*. New York: Dover Publications.
[35] Srinivasan, S. K., & Mehata, K. M. (1988). *Stochastic Processes.* 2nd Edition, New Delhi, McGraw-Hill.
[36] Stewart, I. (2002). *Does God Play Dice?* 2nd Edition. Oxford, Blackwell Publishing.
[37] Stewart, I. (1996). *From Here to Infinity* (2nd ed.). Oxford: Oxford University Press.
[38] Stewart, I. (2012). *In Pursuit of the Unknown*. New York: Basic Books.
[39] Barrow, J. (1992). *Pi in the Sky*. Oxford: Oxford University Press.
[40] Bogdanov, I., & Bogdanov, G. (2009). *Au Commencement du Temps*. Paris : Flammarion.
[41] Bogdanov, I., & Bogdanov, G. (2010). *Le Visage de Dieu*. Paris : Editions Grasset et Fasquelle.
[42] Bogdanov, I., & Bogdanov, G. (2012). *La Pensée de Dieu*. Paris : Editions Grasset et Fasquelle.
[43] Bogdanov, I., & Bogdanov, G. (2013). *La Fin du Hasard*. Paris : Editions Grasset et Fasquelle.
[44] Van Kampen, N. G. (2006). *Stochastic Processes in Physics and Chemistry*. Revised and Enlarged Edition, Sydney, Elsevier.
[45] Bell, E. T. (1992). *The Development of Mathematics*. New York, Dover Publications, Inc., United States of America.
[46] Boursin, J.-L. (1986). *Les Structures du Hasard*. Paris, Editions du Seuil.
[47] Dacunha-Castelle, D. (1996). *Chemins de l'Aléatoire*. Paris, Flammarion.
[48] Dalmedico-Dahan, A., Chabert, J.-L, & Chemla, K. (1992). *Chaos Et Déterminisme*. Paris, Edition du Seuil.
[49] Ekeland, I. (1991). *Au Hasard. La Chance, la Science et le Monde*. Paris, Editions du Seuil.
[50] Gleick, J. (1997). *Chaos, Making a New Science*. New York, Penguin Books.
[51] Dalmedico-Dahan, A., & Peiffer, J. (1986). *Une Histoire des Mathématiques*. Paris, Edition du Seuil.
[52] Gullberg, J. (1997). *Mathematics from the Birth of Numbers*. New York, W.W. Norton & Company.
[53] Science Et Vie. (1999). *Le Mystère des Mathématiques*. Numéro, 984.
[54] Davies, P. (1993). *The Mind of God*. London: Penguin Books.
[55] Gillies, D. (2000). *Philosophical Theories of Probability*. London: Routledge. ISBN 978-0415182768.

[56] Guillen, M. (1995). *Initiation Aux Mathématiques*. Paris : Albin Michel.
[57] Hawking, S. (2002). *On the Shoulders of Giants*. London: Running Press.
[58] Hawking, S. (2005). *God Created the Integers*. London: Penguin Books.
[59] Hawking, S. (2011). *The Dreams that Stuff is Made of*. London: Running Press.
[60] Pickover, C. (2008). *Archimedes to Hawking*. Oxford: Oxford University Press.
[61] Wikipedia, the free encyclopedia. *Probability*. https://en.wikipedia.org/
[62] Wikipedia, the free encyclopedia, *Probability Axioms*. https://en.wikipedia.org/
[63] Wikipedia, the free encyclopedia, *Probability Density Function*. https://en.wikipedia.org/
[64] Wikipedia, the free encyclopedia, *Probability Distribution*. https://en.wikipedia.org/
[65] Wikipedia, the free encyclopedia. *Probability Interpretations*. https://en.wikipedia.org/
[66] Wikipedia, the free encyclopedia, *Probability Measure*. https://en.wikipedia.org/
[67] Wikipedia, the free encyclopedia, *Probability Space*. https://en.wikipedia.org/
[68] Wikipedia, the free encyclopedia, *Probability Theory*. https://en.wikipedia.org/
[69] Wikipedia, the free encyclopedia, *Stochastic Process*. https://en.wikipedia.org/
[70] Gentle, J. (2003). *Random Number Generation and Monté Carlo Methods* (2nd ed.). Sydney: Springer.
[71] Abou Jaoude, A. (2019). The Computer Simulation of Monté Carlo Methods and Random Phenomena. Newcastle upon Tyne: United Kingdom. Cambridge Scholars Publishing.
[72] Abou Jaoude, A. (2019). The Analysis of Selected Algorithms for the Stochastic Paradigm. Newcastle upon Tyne: United Kingdom. Cambridge Scholars Publishing.
[73] Wikipedia, the free encyclopedia. First-Order Reliability Method. https://en.wikipedia.org/
[74] Wikipedia, the free encyclopedia. Reliability Engineering. https://en.wikipedia.org/
[75] Wikipedia, the free encyclopedia. Systems Engineering. https://en.wikipedia.org/
[76] Wikipedia, the free encyclopedia. Stress–Strength Analysis. https://en.wikipedia.org/
[77] Chan Man Fong, C. F., De Kee, D., & Kaloni, P.N. (1997). *Advanced Mathematics for Applied and Pure Sciences*. Amsterdam, Gordon and Breach Science Publishers, The Netherlands.
[78] Abou Jaoude, A. (August 1st 2004). Ph.D. Thesis in Applied Mathematics: *Numerical Methods and Algorithms for Applied Mathematicians*. Bircham International University. http://www.bircham.edu.
[79] Abou Jaoude, A. (October 2005). Ph.D. Thesis in Computer Science: *Computer Simulation of Monté Carlo Methods and Random Phenomena*. Bircham International University. http://www.bircham.edu.
[80] Abou Jaoude, A. (27 April 2007). Ph.D. Thesis in Applied Statistics and Probability: *Analysis and Algorithms for the Statistical and Stochastic Paradigm*. Bircham International University. http://www.bircham.edu.
[81] Stepić, A. I., & Ognjanović, Z. (2014). "*Complex Valued Probability Logics*", Publications De L'institut Mathématique, Nouvelle Série, tome 95 (109), pp. 73–86, DOI: 10.2298/PIM1409073I.
[82] Cox, D. R. (1955). "*A Use of Complex Probabilities in the Theory of Stochastic Processes*", Mathematical Proceedings of the Cambridge Philosophical Society, 51, pp. 313–319.
[83] Weingarten, D. (2002). "*Complex Probabilities on R^N as Real Probabilities on C^N and an Application to Path Integrals*". Physical Review Letters, 89, 335. http://dx.doi.org/10.1103/PhysRevLett.89.240201.
[84] Youssef, S. (1994). "*Quantum Mechanics as Complex Probability Theory*". Modern Physics Letters A 9, pp. 2571–2586.
[85] Fagin, R., Halpern, J., & Megiddo, N. (1990). *A Logic for Reasoning about Probabilities*. Information and Computation 87, pp. 78–128.

[86] Bidabad, B. (1992). *Complex Probability and Markov Stochastic Processes*. Proc. First Iranian Statistics Conference, Tehran, Isfahan University of Technology.
[87] Ognjanović, Z., Marković, Z., Rašković, M., Doder, D., & Perović, A. (2012). *A Probabilistic Temporal Logic That Can Model Reasoning About Evidence*. Annals of Mathematics and Artificial Intelligence, 65, pp. 1-24.
[88] Abou Jaoude, A. (2012). Ph.D. Thesis in Automatic Control and Prognostic: Advanced Analytical Model for the Prognostic of Industrial Systems Subject to Fatigue. EDST-The Lebanese University and Aix-Marseille University-France.
[89] Abou Jaoude, A., K. El-Tawil, S. Kadry, H. Noura, & M. Ouladsine. (November 2010). “Analytic Prognostic Model for a Dynamic System”, International Review of Automatic Control (IREACO), Praise Worthy Prize.
[90] Abou Jaoude, A., S. Kadry, K. El-Tawil, H. Noura, & M. Ouladsine. (April 2011). "Analytic Prognostic for Petrochemical Pipelines", Journal of Mechanical Engineering Research (JMER).
[91] Abou Jaoude, A., & K. El-Tawil. (2103). "Analytic and Nonlinear Prognostic for Vehicle Suspension Systems", American Journal of Engineering and Applied Sciences (AJEAS), Science Publications, vol. 6(1), pp. 42-56.
[92] Abou Jaoude, A., & K. El-Tawil. (2013). "Stochastic Prognostic Paradigm for Petrochemical Pipelines Subject to Fatigue", American Journal of Engineering and Applied Sciences (AJEAS), Science Publications, vol.6(2), pp. 145-160.
[93] El-Tawil, K., & A. Abou Jaoude. (2103). "Stochastic and Nonlinear Based Prognostic Model", Systems Science and Control Engineering (SSCE), Taylor and Francis Publishers, vol. 1(1), pp. 66-81.
[94] Abou Jaoude, A. (2015). “Analytic and Linear Prognostic Model for A Vehicle Suspension System Subject to Fatigue”, Systems Science and Control Engineering (SSCE), Taylor and Francis Publishers, vol. 3(1), pp. 81-98.
[95] El-Tawil, K., S. Kadry, & A. Abou Jaoude. (2009). "Life Time Estimation under Probabilistic Fatigue of Cracked Plates for Multiple Limits States", Numerical Analysis and Applied Mathematics, International Conference (ICNAAM).
[96] El-Tawil, K., Abou Jaoude A., S. Kadry, H. Noura, & M. Ouladsine. (November 2010). "Prognostic Based on Analytic Laws Applied to Petrochemical Pipelines", International Conference on Computer-aided Manufacturing and Design (CMD 2010), China.
[97] Abou Jaoude, A., K. El-Tawil, S. Kadry, H. Noura, & M. Ouladsine. (June 13-15, 2011). “Prognostic Model for Buried Tubes”, International Conference on Advanced Research and Applications in Mechanical Engineering (ICARAME'11), Notre Dame University - Louaize, Zouk Mosbeh, Lebanon.
[98] Abou Jaoude, A., H. Noura, K. El-Tawil, S. Kadry, & M. Ouladsine. (August 29-31, 2012). "Lifetime Analytic Prognostic for Petrochemical Pipes Subject to Fatigue", SAFEPROCESS, 8th IFAC Symposium on Fault Detection, Supervision and Safety of Technical Processes, Mexico City, Mexico.
[99] Abou Jaoude, A., H. Noura, K. El-Tawil, S. Kadry, & M. Ouladsine. (November 15-16, 2012). "Analytic Prognostic Model for Stochastic Fatigue of Petrochemical Pipelines", Australian Control Conference (AUCC 2012), Sydney, Australia.
[100] Abou Jaoude, A. (2013). “Automatic Control and Prognostic: Advanced Analytical Model for the Prognostic of Industrial Systems Subject to Fatigue”, Scholars' Press, Germany.

