Scan to know paper details and author's profile

Two Generalizations of Brouwer Fixed Point Theorem

Bhamini M. P. Nayar

Morgan State University

ABSTRACT

The following fixed point theorems are given: (1) If X is a Hausdorff and compact space and $g : X \to X$ is a one-one continuous function, then g has a fixed point. (2) If X is a compact, Hausdorff and second countable space and $f : X \to X$ is a contraction mapping, then f has a fixed point. Two proofs of Theorem 1 are given, one using sequences and the other using ultrafilters. These theorems generalize the Brouwer Fixed Point Theorem.

Keywords and phrases: Fixed point theorem; Brouwer.

Classification: LCC: QA329

Language: English

Great Britain Journals Press

LJP Copyright ID: 925661
Print ISSN: 2631-8490
Online ISSN: 2631-8504

London Journal of Research in Science: Natural and Formal

Volume 23 | Issue 16 | Compilation 1.0

© 2023. Bhamini M. P. Nayar. This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncom-mercial 4.0 Unported License http://creativecommons..org/licenses/by-nc/4.0/), permitting all noncommercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Two Generalizations of Brouwer Fixed Point Theorem

Bhamini M. P. Nayar

In loving memory of Prof. James E. Joseph

ABSTRACT

The following fixed point theorems are given: (1) If X is a Hausdorff and compact space and g : X → X is a one-one continuous function, then g has a fixed point. (2) If X is a compact, Hausdorff and second countable space and f : X → X is a contraction mapping, then f has a fixed point. Two proofs of Theorem 1 are given, one using sequences and the other using ultrafilters. These theorems generalize the Brouwer Fixed Point Theorem.

Keywords and phrases: Fixed point theorem; Brouwer.

Author: Department of Mathematics, Morgan State University, Baltimore, MD 21251, USA.

London Journal of Research in Science: Natural and Formal

I. INTRODUCTION

Brouwer Fixed Point theorem states that *a continuous map of* D*, the unit disc* $\{(x, y) \in R^2 | x^2 + y^2 \leq 1\}$ *to itself has a fixed point* ([1], [2], [5], [6]). In this article Brower Fixed Point Theorem is extended to compact Hausdorff spaces.

The proof of Brower Fixed Point Theorem is generally given using the concept of homotopy and Retraction Theorem ([1], [2], [5], [6]). A subset A of X is called a *retract* of X, provided that there is a continuous function $f : X \to A$ such that $f(x) = x$ for all $x \in A$ and the function f is called a *retraction* [2]. The argument used in proving the Brower Fixed Point Theorem is that the boundary of the unit disc D, the unit circle, is not a retract of D. It is known that the unit circle, as well as the unit disc, is compact and Hausdorf where as the circle with the point $(0, 1)$ removed is not compact [4]. Since it is known that the unit circle $S = \{(x, y) \in \mathcal{R}^2 | x^2 + y^2 = 1\}$, with continuous function taking each point to its diametrically opposite point does not give a fixed point, and observing that such a function is not one-one, we arrive at the following result. It is well known that a continuous function from a compact space to a Hausdorff space is closed.

Theorem 1. Let X be Hausdorff, compact and $g : X \to X$ be continuous and one-one. Then g has a fixed point.

© 2023 Great Britain Journal Press

Proof. **Proof 1.** If g is constant, then g has a fixed point. In that case g is not on-one. Let g be non-constant. Since X is compact and Hausdorff, $g : X \to X$ is a closed function. Therefore, $g(X)$ is closed. So, $g(X)$ is open in $g(X)$ and is a closed subset of the space X. Choose an $x \in g(X) \subseteq X$ and form the sequence $\{g^n(x)\}$. Note that $g(X)$ being a closed subset of X, for all n , $\{g^n(x)\} \subseteq g(X)$. Moreover, it is compact, and hence is countably compact. So, there is a subsequence $\{g^{n_k}(x)\}$ of $\{g^n(x)\}$ in $g(X)$ such that $\{g^{n_k}(x)\} \to p$, $p \in g(X)$, $g(X)$ being closed. Also since g is one-one and $X - g(X)$ being open, and $\{g^n(x)\} \subseteq g(X)$ no subsequence of $\{g^n(x)\}$ in $X - g(X)$ will converge to $p \in g(X)$. So, $\{g^{n_k+1}(x)\} \to g(p)$, g being continuous. Thus $g(p) = p$, since $\{g^{n_k+1}(x)\}$ is a subsequence of $\{g^{n_k}(x)\}$. Hence there is a fixed point for g in $g(X) \subseteq X$.

Below, another proof for the Theorem 1 using ultrafilters is given.

Proof 2. Consider a space X which is compact and Hausdorff and let $f : X \to X$ be a continuous injection. Suppose $\mathfrak{F}$ is an ultrafilter on X. Since X is compact, $\mathfrak{F} \to x$ for some $x \in X$. Suppose that $\mathfrak{F}$ does not have a fixed point in X. Then for each $x \in X, x \neq f(x)$. The space X is Hausdorff and hence there exist open sets U and V in X such that $x \in U$ and $f(x) \in V, U \cap V = \emptyset$. Note that $\mathfrak{F} \to x$ and hence, there is an $F_1 \in \mathfrak{F}$ such that $F_1 \subseteq U$. Also, since f is continuous, $f(\mathfrak{F})$ is an ultrafilter and $f(\mathfrak{F}) \to f(x)$. Hence there is an $F_2 \in \mathfrak{F}$ such that $f(F_2) \subseteq V$. Let $F = F_1 \cap F_2$. Note that $F \in \mathfrak{F}$ and $F \subseteq U \subseteq X - V$. Also, $f(U) \subseteq V$, since f is continuous. Moreover, $f(U) \subseteq f(X - V) = f(X) - f(V)$. Therefore, $V \cap (f(X) - f(V)) \neq \emptyset$. That is, there exists a $t \in V$ such that $f(t) \notin f(V)$. That is, there is an $a \in V$ such that $a \neq f(t)$ for any $t \in V$ and $f(a) \in f(V)$, for all $a \in V$. Here we arrive at a contradiction since f is one-one and $V \cap (X - V) = \emptyset \Rightarrow f(V) \cap f(X - V) = f(V) \cap (f(X) - f(V)) = \emptyset$. Hence our assumption that for each $x \in X, x \neq f(x)$ is not true. So, there is an $x \in X$ such that $x = f(x)$ and hence f has a fixed point and the proof is complete.

The concepts defined in the rest of the article can be found in any of the referenced books and are not given individual references. A topological space is *metrizable* if there is a metric on X which gives the same collection of open sets as the topological space. In that case we say that the topology and the metric on X are compatible. While every metric space is a topological space, a topological space need not have a metric which provides the same collection of open sets. Investigating conditions on a space which guarantees the existence of a compatible metric is a significant investigative area, called *metrization*.

© 2023 Great Britain Journal Press

A space is *complete* if every Cauchy sequence in the space converges. It is well known that a compact metric space is complete. A space is *separable* if it has a countable dense subset and is *second countable* if it has a countable base. It is well known that a second countable space is separable, but a separable space need not be second countable. However, a separable metric space is second countable. Also, a compact Hausdorff space is regular. A function $f : (X, d) \to (Y, \rho)$ is a *contraction mapping*, if there is an $\alpha \in (0, 1)$ such that $\rho(f(x), f(y)) \leq \alpha d(x, y)$ for every $x, y \in X$. The *Urysohn metrization theorem* states the following:

Theorem 2 [2] Let X be a T_1-space. Then the following are equivalent:

(1) X is separable and metrizable;
(2) X is regular and second countable.

In view of the above result, It is clear that if a second countable topological space X is compact and Hausdorff, then it is metrizable. This guarantees the existence of a metric d on X such that the topology generated by the metric d is the topology on X. With this observation, the following definition is provided.

Definition Let X be a metrizable topological space. A function $f : X \to X$ is *a contraction mapping on* X, if it is a contraction mapping with respect to the metric on X which is compatible with the topology on X.

In view of the above, we have the following Theorem.

Theorem 3 Let X be compact, Hausdorff and second countable and let $f : X \to X$ be a contraction maping. Then f has a fixed point.

Proof. Given that X is compact and Hausdorff. Therefore X is regular and it is also second countable. So, by the Urysohn metrization theorem, the space is metrizable. That is, there is a metric d which is compatible with the topology. Choose $x \in X$ and consider the sequence $\{f^n(x)\}$. Since f is a contraction mapping, $\{f^n(x)\}$ is a Cauchy sequence. The space X being compact and regular, it is complete and hence the sequence $\{f^n(x)\}$ converges, say, to $p \in X$. So, $\{f^{n+1}(x)\} \to f(p)$. Hence $f(p) = p$ and f has a fixed point. The proof is complete.

Note that the unit disc D satisfies the conditions of the space X in Theorems 1 and 3. Hence these theorems genralize the Brouwer Fixed Point Theorem.

© 2023 Great Britain Journal Press

II. CONCLUSION

A word aboout the development of this article: When the work on the current article was started, our attempt was to use one of the characterizations of closed function provided in [3]. It states that: A function $g : X \to Y$ is a closed function if and only if $g(V) - g(X - V)$ is open in $g(X)$ whenever V is an open subset of X, where X and Y are topological spaces. When the space X is compact and Hausdorff, any continuous function $f : X \to X$ will be closed. Originally, we wanted to use this fact, along with the assumption that the space was connected and the function was onto. However, with the assumption that f is an injection, that will make f to be a homeomorphism.

It is well-known that corresponding to each real number t, we can find a point on the unit circle and each point on the unit circle associates with a real number. A function $f(x) = x + k$ is a continuous function on the set of reals, which does not have a fixed point. While the set of reals $\mathfrak{R}$ with usual topology is not compact, the line segment $[0, 2\pi]$, as a subset of the set of reals is compact and every continuous function on $[0, 2\pi]$ has a fixed point. However, the unit circle as a subset of the plane R^2 with Euclidean topology is compact and Hausdorff. A function which takes each point on the circle to its diametrically opposite point does not have a fixed point and it is continuous. These observations highlighted the periodic nature of the function which associates each real number to a point on the unit circle. Thus the assumption that the function on a compact Hausdorff space to be continuous and one-one is made.

The Brower Fixed Point Theorem is one among the significant fixed point theorems in Topology with its possibilities of applications in real world situations. Adams and Franzosa gave applications of Brower Fixed Point Theorem to identify equilibrium price distribution in Economics [1]. Its generalization to set-valued functions, Kakutani's Fixed Point Theorem, and its applications to Game Theory also are detailed in [1]. Considering such prominance, to have a straight forward proof of Borwer Fixed Point Theorem, even in a generalized form, is a significant addition to the literature.

ACKNOWLEDGMENTS

This article was started originally as a joint work by myself and Prof. James E. Joseph before his passing on December 8, 2022. In fact the first proof of Theorem 1 was outlined before he passed and also with the assumption that X as a connected space, using the above mentioned charecterization of a closed function. When I revised it recently, I introduced the function to be one-one and dropped the assumption that the space to be connected. The day before his passing, while discussing that theorem, he indicated to me that we

London Journal of Research in Science: Natural and Formal

© 2023 Great Britain Journal Press

should give a proof using ultrafilters, but we did not discuss any details. I provide here a proof using ultrafilters and also added Theorem 3. He passed away peacefully in his sleep, the next day. Like several signifcant results in Topology, Brower Fixed Point Theorem was in his mind and as always he tried to give simpler proofs for such classical results.

REFERENCES

1. C. Adams and R. Franzosa, Introduction to Topology Pure and Applied, Pearson Prentice Hall (2008).
2. S. W. Davis,Topology, McGraw Hill Higher Education Publications (2006).
3. J. E. Joseph and M. H. Kwack, A Note on Closed Functions, Missouri J. Math. Sci. 18 (1), 59 -61, Winter 2006.
4. M. G. Murdeshwar, General Topology, Wiley Eastern Limited (1986).
5. A. Wilansky, Topology for Analysis, Robert E. Krieger Publishing Company, Inc. Malabar, Florida 1983.
6. A. Willard, General Topology, Addison-Wesley Publishing Compajy (1968).

© 2023 Great Britain Journal Press

This page is intentionally left blank

© 2023 Great Britain Journal Press