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Rena Eldar kizi Kerbalayeva

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I. INTRODUCTION

Let $G \subset R^n$ and $1 \leq s \leq n$; s, n be naturals, where $n_1 + \cdots + n_s = n$. We consider the sufficient smooth function f(x), where the point $x = (x_1, \dots, x_s) \in R^n$ has coordinates

$$x_k = \left(x_{k,1}; \dots; x_{k,n_k}\right) \in R^{n_k} \text{ (k} \in e_s = \{1,\dots,s\}).$$

More precisely,

$$R^n = R^{n_1} \times R^{n_2} \times \cdots \times R^{n_s}.$$

Thus we consider the fixed, non–negative, integral vector l=$(l_1, \dots, l_s)$ such that, $l_k = \left(l_{k.1}; \dots; l_{k,n_k}\right)$, (k$\subset e_s$) that is, $l_{k,j} > 0$, $(j = 1, \dots, n_k)$ for all $k \in e_s$. Here we consider by Q the set of vectors i= $(i_1, \dots, i_s)$ where i_k= 1,2,…, n_k for every $k \in e_s$. The number of set Q is equal to:

$$|Q| = \prod_{k=1}^{s}(1 + n_k).$$

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Therefore, to the vector i= $(i_1, \dots, i_s) \in Q$, we shall correspond the vector $l^i = \left(l_1^{i_1}; \dots; l_s^{i_s}\right)$ of the set of non−negative, integral vectors l=$(l_1, \dots, l_s)$, where

$$l^0 = (0,0, \dots ,0), l_k^1 = \left(l_{k,1}, 0, \dots ,0\right), \dots , l_k^{i_k} = \left(0,0, \dots, l_{k,n_k}\right)$$

for all $k \in e_s$. Then to the vector e^i, we let correspond the vector $\bar{l}^i$=$\left(\bar{l}_1^{i_1}, \bar{l}_1^{i_2}, \dots , \bar{l}_1^{i_s}\right)$, where $\bar{l}_k^{i_k} = \left(\bar{l}_{k,1}^{i_1}, \bar{l}_{k,2}^{i_2}, \dots, \bar{l}_{k,n_k}^{i_k}\right)$ $(k \in e_s)$. Here the largest number $\bar{l}_{k,j}^{i_k}$ is less than $l_{k,j}^{i_k}$ for all $l_{k,j}^{i_k}$>0, when $l_{k,j}^{i_k}$=0 then we assume that $\bar{l}_{k,j}^{i_k} = 0$ for all $k \in e_s$.

Theremore, we consider

$$D^{\bar{l}^i} f = D_1^{\bar{l}_1^{i_1}} \cdots D_s^{\bar{l}_s^{i_s}} f, \quad D_k^{\bar{l}_k^{i_k}} f = D_{k,1}^{\bar{l}_k^{i_k}} \cdots D_{k,n_k}^{\bar{l}_k^{i_k}} f, G_{t^{\varkappa}} = G \cap I_{t^{\varkappa}}(x),$$

$$I_{t^{\varkappa}}(x) = I_{t_1^{\varkappa_1}}(x_1) \times I_{t_2^{\varkappa_2}}(x_2) \times \cdots \times I_{t_s^{\varkappa_s}}(x_s),$$

$$I_{t_k^{\varkappa_k}}(x_k) = \left\{ y_k : |y_k - x_k| < \frac{1}{2} t_k^{|\varkappa_k|}, \ k \in e_s \right\}$$

and

$$|\beta_k| = \sum_{j=1}^{n_k} \beta_{k,j}^{i_k} ; \ |\beta_k^{i_k}| = \sum_{j=1}^{n_k} \beta_{k,j}^{i_k} \ \frac{dt_k}{t_k} = \prod_{j \in e_k^i} \frac{dt_{k,j}}{t_{k,j}},$$

we take 0<$\beta_{k,j}^{i_k} = l_{k,j}^{i_k} - \bar{l}_{k,j}^{i_k} \le 1$, when $l_k^{i_k} > 0$, but when $l_{k,j}^{i_k} = 0$, then $\beta_{k,j}^{i_k} = 0$; $t = (t_1, \dots, t_s)$, $t_k = \left(t_{k,1}, \dots, t_{k,n_k}\right), \omega = (\omega_1, \dots, \omega_s), \omega_k = \left(\omega_{k,1}, \dots, \omega_{k,n_k}\right)$ and we take

$$\omega_{k,j} = 1, \text{ when } k \in e^i,$$

or we give

$$\omega_{k,j} = 0, \text{ when } k \in e_s / e^i,$$

$$e^i = supp\bar{l}^i = suppl^i = supp\omega, 1 \le \theta \le \infty; \ 1 \le p < \infty.$$

Here $t_0 = \left(t_{0,1}, \dots, t_{0,s}\right), t_{0,k} = \left(t_{0,k,1}, \dots, t_{0,k,n_k}\right)$ −is fixed vector and $\varkappa \in (0, \infty)^n$, a$\in [0, 1], \tau \in [1, \infty], [t_k]_1 = \min\{1, t_k\}, \ k \in e_s$. Here

$$\Delta^{\omega}(t) f = \Delta_1^{\omega_1}(t_1) \cdots \Delta_s^{\omega_s}(t_k) f,$$

when $2\omega = (2, 2, \dots, 2)$, and

$$\Delta_{k}^{\omega_k}(t_k)f = \Delta_{k,1}^{\omega_{k,1}}(t_{k,1}) \cdots \Delta_{k,\,n_k}^{\omega_{k,\,n_k}}(t_{k,n_k})f, (k \in e_s),$$

following $\Delta_{k,j_k}^{\omega_{k,\,j_k}}(t_{k,j_k})f$ are finite difference function, which has direction with variables t_{k,j_k} and with order $\omega_{k,\,j_k}$, by step $t_{k,\,j_k}$ for $j = 1, \dots, n_k$ and for all and $k \in e_s$, following

$$\Delta_{k,\,j_k}^{1}(t_{k,j_k})f(\cdots, x_{k,\,j_k}, \cdots) =$$
$$f(\cdots, x_{k,\,j_k} + t_{k,\,j_k}, \cdots) - f(\cdots, x_{k,j_k}, \cdots),$$

and

$$\Delta_{k,\,j_k}^{\omega_{k,\,j_k}}(t_{k,j_k})f(\cdots, x_{k,\,j_k}, \cdots) =$$
$$\Delta_{k,\,j_k}^{1}(t_{k,j_k})\left\{\Delta_{k,\,j_k}^{\omega_{k,\,j_k}-1}(t_{k,\,j_k})f(\cdots, x_{k,\,j_k}, \cdots)\right\},$$

but when $\omega_{k,\,j_k} = 0$, then

$$\Delta_{k,\,j_k}^{0}(t_{k,\,j_k})f(\cdots, x_{k,\,j_k}, \cdots) = f(\cdots, x_{k,\,j_k}, \cdots).$$

[10, 24, 25, 27]

Let us assume that we have a basis functions $\Phi = \{\varphi_n\}_{n=1}^{\infty}$. Given function f(x) can be rewritten with this basis: $f(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x)$. Hence we obtain

$$\langle \varphi_k, f \rangle = \langle \varphi_k, \sum_{n=1}^{\infty} c_n \varphi_n(x) \rangle = \sum_{n=1}^{\infty} c_n \langle \varphi_k, f \rangle =$$
$$\sum_{n=1}^{\infty} c_n \langle \varphi_k, \varphi_n \rangle = \langle \varphi_k, \sum_{k=1}^{\infty} c_k \varphi_k(x) \rangle.$$

If introducing basis is an orthogonal basis, then we get $\langle \varphi_k, \varphi_n \rangle = n_k \delta_{kn}$, where δ_{kn} is the Kronecker delta:

$$\delta_{jn} = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}.$$

Then we hold

$$\langle \varphi_k, f \rangle = \sum_{n=1}^{\infty} c_n \langle \varphi_k, \varphi_n \rangle = \sum_{n=1}^{\infty} c_n n_k \delta_{kn}.$$

That is,

$$c_k = \frac{\langle \varphi_k, f \rangle}{n_k} = \frac{\langle \varphi_k, f \rangle}{\langle \varphi_k, \varphi_k \rangle}, k = 1, 2, \ldots$$

Let $\{\varphi_n\}_{n=0}^{\infty}$ be a linearly independent sequence of continuous functions defined for $x \in R^n$. The an orthogonal basis of functions can be found following

$$\varphi_0 = f_0, \varphi_k = f_k - \sum_{n=0}^{n-1} \frac{\langle f_k, \varphi_n \rangle}{\|\varphi_n\|^{\tau}}, n = 1, 2, \ldots$$

[5, 7, 21, 26]

Definition. We denote by

$$\mathcal{L}_{p,a,\varkappa,\tau}(G) \tag{1}$$

normed Lorentz–Morrey space of locally summability, measurable functions f, on G, with finite norm ($N^i > l^i > m^i \geq 0$, i=1,2,…,n)

$$\|f\|_{p,a,\varkappa,\tau:G} = \|f\|_{\mathcal{L}_{p,a,\varkappa,\tau}(G)} =$$

$$\begin{cases} \left\{\int_0^{\infty} \left[\prod_{k\in e_s} [t_k]_1^{-\frac{|\varkappa_k|a}{p}-1} \times \|f^*\|_{p,G_{t^\varkappa}(x)}\right]^{\tau} \prod_{k\in e_s} \frac{dt_k}{t_k}\right\}^{1/\tau}, \\ sup_{0<t<\infty} \left(\prod_{k\in e_s} [t_k]_1^{-\frac{|\varkappa_k|a}{p}} \times \|f^*\|_{p,G_{t^\varkappa}(x)}\right) \end{cases} \tag{2}$$

where $|\varkappa_k| = \sum_{j=1}^{n_k} \varkappa_{k,j}$; $[t_k]_1 = \min\{1, t_k\}$ and $f^*(t)$ is the decreasing rearrangement of f [9].

The properties of this space are main objects of Analysis. Let us give some characterization of $\mathcal{L}_{p,a,\varkappa,\tau}(G)$:

1) $\|\cdot\|_{p,a,\varkappa,\tau:G}$ is a qiasi-norm.

2) We must note that, for every $\tau > 0$

$$\mathcal{L}_{p,a,\varkappa,p}(G) = \mathcal{L}_{p,a,\varkappa}(G)$$

3) The space $\mathcal{L}_{p,a,\varkappa,\tau}(G)$ is complete.

4) For c>0 we have

$$\|f\|_{p,a,c\varkappa,\tau:G}=\frac{1}{C^{\frac{s}{\tau}}}\|f\|_{p,a,\varkappa,\tau:G}.$$

5) For any $\varkappa=(\varkappa_1,\dots,\varkappa_n)>0$ we get:

a)$\|f\|_{p,0,\varkappa,\infty:G}=\|f\|_{p,G}$;

b)$\|f\|_{p,1,\varkappa,\tau:G}\geq\|f\|_{\infty,G}$.

6) If $p\leq q,\frac{1-b}{q}\leq\frac{1-a}{p}$, $1\leq\tau_1\leq\tau_2\leq\infty$ then

$$\mathcal{L}_{q,b,\varkappa,\tau_1}(G)\subset_{>}\mathcal{L}_{p,a,\varkappa,\tau_2}(G)$$

and

$$\|f\|_{p,a,\varkappa,\tau_2:G}\leq\|f\|_{q,b,\varkappa,\tau_1:G}. \quad (3)$$

[2, 4, 8, 15, 16, 18, 23]

Some relations between this norm and some corresponding sums of Fourier coefficients are introduced for the case with a general orthonormal bounded system.

Let us take following well-known inequalities for 1<p<∞

$$c_1\|\bar{f}\|^p_{\mathcal{L}_{p,a,\varkappa,\tau}(G)}\leq\sum_{i\in Q}\prod_{k\in e_s}[t_k]_1^{-\frac{|\varkappa_k|a}{p}}\times$$

$$|r_k|_p\leq\frac{c_2}{\prod_{k\in e_s}[t_k]_1^{-\frac{|\varkappa_k|a}{p}}}\left\|D^{\bar{l}^i}f\right\|^p_{\mathcal{L}_{p,a,\varkappa,\tau}(G)} \quad (4).$$

In addition, here

$$\overline{f(t)}=\frac{1}{\prod_{k\in e_s}[t_k]_1^{-\frac{|\varkappa_k|a}{p}}}\times\int_0^\infty f(s)\prod_{k\in s_k}\frac{ds_k}{s_k}$$

and $f^{'}(t)$ is the derivative of the function f(t). That is, $\{a_k\}_{k=1}^\infty$ are the Fourier coefficients of the function f. Here, $\{a_k{}^*\}_{k=1}^\infty$ is the nonincreasing rearrangement of the sequence $\{|a_k|\}_{k=1}^\infty$.

Let us give generalized Lorentz-Morrey space such that

$$\|f\|_{\Lambda_{p,a,\varkappa,\tau}(\omega)} = \begin{cases} \left\{\int_0^\infty \left[\prod_{k\in e_s}[t_k]_1^{-\frac{|\varkappa_k|a}{p}-1} \times \|f^*\|_{p,G_{t^\varkappa}(x)}\omega(t)\right]^\tau \prod_{k\in e_s}\frac{dt_k}{t_k}\right\}^{1/\tau}, for\ 0<\tau<\infty \\ sup_{0<t<\infty}\left(\|f^*(t)\omega(t)\|_{p,G_{t^\varkappa}(x)}\right), for\ \tau=\infty \end{cases}$$

and where $\omega(t)$ positive, having some additional growth property and $\|f\|_{\Lambda_{p,a,\varkappa,\tau}(\omega)} < \infty$.

Therefore, in this problem, if $\sum_{n=1}^N c_n\, \varphi_n$ converges, then

$$\int_0^\infty \left(f(x) - \sum_{n=1}^N c_n\, \varphi_n\right)^\tau \to 0, N \to \infty.$$

Definition: Let $f(x) = f(x_1, \dots, x_s)$ be integrability function with s variables $x_1, \dots, x_s$ defining on the R^n. The Fourier series expansion of the function f is following

$$f(\sigma) = f(\sigma_1, \dots, \sigma_s) = \int\int \cdots \int e^{-i(x_1\sigma_1+\cdots+x_s\sigma_s)} \prod_{k\in e_s} x_k.$$

Then we hold

$$f(x) = \frac{1}{2\pi}\int_0^\infty (\cdots \frac{1}{2\pi}\left\{\int_0^\infty f(x_1, \dots, x_s)\, e^{ix_s\cdot\sigma_s} d\sigma_s\right\} \times$$

$$e^{ix_{s-1}\cdot\sigma_{s-1}} d\sigma_{s-1} \cdots) e^{ix_1\cdot\sigma_1}\frac{d\sigma_1}{\sigma_1}. \quad (5)$$

We can imagine writing the Fourier series as following

$$\sum_{i\epsilon Q} c_{\sigma_1,\cdots,\sigma_s} \prod_{k\in e_s} e^{2\pi\sigma_k\cdot x_k}.$$

The Fourier series expansion in n dimensional is approximated following

$$f(x) = \sum_{i \in Z^n} c_i \prod_{k \in e_s} e^{2\pi\sigma_k \cdot x_k}.$$

The Fourier coefficients ($\hat{f} = c_n$) can be defined by the integral

$$\hat{f} = \int_0^\infty \int_0^\infty e^{-2\pi i \sigma_1 x_1} e^{-2\pi i \sigma_2 x_2} \dots e^{-2\pi i \sigma_s x_s} f(x_1, \cdots x_s) \prod_{k \in e_s} x_k$$

$$r_k = r_k(f) = \int_0^\infty f(x) \varphi_k(x) dx, k \in N.$$

[11, 12, 22]

II. SOME MAIN RESULTS

Theorem (Generalized Parseval): Let $f \in \mathcal{L}_{p,a,\varkappa,\tau}(G)$. Then

$$\|f\|^\tau_{\mathcal{L}_{p,a,\varkappa,\tau}(G)} = \sum_{k=1}^\infty |r_k|^\tau$$

where

$$r_k = r_k(f) = \int_0^\infty \left| \prod_{k \in e_s} [t_k]_1^{-\frac{|\varkappa_k| a}{p} - 1} \times f(x) \right| \times$$

$$\prod_{k \in e_s} e^{2\pi\sigma_k \cdot x_k} \prod_{k \in e_s} dx_k$$

are the Fourier coefficients of the functions f with respect to the trigonometric system.

Proof: Inverse of Fourier transformation is

$$f(x) = \sum_{k=1}^\infty r_k(\sigma) \prod_{k \in e_s} e^{2\pi\sigma_k \cdot x_k}$$

Use these two properties to rewrite the left-hand side of this theorem:

$$\int_0^\infty |f(x)|^\tau \prod_{k \in e_s} dx_k = \int_0^\infty |f(x)| \cdot |f(x)| \cdots |f(x)| \prod_{k \in e_s} x_k =$$

$$\sum_{k=1}^\infty |r_k| \{ \cdots \sum_{k=1}^\infty |r_k(\sigma)| \prod_{k \in e_s} e^{2\pi\sigma_k \cdot x_k} \}.$$

Then we hold

$$\int_0^\infty |f(x)|^\tau \prod_{k\in e_s} dx_k = \sum_{k=1}^\infty |r_k| \left\{ \cdots \sum_{k=1}^\infty |r_k(\sigma)| \prod_{k\in e_s} e^{2\pi\sigma_k \cdot x_k} \right\} =$$

$$= \sum_{k=1}^\infty |r_k| \cdots \sum_{k=1}^\infty |r_k^*|.$$

Taking generalized Cauchy-Schwarz and Holder inequalities we have

$$\int_0^\infty |f(x)|^\tau \prod_{k\in e_s} dx_k = \sum_{k=1}^\infty |r_k| \cdots \sum_{k=1}^\infty |r_k^*| = \sum_{k=1}^\infty |r_k|^\tau.$$

We must note that, Bessel inequality holds for any general orthonormal system. Let the function f be periodic with period 1 and integrable on $[0, \infty)$ and $\Phi = \{\varphi_n\}_{n=1}^\infty$ be an orthogonal system. The numbers

$$r_n = r_n(f) = \int_0^\infty |f(x)\varphi_k(x)| \prod_{k\in e_s} dx_k, n \in N,$$

are called the Fourier coefficients of the function f with respect to the system $\Phi = \{\varphi_n\}_{n=1}^\infty$.

Theorem (Bessel F.): Let $\Phi = \{\varphi_k\}_{k=1}^\infty$ are orthonormal system in $\mathcal{L}_{p,a,\varkappa,\tau}(G), f \in \mathcal{L}_{p,a,\varkappa,\tau}(G)$, and $r_k = r_k(f) = \int_0^\infty f(x)\varphi_k(x) \prod_{k\in e_s} dx_k$, $k \in \{1, \cdots, \infty\}$ are the Fourier coefficients of the function f. Then

$$\sum_{k=1}^\infty |r_k|^\tau \leq \int_0^\infty \left| \prod_{k\in e_s} [t_k]_1^{-\frac{|\varkappa_k|a}{p}-1} \times f(x) \right|^\tau \prod_{k\in e_s} \frac{dx_k}{x_k} = \|f\|^\tau_{\mathcal{L}_{p,a,\varkappa,\tau}(G)}.$$

Proof of theorem: Let us rewrite following

$$\sum_{k=1}^\infty |r_k|^\tau = \sum_{k=1}^\infty \left[\int_0^\infty f(x)\varphi_k(x) \prod_{k\in e_s} dx_k \right]^\tau.$$

If we introduce infinite sum

$$f \;\; = \sum_{k=1}^\infty \int_0^\infty (f(x)\varphi_k(x))\varphi_k(x) \prod_{k\in e_s} dx_k.$$

We know that, this series converges. With aid to Parseval's identity we have following

$$0 \leq \left\| f - \sum_{k=1}^{\infty} \int_{0}^{\infty} \left(f(x)\varphi_k(x)\right)\varphi_k(x) \prod_{k\in e_s} dx_k \right\|^{\tau} =$$

$$\|f\|^{\tau} - C_{\tau}^{1} \sum_{k=1}^{\infty} \int_{0}^{\infty} f(x) \cdot \left(\left(f(x)\varphi_k(x)\right)\varphi_k(x)\right) \prod_{k\in e_s} dx_k +$$

$$\cdots + (-1)^{\tau+1} C_{\tau}^{\tau} \sum_{k=1}^{\infty} \int_{0}^{\infty} |f(x)\varphi_k(x)|^{\tau} \prod_{k\in e_s} dx_k =$$

$$\|f\|^{\tau} - C_{\tau}^{1} \sum_{k=1}^{\infty} \int_{0}^{\infty} f(x)\, |f(x)\varphi_k(x)|^{\tau} \prod_{k\in e_s} dx_k + \cdots$$

$$(-1)^{\tau+1} C_{\tau}^{\tau} \sum_{k=1}^{\infty} \int_{0}^{\infty} |f(x)\varphi_k(x)|^{\tau} \prod_{k\in e_s} dx_k =$$

$$\|f\|^{\tau} + \sum_{k=1}^{\infty} \int_{0}^{\infty} |f(x)\varphi_k(x)|^{\tau} \prod_{k\in e_s} dx_k$$

or $0 \leq \|f\|^{\tau}$. (if τ is even)

Let us introduce Rietz F. and Ficher E. theorem for Lorentz-Morrey type spaces with many groups of variables, which is the result that given space is complete and that is, every Cauchy sequence of function in $\mathcal{L}_{p,a,\varkappa,\tau}(G)$ convergence to a function in $\mathcal{L}_{p,a,\varkappa,\tau}(G)$.

Theorem (Rietz F. and Ficher E.): Let $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ are orthonormal system in $\mathcal{L}_{p,a,\varkappa,\tau}(G)$ and $\{a_k\}_{k=1}^{\infty}$ be an arbitrary sequence of $\sum_{k=1}^{\infty}|a_k|^{\tau} < \infty$ Then there exists a function $f \in \mathcal{L}_{p,a,\varkappa,\tau}(G)$ for which the numbers a_n are its Fourier coefficients in this system and following inequality exits

$$\|f\|_{\mathcal{L}_{p,a,\varkappa,\tau}(G)}^{\tau} \leq \sum_{k=1}^{\infty} |r_k|^{\tau}.$$

Proof of the theorem: In order to proof this theorem I have to proof that, given space is complete. It has been proved in [19].

From this theorem we hold following theorem.

Teorem: (F. Hausdorf and W. Yong)

1) If $1 < \tau \leq 2, f \in \mathcal{L}_{p,a,\varkappa,\tau}(G)$ and

$$r_k = r_k(f) = \int_0^\infty \left| \prod_{k\in e_s} [t_k]_1^{-\frac{|\varkappa_k|a}{p}-1} \times f(x) \right| \prod_{k\in e_s} e^{2\pi\sigma_k\cdot x_k} \prod_{k\in e_s} dx_k$$

then we get

$$\left(\sum_{k\in Z^n} |r_k|^\rho \right)^{1/\rho} \leq \|f\|_{\mathcal{L}_{p,a,\varkappa,\rho}(G)}.$$

2) If $\rho \geq 2$ and $\{r_k\}_{k\in Z} \in \mathcal{L}_{p,a,\varkappa,\rho}(G)$ then the trigonometric series $\sum_{k\in Z} r_k e^{2\pi ikx}$ converges in the metric $\mathcal{L}_{p,a,\varkappa,\rho}(G)$ to some function f and it holds that

$$\|f\|_{\mathcal{L}_{p,a,\varkappa,\rho}(G)} \leq \left(\sum_{k\in Z^n} |r_k|^\rho \right)^{1/\rho}.$$

Where $\rho = \frac{\tau}{\tau-1}$.

Teorem (Paley R.): Let $\{\varphi_k\}_{k=1}^\infty$ be the orthonormal system on R^n such that $|\varphi_k(t)| \leq M$ for all $k \in N$ and $x \in R^n$ and $r_i = r_i(f) = \int_0^\infty f(x)\varphi_i(x) \prod_{k\in e_s} dx_k$, $k \in N$. Then we have

1) $(\sum_{i=1}^\infty |r_i|^\tau k^{\tau-2})^{1/\tau} \leq c_3 M^{\frac{2-\tau}{\tau}} \|f\|_{\mathcal{L}_{p,a,\varkappa,\tau}(G)}$,

where $1 < \tau \leq 2$ and $f \in \mathcal{L}_{p,a,\varkappa,\tau}$.

2) $\|f\|_{\mathcal{L}_{p,a,\varkappa,\tau}(G)} \leq c_4 M^{\frac{\tau-2}{\tau}} (\sum_{i=1}^\infty |r_k|^\tau \times k^{\tau-2})^{1/\tau} < \infty$

where $2 \leq \tau < \infty$ and the sequence $\{r_k\}_{k=1}^\infty$ satisfies the following condition

$$\left(\sum_{i=1}^\infty r_i \times k^{\tau-2} \right)^{1/\tau} < \infty$$

and the function f is given by the formula $f = \lim_{i\to\infty} [\sum_{i=1}^\infty r_i \varphi_k]$.

Proof: Taking inequality (2.7.3) in [17] we hold

$$\left(\sum_{k=1}^{\infty}|r_k|^{\tau}\,k^{\tau-2}\right)^{1/\tau} \le \sum_{k=1}^{\infty}\{|r_k|^{\tau}\}^{1/\tau}\sum_{k=1}^{\infty}\{k^{\tau-2}\}^{1/\tau} =$$

$$= \sum_{k=1}^{\infty}|r_k|\sum_{k=1}^{\infty}\{k^{\tau-2}\}^{1/\tau}.$$

Then taking Bessel theorem we get

$$\sum_{k=1}^{\infty}|r_k| = \|f\|_p$$

and using Hardy-Littlewood inequality follows

$$\sum_{k=1}^{\infty}\{k^{\tau-2}\}^{1/\tau} = \sum_{k=1}^{\infty} k^{\frac{\tau-2}{\tau}} =$$

$$\sum_{k=1}^{\infty}\left(k^{\frac{2-\tau}{\tau}}\right)^{-1/\tau} \le c_3 M^{\frac{2-\tau}{\tau}}.$$

Then we get given first assumption.

2) Taking the Hardy-Littlewood inequality and Minkowski's inequality we hold

$$c_4 M^{\frac{\tau-2}{\tau}}\left(\sum_{i=1}^{\infty}|r_k|^{\tau}\times k^{\tau-2}\right)^{1/\tau} \ge c_4 M^{\frac{\tau-2}{\tau}}\left(\sum_{i=1}^{\infty}|r_k|^{\tau}\times(|k|+1)^{\tau-2}\right)^{1/\tau} \ge$$

$$\left(\sum_{i=1}^{\infty}|r_k^{*}|^{\tau}\times(|k|+1)^{\tau-2}\right)^{1/\tau} \ge \int_0^{\infty}|f(x)|^{\tau}\prod_{k\in e_s}dx_k.$$

Theorem A: Let us suppose that the function f satisfying following conditions

$$|f(x_1+t_1,x_2,\cdots x_s)-f(x_1,x_2,\cdots x_s)| \le C|t_1|^{\alpha}$$

$$|f(x_1,x_2+t_2,\cdots x_s)-f(x_1,x_2,\cdots x_s)| \le C(x_1)|t_2|^{\alpha}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$|f(x_1,x_2,\cdots x_s+t_s)-f(x_1,x_2,\cdots x_s)| \le C(x_1,\cdots x_{s-1})|t_s|^{\alpha}$$

and

$$\int_{0}^{\infty} C(x_1) \prod_{k=1,\cdots n_k} dx_{1,k}\, , k\epsilon e_s < \infty$$

$$\int_{0}^{\infty} C(x_1,\cdots,x_{s-1}) \prod_{k\epsilon e_s} \prod_{k=1,\cdots n_k} dx_{1,k} < \infty, 0 < \alpha \le 1.$$

Then (5) holds, if we take limit for $N_s \to \infty, \dots, N_1 \to \infty$:

$$f(x) = \frac{1}{2\pi} \lim_{N_1\to\infty} \int_{-N_1}^{N_1} (\cdots \frac{1}{2\pi} \lim_{N_{s-1}\to\infty} \left\{ \int_{-N_{s-1}}^{N_{s-1}} \lim_{N_s\to\infty} \int_{-N_1}^{N_1} f(x_1,\dots,x_s)\, e^{ix_s\cdot\sigma_s} d\sigma_s \right\} \times$$

$$e^{ix_{s-1}\cdot\sigma_{s-1}} d\sigma_{s-1} \cdots) e^{ix_1\cdot\sigma_1} d\sigma_1. \qquad (6)$$

Proof: Taking

$$f_1(\sigma_1, x_1,\cdots,x_s) = \int_{0}^{\infty} f(x_1,\cdots,x_s)\, e^{ix_1\cdot\sigma_1} \prod_{k=1,\cdots n_k} dx_{1,k}$$

and with aid of Fubini's theorem the function $f(x_1,\cdots,x_s)$ is summarized for all $x_2,\cdots,x_s$. Following taking first condition we get

$$f(x_1,\cdots,x_s) = \lim_{N_1\to\infty} \frac{1}{2\pi} \int_{0}^{N_1} f_1(\sigma_1, x_1,\cdots,x_s) e^{i\sigma_1 x_1} \prod_{k=1,\cdots n_k} dx_{1,k}.$$

Indeed, the function $f_1(\sigma_1, x_1,\cdots,x_s)$ is summarized for all $x_3,\cdots,x_s$. In addition, with aid of given condition we have

$$|f(\sigma_1, x_2 + t_2,\cdots x_s) - f(x_1, x_2,\cdots x_s)| \le$$

$$\int_0^\infty |f(\sigma_1, x_2 + t_2,\cdots x_s) - f(x_1, x_2,\cdots x_s)| \le$$

$$|t_\alpha|^\alpha \int_0^\infty C(x_1) \prod_{k=1,\cdots n_k} dx_{1,k}.$$

Then we hold following

$$f_2(\sigma_1,\sigma_2,x_1,\cdots,x_s)=\int_0^{\infty} f_1(x_1,\cdots,x_s)\, e^{ix_2\cdot\sigma_2}\prod_{k=1,\cdots n_k} dx_{2,k}.$$

Then next expression is real

$$f_1(\sigma_1,x_1,\cdots,x_s)=\lim_{N_2\to\infty}\frac{1}{2\pi}\int_{N_2}^{N_2} f_2(\sigma_1,\sigma_2,x_1,\cdots,x_s)\, e^{ix_2\cdot\sigma_2}\prod_{k=1,\cdots n_k} dx_{2,k}.$$

Where

$$f(x_1,\cdots,x_s)$$

$$=\lim_{N_1\to\infty}\frac{1}{2\pi}\int_0^{N_1}\{\lim_{N_2\to\infty}\frac{1}{2\pi}\int_0^{N_2} f_2(\sigma_1,\sigma_2,x_3,\cdots,x_s)\, e^{i\sigma_2 x_2}\prod_{k=1,\cdots n_k} d\sigma_{2,k}\prod_{k=1,\cdots n_k} d\sigma_{1,k}.$$

Then continuing such way we get our assumption. [3, 6, 13, 14, 20, 28]

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