On the Geometrical Structure of Natural Numbers

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This work studies the natural powers of prime numbers as the building blocks of a Euclidian vector semispace. Some vectors generate the composite natural numbers by defining an appropriate geometrical norm. One also studies the structure of extended Mersenne numbers within this geometric point of view.

Further geometric applications and extensions of the powers of natural numbers are also studied with the help of inward vector operations. Two research lines follow the first discussion on the geometrical aspects of natural numbers: the extension of the Fermat theorem and the Euler-Riemann function.

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I. INTRODUCTION

In the current literature, a large volume of work describes the structure of vector spaces; see, for example, [1]. Lattices, a close concept defined over the set of natural numbers, appear as a helpful framework in several physical applications [2]. Also, a similar setup has been developed in a mathematical application context and named natural vector (semi-)[1]spaces [3-9]. Previous work on natural numbers varied problems has also been developed in this laboratory [11-22].

This paper will try to find out how starting from natural numbers, more concretely from natural prime numbers, an infinite dimensional vector space can be built up so that a conveniently defined vector norm can generate the set of natural composite numbers. Also, one can consider another kind of vector space structured from the natural powers of prime numbers, which one can connect with Mersenne numbers and the recursive generation of natural numbers. Finally, the previous framework will generate information not only on Fermat's theorem and his extension but also on the Euler-Riemann zeta function.

Essentially the present paper will possess four parts to develop the above-proposed scheme.

First, the prime numbers' powers and the accompanied structure will be minutely described. The second part will describe the infinite-dimensional natural vector space, whose vector elements one can use to generate all the natural numbers. Third, the extension of Mersenne numbers will be analyzed, along with their twins, discussing their prime and composite nature and their relationship to the recursive generation of natural numbers. Finally, with some simple operations performed over the previously constructed natural vectors, one will display information about the connection of the

[1] From now on, these constructs will be called *natural* vector spaces. They, besides the name semispaces, can also be called *orthants*.

described geometrical structure of natural numbers and the extension of Fermat's theorem, the Euler-Riemann function.

II. PRIME POWER SETS: NATURAL POWERS OF PRIME NUMBERS

Before describing a geometrical framework generating the natural number set, this section discusses some aspects of prime numbers and their natural powers.

2.1 The Set of Prime Numbers

Suppose that one defines the entire set of natural prime numbers as:

$$\mathbb{P}=\{2\}\cup\mathbb{T}, \tag{1}$$

at that point, the set $\mathbb{T}$ holds all *odd* prime numbers.

Without loss of generality, one can also suppose the set $\mathbb{P}$ ordered.

2.2 Powers of Prime Numbers

2.2.1 Prime Power Sets

One can also construct the natural power set associated with every prime number $p\in\mathbb{P}$

That is, after choosing any prime number *p*, one can structure a set of its natural powers $\mathbf{p}^{\mathbb{N}}$, using:

$$\forall p\in\mathbb{P}\wedge\forall n\in\mathbb{N}:\mathbf{p}^{\mathbb{N}}=\left\{p,p^2,p^3,\ldots p^n,\ldots\right\}; \tag{2}$$

one can call such a set a *prime power set*. The prime number *p* used to construct the set $\mathbf{p}^{\mathbb{N}}$ can be named the *prime source (or origin)* of the power set.

If one studies the prime power set of any pair of prime numbers, the corresponding prime power sets are disjoint. Then one can write:

$$\forall\left\{p,q\right\}\in\mathbb{P}\Rightarrow\mathbf{p}^{\mathbb{N}}\cap\mathbf{q}^{\mathbb{N}}=\varnothing. \tag{3}$$

2.2.2 The Union of Prime Power Sets

It is evident that the cardinality of every prime power set is equal to the cardinality of the whole set of natural numbers, that is:

$$\forall p\in\mathbb{P}:Card\left(\mathbf{p}^{\mathbb{N}}\right)=Card\left(\mathbb{N}\right). \tag{4}$$

While $\mathbb{U}$ the union of all the prime power sets is defined as:

$$\mathbb{U}=\bigcup_{\forall p\in\mathbb{P}}\mathbf{p}^{\mathbb{N}}, \tag{5}$$

when observed as a collection of natural numbers, produces what could be named a prime power set *paradox*.

Because not all natural numbers are contained in $\mathbb{U}$, therefore:

$$\mathbb{U} \subset \mathbb{N}, \tag{6}$$

then one might admit that:

$$Card(\mathbb{U}) \leq Card(\mathbb{N}), \tag{7}$$

but at the same time, by construction, every prime power subset contained in $\mathbb{U}$ has the cardinality of the natural number set.

Moreover, one can also write:

$$\mathbb{P} \subset \mathbb{U}. \tag{8}$$

2.2.3 Natural Powers of the Prime Set

The nature of the set of prime powers can be made more explicit by considering the set $\mathbb{U}$ constructed differently. By the definitions of the previous sections, one can also think of defining the *natural powers of the prime set*: $\mathbb{P}^r$, as follows:

$$\forall r \in \mathbb{N} : \mathbb{P}^r = \left\{2^r, 3^r, 5^r, \ldots p^r, \ldots\right\}. \tag{9}$$

The sequence of such powers starts at:

$$r = 1 : \mathbb{P}^1 \equiv \mathbb{P}, \tag{10}$$

then the definition above is the same as supposing that there are several sets $\mathbb{P}^r$, as one can compute any natural power of the elements of the prime set $\mathbb{P}$. As many as the cardinality of the natural number set $\mathbb{N}$. Every set $\mathbb{P}^r$ associated with the power of any prime number has the cardinality of the prime set itself:

$$\forall r \in \mathbb{N} : Card\left(\mathbb{P}^r\right) = Card(\mathbb{P}). \tag{11}$$

Then, one can construct the union of all the powers of the primes sets $\mathbb{P}^r$, which shall be coincident with the union of all prime power sets:

$$\mathbb{U} = \bigcup_{\forall r \in \mathbb{N}} \mathbb{P}^r. \tag{12}$$

However, the cardinality of the union of powers of the prime set is at least the one corresponding to the natural number set:

$$Card(\mathbb{U}) \geq Card(\mathbb{N}), \tag{13}$$

a result that paradoxically contradicts the previous cardinality considerations on the union set $\mathbb{U}$.

This paradoxical situation appears again from this alternative point of view.

2.3 The Matrix of Powers of Prime Numbers

One can reorder the two ways of constructing the union of the powers of prime numbers into an array $\mathbf{U}$ of dimensions: $\left(Card\left(\mathbb{P}\right)\times Card\left(\mathbb{N}\right)\right)$, with elements defined in the following way:

$$\mathbf{U}=\left\{u_{IJ}=\left(p_I\right)^J \middle| p_I\in\mathbb{P}\middle| I=1,Card\left(\mathbb{P}\right);J=1,Card\left(\mathbb{N}\right)\right\}, \tag{14}$$

where the rows of the matrix $\mathbf{U}$ are made by the ordinal index of the I-th prime number, and the columns of the matrix $\mathbf{U}$ follow the natural powers to which the prime number in question is raised.

One can write a partial matrix $\mathbf{U}$ example using this finite sample:

$$\begin{pmatrix} \dots & 2^3 & 2^2 & 2 \\ \dots & 3^3 & 3^2 & 3 \\ \dots & 5^3 & 5^2 & 5 \\ \dots & \dots & \dots & \dots \end{pmatrix}, \tag{15}$$

noting that one has reversed the ordering of columns from the customary way: using left to right order, while the usual up-to-down order for rows is maintained. The reason for reversing the column order will be evident when the related natural vector spaces are constructed below in the next section.

2.4 Ordering the Prime Power Sets and the Powers of the Prime Set

Considering the set $\mathbb{P}$ ordered, then the set $\mathbb{U}$ can be supposedly (supra)ordered, so one can write:

$$\mathbb{U}=\left\{\mathbf{2}^{\mathbb{N}},\mathbf{3}^{\mathbb{N}},\mathbf{5}^{\mathbb{N}},\dots,\mathbf{p}^{\mathbb{N}},\dots\right\}\Rightarrow\mathbf{2}^{\mathbb{N}}\prec\mathbf{3}^{\mathbb{N}}{}_{,}\prec\mathbf{5}^{\mathbb{N}}\dots\prec\mathbf{p}^{\mathbb{N}}\dots \quad , \tag{16}$$

with the symbol $\prec$ indicating a prime power set's (supra)order.

Of course, one can also (supra)order the set $\mathbb{U}$ according to its alternative construction as the union of the powers of the prime set, using the order of the natural powers of the prime set:

$$\mathbb{U}=\left\{\mathbb{P}^1,\mathbb{P}^2,\mathbb{P}^3,\dots\mathbb{P}^r,\dots\right\}\Rightarrow\mathbb{P}^1\prec\mathbb{P}^2\prec\mathbb{P}^3\prec\dots\mathbb{P}^r\prec\dots \tag{17}$$

2.5 The Complement of the Union of Prime Power Sets

The whole set of prime powers sets: $\mathbb{U}$, when compared with the set of natural numbers, permits writing the construction of a new set $\mathbb{K}$, say, which one can identify as the complement of the set $\mathbb{U}$, and defined like:

$$\mathbb{K}=\mathrm{C}\left[\mathbb{U}\right]\Rightarrow\forall x\in\mathbb{K}:x\notin\mathbb{U}. \tag{18}$$

Therefore, this is the same as considering the set $\mathbb{K}$ formed by **all** natural composite numbers, which are **not** expressible as a **unique** prime number power.

One must take into account that with this definition of the set $\mathbb{K}$, one can write:

$$\mathbb{N} = \mathbb{U} \bigcup \mathbb{K} \wedge \mathbb{U} \bigcap \mathbb{K} = \varnothing. \tag{19}$$

2.6 Some Thoughts about Prime Number Powers and Composite Natural Numbers

The nature of the elements of the prime power sets is such that one can separate them from the composite natural numbers.

Powers of prime numbers are composite numbers whose divisors are restricted to be the prime source or some of its adequate powers.

Thus, the prime powers have some *intermediate* character between the prime numbers and the composite numbers, which one can suppose constructed at least with the product of ***two** distinct* prime number powers.

Therefore, it seems adequate to partition natural numbers into three different classes: primes, prime powers, (without the first power) and composites.

III. NATURAL PRIME POWER SPACES

These preliminary considerations about composite natural numbers, primes, and their power sets, permit the definition of a vector space structure defined over the set $\mathbb{U}$, the union of prime power sets, as defined in equations (16) and (17).

3.1 Introduction

First, one can outline a *natural prime power space* as a set of infinite-dimensional vectors possessing a structure in the form of a row[2] (or column), with their components chosen over the elements belonging to the union of the prime power sets $\mathbb{U}$.

One can note an infinite-dimensional natural prime power space with the symbol: $\mathrm{V}_\infty(\mathbb{U})$. Also, to shorten this verbose naming, one can refer along this paper to a (natural) (prime) power (vector) space. That is: a *power space*, the words in parenthesis elided for literary convenience.

3.2 Canonical Basis Set

As it is well-known, one defines the canonical basis set of such a power space as an infinite set of infinite-dimensional vectors forming the row vector set **E**:

$$\forall I \in \mathbb{N} : \langle \mathbf{e}_I | = (0,0,\ldots,0,1,0\ldots,0) \Rightarrow \langle \mathbf{e}_I | = \{ e_{JI} = \delta_{JI} | \forall J \in \mathbb{N} \}. \tag{20}$$

[2] Here, Dirac's bra symbol: $\langle \mathbf{v} |$ is used to denote row vectors. Column vectors, if needed, are written with the ket symbols: $| \mathbf{v} \rangle$. That means the relationship: $\langle \mathbf{v} | = | \mathbf{v} \rangle^T$ and $| \mathbf{v} \rangle = \langle \mathbf{v} |^T$, holds; with the supraindex T meaning transposition. The present work will represent prime number collections as row vectors.

All the components in one vector belonging to the infinite canonical basis set are null $\{0\}$, except for a unique unit $\{1\}$ element at some fixed position associated with a natural order.

One can order the canonical basis vectors set: $\mathbf{E}=\{\langle\mathbf{e}_I\| \forall I \in \mathbb{N}\}$ according to the order of natural numbers. This ordering corresponds to a construction where the *I*-th canonical vector has the unit $\{1\}$ coincident with the *I*-th vector component.

From the usual definition of the Euclidean scalar product point of view, the canonical basis set is orthonormalized, and one can write:

$$\forall\{I,J\}\in\mathbb{N}:\langle\mathbf{e}_I|\mathbf{e}_J\rangle=\delta_{IJ}, \tag{21}$$

where $\forall\{I,J\}\in\mathbb{N}:\delta_{IJ}$ is a Kronecker's delta.

Suppose a square infinite-dimensional unit matrix, whose elements are Kronecker's delta symbols:

$\mathbf{I}_\infty=\{I_{\infty;IJ}=\delta_{IJ}|\forall\{I,J\}\in\mathbb{N}\}$, the rows (or columns) of such a matrix corresponding to the canonical basis set.

3.3 Canonical Prime Vectors

One can associate every vector of the canonical basis set $\mathbf{E}$ with an ordered prime number just using the simple rule, where the unit component position is substituted by a prime number, like in the ordered sequence:

$$\begin{aligned}&\langle\mathbf{2}_1|=(0,\ldots,0,0,0,2);\langle\mathbf{3}_2|=(0,\ldots,0,0,3,0);\langle\mathbf{5}_3|=(0,\ldots,0,5,0,0);\ldots\\&\ldots\langle\mathbf{p}_I|=(0,\ldots,0,p_I,0,\ldots0);\ldots\end{aligned} \tag{22}$$

where the vector subindices correspond to the natural order of the prime numbers, that is: $p_I\in\mathbb{P}$ corresponds to the *I*-th prime number. The row vector set: $\mathbf{P}=\{\langle\mathbf{p}_I\|\forall I\in\mathbb{N}\}$ can be called the *canonical prime* vector set. The canonical prime vector set forms an orthogonal set:

$$\forall\{I,J\}\in\mathbb{N}:\langle\mathbf{p}_I|\mathbf{p}_J\rangle=\delta_{IJ}\,p_I^2, \tag{23}$$

whose Euclidean norms coincide with the squared prime number associated with the involved canonical prime vector.

The canonical prime vectors can be alternatively defined as products of a canonical base vector by the corresponding ordered natural prime number, that is:

$$\forall I\in\mathbb{N}:\langle\mathbf{p}_I|=(0,\ldots,0,p_I,0,\ldots0)=p_I\langle\mathbf{e}_I|. \tag{24}$$

3.4 Dimension and Inner Dimension

The above-described construction of the power space original or the prime canonical vectors supposes every vector being of *infinite* dimension, that is, for instance:

$$\forall\langle\mathbf{e}_I|\in\mathbf{E}\wedge\forall\langle\mathbf{p}_I|\in\mathbf{P}\Rightarrow\langle\mathbf{e}_I|\in\mathrm{V}_\infty(\mathbb{U})\wedge\langle\mathbf{p}_I|\in\mathrm{V}_\infty(\mathbb{U}). \tag{25}$$

Then one can assume the existence associated with some power space elements of a vector's *inner* dimension, made by the number of components different from zero within a given power space vector.

Considering this, the inner dimension of each original or prime canonical vector is one.

3.5 Homotheties of the Canonical Prime Vectors

Also, one can study the homotheties of the prime canonical basis set vectors by the powers of the prime source number involved in each case. One can express these homothetic vectors as:

$$\forall p_I^r\in\mathbb{U}:\langle\mathbf{p}_I^r|=(p_I)^{r-1}\langle\mathbf{p}_I|=\left(0,\ldots,0,(p_I)^r,0,\ldots,0\right)=(p_I)^r\langle\mathbf{e}_I|. \tag{26}$$

According to the previous point of view, the inner dimension of the homothetic prime vectors is also the unit. They are monodimensional, say. Also, these vectors correspond to an orthogonal set whose Euclidian norms are the squared prime power:

$$\forall\{I\neq J\}\in\mathbb{N}:\langle\mathbf{p}_I^r|\mathbf{p}_J^s\rangle=0\wedge\forall\{I,r\}\in\mathbb{N}:\langle\mathbf{p}_I^r|\mathbf{p}_I^r\rangle=p_I^{2r}. \tag{27}$$

3.6 The Sum of Vectors in Power Spaces

One can define the *pseudosum* between two prime canonical basis set vectors, or their homotheties, as the commutative operation expressed by:

$$\forall I\neq J:\langle\mathbf{p}_I^r|\oplus\langle\mathbf{q}_J^s|=(1-\delta_{IJ})\left(0,\ldots,0,(p_I)^r,0,\ldots,(q_J)^s,0,\ldots,0\right), \tag{28}$$

which requires a different symbol from the usual vector addition.

Because, in the natural power space definition case, the sum of two (or more) vectors possessing in the same position a non-zero component is forbidden, which is the same as saying it doesn't exist by definition.

Considering this, the pseudosum of two canonical prime vectors, or their homotheties, results in a vector with an inner dimension of two.

Of course, one can extend the pseudosum to any number of vectors belonging to the homothetic prime canonical basis set up to the chosen inner dimension of the resultant vector.

For example, one can express the construction of a vector of inner dimension *N* as:

$$\{I_K|K=1,N\}\subset\mathbb{N}\wedge\{r_K|K=1,N\}\subset\mathbb{N}:\langle\mathbf{x}_N|=\bigoplus_{K=I_1}^{I_N}\langle\mathbf{p}_K^{r_K}|, \tag{29}$$

provided that the set of subindices corresponds to *different* prime order numbers. In this way, one can generate any power space vector within a chosen inner dimension.

3.7 Projection of a Pseudospace Vector into the Internal Space

One can project any natural power space row vector with inner space dimension N into an N-dimensional power space by using a projector matrix of dimension $(\infty \times N)$ constructed as:

$$\mathbf{J} = \left(\left| \mathbf{e}_{I_N} \right\rangle, \left| \mathbf{e}_{I_{N-1}} \right\rangle, \ldots, \left| \mathbf{e}_{I_2} \right\rangle, \left| \mathbf{e}_{I_1} \right\rangle \right). \tag{30}$$

The columns of the projection matrix $\mathbf{J}$ correspond to the canonical basis set vectors, coincident with the non-null components positions of the power space vector of inner dimension N.

Then for any power space vector, there exists a projector matrix $\mathbf{J}$ that transforms the infinite-dimensional vector into a finite-dimensional one:

$$\forall \left\langle \mathbf{v}_\infty \right| \in \mathrm{V}_\infty(\mathbb{U}) \Rightarrow \exists \mathbf{J}_{(\infty \times N)} : \left\langle \mathbf{v}_\infty \right| \mathbf{J}_{(\infty \times N)} = \left\langle \mathbf{v}_N \right| \in \mathrm{V}_N(\mathbb{U}). \tag{31}$$

3.8 Linear Combinations of the Power Space Canonical Basis Set

One can restate the paragraphs above with more straightforward vector space algebra.

Taking into account that one might express any power space vectors as a linear combination of vectors of the canonical basis set with coefficients, the coordinates with respect to such a basis set, taken from the union of prime powers set $\mathbb{U}$:

$$\Pi = \left\{ p_{I_K}^{r_K} \middle| K = 1, N \right\} \subset \mathbb{U} \Rightarrow \sum_{K=1}^{N} p_{I_K}^{r_K} \left\langle \mathbf{e}_{I_K} \right| = \left\langle \mathbf{v} \right| \in \mathrm{V}_\infty(\mathbb{U}). \tag{32}$$

It is simple to consider the set $\prod$, made by arbitrarily chosen elements of the set $\mathbb{U}$, as the coordinates of the vector $\left\langle \mathbf{v} \right|$ concerning the canonical basis set $\mathbf{E}$.

The projection of the vector $\left\langle \mathbf{v} \right|$ results in an N-dimensional vector, with inner dimension N, coincident with the cardinality of $\prod$.

That is one can write:

$$\left\langle \mathbf{v} \right| \mathbf{J} = \left\langle \mathbf{u} \right| \in \mathrm{V}_N(\mathbb{U}) \rightarrow \left\langle \mathbf{u} \right| = \left(p_{I_N}^{r_N}, \ldots, p_{I_2}^{r_2}, p_{I_1}^{r_1} \right). \tag{33}$$

3.9 Geometrical Norm of a Projected Power Space Vector

To every projected power space N-dimensional vector, one can associate a *geometrical norm* by the easily defined algorithm:

$$\forall\langle\mathbf{v}_N| = (v_N, v_{N-1}, \ldots, v_K, \ldots, v_2, v_1) \in \mathbf{V}_N(\mathbb{U}) \Rightarrow$$
$$\mathrm{G}\left[\langle\mathbf{v}_N|\right] = \prod_{K=1}^{N} v_K \rightarrow \mathrm{G}\left[\langle\mathbf{v}_N|\right] \in \mathbb{K} \quad (34)$$

It is easy to see that the natural values of the geometrical norm of a projected vector of a power space generate the *composite natural numbers* contained in the set $\mathbb{K}$.

To these elements, one can add the prime powers contained in the set $\mathbb{U}$ when also considering the geometrical norm of the homotheties of the canonical basis set.

3.10 The Odd and Even Natural Power Spaces

The constructed vectors of a natural power space can have or do not have a non-null component at the position of the first prime number $\{2\}$, associated in turn with the $\langle\mathbf{e}_1|$ canonical or prime canonical vector $\langle\mathbf{2}_1|$. All the vectors lacking the components in the first position produce, via the corresponding projections and their geometric norms, prime power, or composite **odd** natural numbers. The projected odd vectors have the form:

$$\langle\mathbf{o}| = (\langle\mathbf{v}|, 0).$$

Therefore, all vectors with this characteristic create an odd natural power subspace. Calculus of their vector projections' geometric norms to the inner space yields odd composite numbers.

To every odd natural power subspace, there exists an **even** power subspace. One can obtain the vectors belonging to such an even structure from the subspace odd vectors by adding an element of the power set $\mathbf{2}^{\mathbb{N}}$ in the first zero component. That is, for example:

$$\langle\mathbf{e}| = (\langle\mathbf{v}|, 2^r) \quad (35)$$

Therefore, to every odd power subspace, one can associate an infinite sequence of even power subspaces, where every vector, instead of a zero element within the first component, has an element of the power set $\mathbf{2}^{\mathbb{N}}$.

Then, while the geometric norm of the odd vectors generates odd composite numbers, the same operation on even vectors generates even composite numbers.

3.11. The In and Out Natural Power Spaces

One can present the same odd-even scheme symbolically:

$$\left\{Odd\left(\sim \exists\mathbf{2}^{\mathbb{N}}\right) \leftrightarrow Even\left(\exists\mathbf{2}^{\mathbb{N}}\right)\right\}, \quad (36)$$

that resumes the construction of odd and even spaces to any power space vector.

However, this possibility will also indicate that, from the point of view of the natural numbers structure, the power set $\mathbf{2}^{\mathbb{N}}$ might not necessarily be considered prevalent in front of any other odd prime power set $\mathbf{p}^{N}$.

In decimal representation, even numbers have been made significant when compared to odd numbers, because humans and other livings are two-sided. However, other creatures might be three, five, seven, or multiple-sided ... So, if they could be aware of it, the power sets: $\mathbf{3}^{\mathbb{N}}, \mathbf{5}^{\mathbb{N}}, \mathbf{7}^{\mathbb{N}}$, or $\mathbf{p}^{\mathbb{N}}$ ... can be as crucial as the set $\mathbf{2}^{\mathbb{N}}$.

In the present construction of the power spaces, one can contemplate power subspaces with the position, the *I*-th, say, corresponding to the *I*-th prime number p_I power set $\mathbf{p}_I^{\mathbb{N}}$, holding a zero $\{0\}$ instead of the corresponding prime powers.

Therefore, infinite power subspaces with the corresponding prime number zeroed from their ordered position exist. So, one can see the situation corresponding to the even numbers set is just one of many infinite possibilities.

One can thus propose a general binary classification of the kind:

$$\left\{Out\left(\sim \exists \mathbf{p}_I^{\mathbb{N}}\right) \leftrightarrow In\left(\exists \mathbf{p}_I^{\mathbb{N}}\right)\right\}, \tag{37}$$

when the prime number chosen in the above classification is $\{2\}$, the number sorting coincides with the customary *Odd-Even* one in the equation (36).

Such possibilities come from the fact that every prime number, with its power set, corresponds to one direction independent of the other in the power space. If the defined prime power space is isotropic, no direction of space has to be prevalent from the rest. One can choose any direction, and thus the same partition as customarily done with the even-odd classification of natural numbers can create two power space classes.

Of course, using large prime numbers to create alternative power subspaces to the even-odd structure, although on the same footing, might not be as enjoyable as those power subspaces created with the first prime numbers less than 20, say: $\{2,3,5,7,11,13,17,19\}$. And from this possibility, a less reduced one involving less than 10 four primes looks handy.

3.11.1 General Classification of Composite Numbers

Resuming what one discussed in the previous section: if the power set elements position $\mathbf{2}^{\mathbb{N}}$ produces composite numbers of even or odd structures, the elements of any power set $\mathbf{p}^{N}$ can do the same. They partition the power spaces into two classes, one divisible by some elements of the power set and others not. Precisely the same behavior as the even – odd numbers.

One can conclude that the even – odd classification is just a conventional particular way to study composite numbers. There are equivalent independent infinite ways to do the same partition with any prime number.

However, one can imagine a general partition of composite numbers corresponding to constructing them into classes that use the in – out binary classification discussed so far, but not referring to one prime number only, but two, three, ... and so forth.

A simple example will be the class of composite numbers not divisible by the elements of two prime power sets: $\left\{\mathbf{p}^{\mathbb{N}};\mathbf{q}^{\mathbb{N}}\right\}$, formed by the power space vectors, wherein the proper component positions of the primes $\left\{p,q\right\}$, one encounters zeros instead. Therefore, one can imagine a ternary, quaternary, ... composite number classification.

3.12 Structure of the Matrix of Prime Numbers Powers

In the equation , the already defined matrix **U**, holding all the powers of prime numbers, as defined in section *2.3*, can be revisited using the row and column structure of the prime powers described so far.

Therefore, one can write the prime powers matrix in two manners. Using the column set and the supraindex T as the transposition symbol to spare typing space:

$$\forall J \in \mathbb{N}: \left|\mathbf{p}^{[J]}\right\rangle = \left(2^J, 3^J, 5^J, \ldots p^J, \ldots\right)^T, \tag{38}$$

one can write the matrix **U** in the form of a row made of columns:

$$\mathbf{U} = \left(\ldots\left|\mathbf{p}^{[J]}\right\rangle, \ldots\left|\mathbf{p}^{[3]}\right\rangle, \left|\mathbf{p}^{[2]}\right\rangle, \left|\mathbf{p}^{[1]}\right\rangle\right). \tag{39}$$

Also, defining the matrix **U** row vector set as:

$$\forall p_I \in \mathbb{P}: \left\langle\mathbf{p}_I^{\mathbb{N}}\right| = \left(\ldots, p_I^J, \ldots, p_I^3, p_I^2, p_I^1\right), \tag{40}$$

one can also write the matrix **U** like a column made of rows:

$$\mathbf{U} = \left(\left\langle\mathbf{2}^{\mathbb{N}}\right|, \left\langle\mathbf{3}^{\mathbb{N}}\right|, \left\langle\mathbf{5}^{\mathbb{N}}\right|, \ldots, \left\langle\mathbf{p}_I^{\mathbb{N}}\right|, \ldots\right)^T. \tag{41}$$

3.12.1 Alternative Basis Sets of $V_\infty\left(\mathbb{U}\right)$.

The vectors defined in equations (38) and (40) correspond to infinite sets of infinite vectors belonging to $V_\infty\left(\mathbb{U}\right)$. However, vectors in the equation (38) are in column form for the convenience of constructing the matrix **U** columns. One can easily transpose them, forming rows, as the vectors used in previous sections. In each of the (38) and (40) vector series, every vector is linearly independent of the rest. Thus, one can alternatively use both sequences as basis sets of the space $V_\infty\left(\mathbb{U}\right)$.

As an example, take two vectors of the sequence : $\left\{\left\langle\mathbf{p}_I\right|, \left\langle\mathbf{p}_K\right|\right\}$. To show the linear independence of the vectors is sufficient to see that any (2×2) determinant with one row from the first vector and another from the second is not null:

$$\forall I \neq K \wedge \forall R \neq S : Det\begin{vmatrix} p_I^R & p_I^S \\ p_K^R & p_K^S \end{vmatrix} = p_I^R p_K^S - p_I^S p_K^R \neq 0. \quad (42)$$

One can say the same about the sequence vectors.

Consequently, the columns and rows of matrix $\mathbf{U}$ are two sets of linearly independent vectors, and one can say that because of this, the determinant of this matrix is non-null: $Det|\mathbf{U}| \neq 0$.

IV. NATURAL POWER OF PRIMES SPACES

4.1 Mersenne Numbers: Definition and Connection with Binary Representation.

Mersenne numbers are natural numbers that, in a decimal base, can be calculated as the set of powers of 2 minus one unit:

$$\forall r \in \mathbb{N} : M_{(-)}(r) = 2^r - 1. \quad (43)$$

They have another interesting property. When represented in binary form can be written as the binary unity vector of dimension *r*: $\langle \mathbf{1}_r |$, see, for example [10,14]. So, the unity vector $\langle \mathbf{1}_r |$ representing the vertex of an *r*-dimensional Boolean hypercube, maximally far away from the corresponding origin or binary zero vector of the same dimension: $\langle \mathbf{0}_r |$. The many applications and properties of zero and unity vectors [18-22] and the Mersenne numbers have been studied in our laboratory as two elements forming part of Boolean hypercube vertices.

4.2. Mersenne Numbers in the Prime Power Space

Due to the definition of Mersenne numbers, one can quickly write the whole set $\mathbf{M}_{(-)}^{\mathbb{N}}$ of Mersenne numbers with a simple algorithm using the vectors of the prime power spaces. For example:

$$\langle \mathbf{M}_{(-)}^{\mathbb{N}} | = \langle \mathbf{2}^{\mathbb{N}} | - \langle \mathbf{1}_{\mathbb{N}} |. \quad (44)$$

Where one uses row vectors to contain the implied sets. That is: $\langle \mathbf{1}_{\mathbb{N}} |$ contains a vector of the dimension of the natural number cardinality, made entirely by ones $\{1\}$. The already described vector $\langle \mathbf{2}^{\mathbb{N}} |$ contains the ordered set of powers of 2, acting as coordinates associated with the canonical basis set:

$$\langle \mathbf{2}^{\mathbb{N}} | = \left(\ldots, 2^K, \ldots, 2^3, 2^2, 2\right) = \bigoplus_{K=1}^{\infty} 2^K \langle \mathbf{e}_K |. \quad (45)$$

Finally, the vector $\langle \mathbf{M}_{(-)} |$ holds all the elements forming the set of Mersenne numbers, which also can be written as:

$$\left\langle \mathbf{M}_{(-)}^{\mathbb{N}} \right| = \underset{K=1}{\overset{\infty}{\oplus}} M(K) \langle \mathbf{e}_K | = \underset{K=1}{\overset{\infty}{\oplus}} \left(2^K - 1\right) \langle \mathbf{e}_K | \\ = \underset{K=1}{\overset{\infty}{\oplus}} 2^K \langle \mathbf{e}_K | - \underset{K=1}{\overset{\infty}{\oplus}} \langle \mathbf{e}_K | = \left\langle \mathbf{2}^{\mathbb{N}} \right| - \langle \mathbf{1}_{\mathbb{N}} | . \tag{46}$$

4.3. Mersenne Twins

In the same way that Mersenne numbers are defined, one can construct a set of Mersenne twins, see reference [14], which one can describe as:

$$\forall r \in \mathbb{N} : M_{(+)}(r) = 2^r + 1, \tag{47}$$

and the prime power space form is immediately written like this:

$$\left\langle \mathbf{M}_{(+)}^{\mathbb{N}} \right| = \left\langle \mathbf{2}^{\mathbb{N}} \right| + \langle \mathbf{1}_{\mathbb{N}} | . \tag{48}$$

Mersenne twins also have a characteristic binary representation, which one can write as the $(r+1)$ -dimensional bit string: $\left(1, \left\langle \mathbf{0}_{(r-1)} \right|, 1\right)$. Note that the canonical basis set **E**, now with every vector taken in binary form as a bit string, is the basis that permits the expression of natural numbers as sums of powers of two.

And this one can make it as in the previous equations used to express the Mersenne numbers in terms of the binary canonical basis set.

4.4. The Linear Independence of Mersenne Vectors ad other Questions

The vector pair defined in equations (44) and (48) constitute two linearly independent vectors as any (2×2) determinant with two elements of the Mersenne vector and two taken from the Mersenne twin vector is non-zero:

$$\forall p \neq q : Det \begin{vmatrix} 2^p - 1 & 2^q - 1 \\ 2^p + 1 & 2^q + 1 \end{vmatrix} = 2\left(2^p - 2^q\right) \neq 0 . \tag{49}$$

Therefore, the vector pair: $\left\{ \left\langle \mathbf{M}_{(-)}^{\mathbb{N}} \right|, \left\langle \mathbf{M}_{(+)}^{\mathbb{N}} \right| \right\}$ is linearly independent. The elements of such vectors play different roles in natural number generation [14,19,20] and related subjects [21,22].

Moreover, several elements of $\left\langle \mathbf{M}_{(-)}^{\mathbb{N}} \right|$ are prime and well-studied; see, for example [23]. The same occurs with the twin counterpart vector $\left\langle \mathbf{M}_{(+)}^{\mathbb{N}} \right|$ but has not been so exhaustively studied, possibly because of its binary representation and scarcity of primes in the sequence compared with the Mersenne counterpart.

As evident upon observing both vectors' structure, the basic vector in natural number generation corresponds to the elements of the power vector $\left\langle \mathbf{2}^{\mathbb{N}} \right|$, as one has previously analyzed [14,19].

The use of vector $\left\langle \mathbf{2}^{\mathbb{N}} \right|$ elements to recursively generate natural numbers has been discussed in deep [19] and will not be repeated.

4.5. Extended Mersenne Numbers

Although the representation of numbers constructed with a similar algorithm to Mersenne's does not present a simplistic binary representation, they can also be a source of large prime numbers. Some previous study has been performed [14,18]. Still, it is interesting to rewrite in the present notation and framework the possibility of writing extended Mersenne numbers involving the rest of the primes.

One can observe that, for instance, one can construct a set of odd numbers using the prime number 3 as a source, like:

$$\forall r \in \mathbb{N}: L(r) = 3^r \pm 2, \tag{50}$$

which can be associated with a vector structure similar to the one used in the Mersenne twin case, as one can write instead of the equation (50), the expression:

$$\left\langle \mathbf{L}_3^{\mathbb{N}} \right| = \left\langle \mathbf{3}^{\mathbb{N}} \right| \pm 2 \left\langle \mathbf{1}_{\mathbb{N}} \right|. \tag{51}$$

Some of these extended Mersenne twins can be prime, like the large one obtained in the following way:

$$3^{123} + 2 = \\ 48519278097689642681155855396759336072749841943521979872829,$$

which is composed of 59 digits. The corresponding twin with a negative form is a composite number.

This situation permits constructing a superfamily of extended Mersenne numbers using the notation that one has used.

Then the sequence:

$$\forall p \in \mathbb{P}: \left\langle \mathbf{L}_{p\ell}^{\mathbb{N}} \right| = \left\langle \mathbf{p}^{\mathbb{N}} \right| \pm (\ell = 2, p-1; 2) \left\langle \mathbf{1}_{\mathbb{N}} \right|, \tag{52}$$

provides a set of extended Mersenne vectors depending on the number of possible vectors homothetic to the unity vector $\left\langle \mathbf{1}_{\mathbb{N}} \right|$.

V. EXTENDED APPLICATIONS OF NATURAL NUMBERS POWER SETS

In the present section, one will study two applications of natural number powers to show the far-reaching potential of what has been commented on until now.

5.1 Fermat's Theorem and its Extension.

This section will discuss the extension of the Fermat theorem to higher dimensions than the usual one having the origin of the so-called Pythagorean triples. One can find previous work in our laboratory in the references [13,15,16].

5.1.1 Sums of Powers of the Natural Numbers Set

Let's recall the natural number set $\mathbb{N}$. The symbol $\mathbb{N}^{[P]}$ means the *P*-th power of *all* the elements of the natural set $\mathbb{N}$.

That is, if $\mathbb{N}=\{1,2,3\ldots,n,\ldots\}$ then: $\mathbb{N}^{[P]}=\{1,2^P,3^P\ldots,n^P,\ldots\}$.

Therefore: $\mathbb{N}^{[P]}\subset\mathbb{N}$. In the same way, as one has discussed in the first paragraph of this work referred to the prime number set $\mathbb{P}$.

The exciting thing about this definition is the possibility to sum the elements of every natural power set an indefinite number of times. The present sum formalism now will differ from the earlier definition of the pseudosum connected with powers of prime numbers and acting on homothetic canonical basis vectors.

One can construct sets like: $\mathbb{S}_{[Q]}^{[P]}$, that is, sets of *Q* natural numbers, all *P* powered and ordered. One can write to express this operation of summing equal powers of *Q* natural numbers as:

$$\forall\{P,Q\}\in\mathbb{N}:\mathbb{S}_{[Q]}^{[P]}=\left\{s_K^P\,\middle|\,K=1,Q\right\}\subset\mathbb{N}^{[P]}\Rightarrow S_{[Q]}^{[P]}=\left\langle\mathbb{S}_{[Q]}^{[P]}\right\rangle=\sum_{K=1}^{Q}s_K^P\,. \tag{54}$$

However, this does not imply that all the elements of the resultant sums $S_{[Q]}^{[P]}$ belong necessarily to $\mathbb{N}^{[P]}$.

For example, one can write the two-dimensional Fermat's theorem as:

$$\forall P\in\mathbb{N}\wedge P>2:S_{[2]}^{[P]}=\left\langle\mathbb{S}_{[2]}^{[P]}\right\rangle\notin\mathbb{N}^{[P]}. \tag{55}$$

But it is empirically true, see for instance [13], that one can also write the *Q*-dimensional version of the usual Fermat's theorem:

$$\forall Q\in\mathbb{N}\wedge Q\geq 2:\exists S_{[Q]}^{[2]}=\left\langle\mathbb{S}_{[Q]}^{[2]}\right\rangle\Rightarrow S_{[Q]}^{[2]}\in\mathbb{N}^{[2]}. \tag{56}$$

The same is empirically admissible [15] for higher powers and sums, like:

$$\forall P\in\mathbb{N}\wedge P>3:S_{[3]}^{[P]}=\left\langle\mathbb{S}_{[3]}^{[P]}\right\rangle\notin\mathbb{N}^{[P]}, \tag{57}$$

and also, it can be written that:

$$\forall Q \in \mathbb{N} \wedge Q \geq 3 :\Rightarrow \exists S_{[Q]}^{[3]} = \left\langle \mathbb{S}_{[Q]}^{[3]} \right\rangle \in \mathbb{N}^{[3]}, \tag{58}$$

it corresponds to behavior similar to Fermat's theorem but involves sums of $\mathbb{N}^{[3]}$ elements.

Perhaps this comportment can be generalized in the terms:

$$\forall P \in \mathbb{N} \wedge P > N : S_{[N]}^{[P]} = \left\langle \mathbb{S}_{[N]}^{[P]} \right\rangle \notin \mathbb{N}^{[P]} \tag{59}$$

and

$$\forall Q \in \mathbb{N} \wedge Q \geq N :\Rightarrow \exists S_{[Q]}^{[N]} = \left\langle \mathbb{S}_{[Q]}^{[N]} \right\rangle \in \mathbb{N}^{[N]}. \tag{60}$$

Although such two statements have not been fully proved, even empirically, one shall understand them as conjectures.

There is an exception when $N = 4$ where it seems, after a large sample of tested cases, that:

$$\exists S_{[4]}^{[5]} = \left\langle \mathbb{S}_{[4]}^{[5]} \right\rangle \in \mathbb{N}^{[5]}, \tag{61}$$

from vast computational experience, it seems that one can also write:

$$\forall P > 5 : S_{[4]}^{[P]} = \left\langle \mathbb{S}_{[4]}^{[P]} \right\rangle \notin \mathbb{N}^{[P]}. \tag{62}$$

Provisionally, in the light of the gathered empirical results, one can write a generalized Fermat theorem provided that the condition $\forall N \geq 5$ holds:

$$\forall P \in \mathbb{N} \wedge P > N > 5 : S_{[N]}^{[P]} = \left\langle \mathbb{S}_{[N]}^{[P]} \right\rangle \notin \mathbb{N}^{[P]} \tag{63}$$

and

$$\forall Q \in \mathbb{N} \wedge Q \geq N :\Rightarrow \exists S_{[Q]}^{[N]} = \left\langle \mathbb{S}_{[Q]}^{[N]} \right\rangle \in \mathbb{N}^{[N]}. \tag{64}$$

Due to the combinatorial explosion, expression (64) appears very hard to test for large natural numbers.

5.1.2. Minkowski Spaces and Extended Fermat Theorem

Natural vectors might possess a zero norm in Minkowski spaces; see, for example, [12,16,25]. Therefore, one might define the Fermat theorem in a $[2+1]$ space, that is, a three-dimensional space, not a two-dimensional one...

The notation $[N+M]$ in the context of Minkowski spaces corresponds to one of such structures possessing *N*-dimensional Euclidian and *M*-dimensional Minkowskian parts.

Concerning understanding the idea of the Minkowskian part, one supposes to write the metric vector in one of these spaces as the row vector: $\left(\langle \mathbf{1}_N | ; -\langle \mathbf{1}_M | \right)$.

In the following extended Fermat theorem discussion, one will use Minkowski spaces of the kind: $[N+1]$.

Let us suppose that one can describe the possible Fermat vectors by the symbol: $\{[N+1],P\}$ involving the dimension of the space and the power of the natural number set.

Then a set of vectors can be defined, like: $\mathbb{X}\{[N+1],P\}$. One can construct vectors with elements of the natural power set $\mathbb{N}^{[P]}$ such that: (1) do not contain 0 nor 1, (2) are all different, and (3) possess an ascending order.

Such vectors might be called perfect whole vectors [9,24]. One might define them as:

$$\begin{gathered}\langle\mathbf{x}|\in\mathbb{X}\{[N+1],P\}\Rightarrow\langle\mathbf{x}|=\left(x_1^P;x_2^P;x_3^P;\ldots x_N^P;x_{N+1}^P\right)\\ \wedge\left\{x_I^P\,|\,I=1,N+1\right\}\subset\mathbb{N}^{[P]}\wedge x_1^P<x_2^P<x_3^P<\ldots x_N^P<x_{N+1}^P\end{gathered}\quad(65)$$

One can suppose these vectors to belong to a vector space $\mathrm{V}_{(N+1)}\left(\mathbb{N}^{[P]}\right)$ where scalar products are subject to a metric vector which one can write like $\left(\langle\mathbf{1}_N|;-1\right)$.

Then a Minkowski norm over the set $\mathbb{X}\{[N+1],P\}$ is simply well-defined corresponding to the algorithm:

$$M\left[\langle\mathbf{x}|\right]=\langle\langle\mathbf{m}|*\langle\mathbf{x}|\rangle=\left(\sum_{I=1}^{N}x_I^P\right)-x_{N+1}^P\Leftarrow\langle\mathbf{m}|=\left(\langle\mathbf{1}_N|;-1\right),\quad(66)$$

such that one can define a Fermat vector of order P as:

$$\langle\mathbf{f}|\in\mathbb{X}\{[N+1],P\}\wedge M\left[\langle\mathbf{f}|\right]=0.\quad(67)$$

If one finds a Fermat vector, all its homotheties are Fermat vectors. That is, the following property holds:

$$\forall\langle\mathbf{f}|:M\left(\langle\mathbf{f}|\right)=0\Rightarrow\forall\lambda\in\mathbb{N}:M\left(\lambda\langle\mathbf{f}|\right)=0.\quad(68)$$

There must be a *simple* and general reason providing the clue about the growing scarcity or lack of Fermat vectors when the parameters N and P become large. If found, it could also be a crucial mathematical result that can illuminate the nature of the extension of the Fermat theorem.

5.1.3. Copies of the Whole Perfect Vectors

One can define copies of all the perfect whole [9,24] vectors in all the vector spaces defined over the powers of the natural number set $\mathbb{N}^{[P]}$.

One can consider the vector sets $\mathbb{X}\{[N+1],P\}$, starting from $N=2$ and with larger values, as copies of themselves. Meaning that, provided the arrangement of increasing powers:

$$\mathbb{X}\{[N+1],1\} \to \mathbb{X}\{[N+1],2\} \to \dots\ \mathbb{X}\{[N+1],P\} \to \dots, \tag{69}$$

a one-to-one correspondence connects every element of all the sets in the sequence, involving all dimensions and powers. But such a correspondence does not imply all the vectors within the sequence fulfill the property of being Fermat vectors.

5.2 Inward Inverse of a Prime Power Vector and the Euler-Riemann's Zeta Function

This section will present the connection between the vectors made of natural number powers with the Euler-Riemann function [1,26,27].

5.2.1. Inward Powers of the Prime Number Vector

Among the vector structures, which one can construct within the vector space $V_\infty(\mathbb{U})$, appear the vectors whose elements are sequences of the same power, *r* say, associated with the prime number set $\mathbb{P}$. One can write such vectors as the columns of the matrix **U**, as defined in the equation (38),or use the transposed vectors directly in the form of the rows instead:

$$\forall r \in \mathbb{N}: \langle \mathbf{p}^{[r]} | = (\dots, p^r, \dots, 5^r, 3^r, 2^r) \in V_\infty(\mathbb{U}) . \tag{70}$$

One can obtain such kind of vectors from the initial prime number vector holding the ordered prime number set $\mathbb{P}$ as its elements:

$$\langle \mathbf{p} | = (\dots, p, \dots, 5, 3, 2) \in V_\infty(\mathbb{U}) \tag{71}$$

through the inward power operation:

$$\langle \mathbf{p}^{[2]} | = \langle \mathbf{p} | * \langle \mathbf{p} | = (\dots, p \cdot p, \dots, 5 \cdot 5, 3 \cdot 3, 2 \cdot 2) = (\dots, p^2, \dots, 5^2, 3^2, 2^2), \tag{72}$$

where one multiplies the vector elements in the same position. Thus, the repetition of the inward power produces the vectors of the equation :

$$\langle \mathbf{p}^{[r]} | = \mathop{*}_{K=1}^{r} \langle \mathbf{p} | . \tag{73}$$

5.2.2. Inverse Powers of the Prime Number Vector

The vector $\langle \mathbf{p} |$ was described in a previous paper as whole and perfect [9,24], meaning a vector with components ordered and non-zero. One has used a similar arrangement to study the vectors in extending the Fermat theorem in paragraph 5.1.

In this case, one can define a more general possibility by constructing the inward inverse of a given vector. In the present case, the inward inverses of the perfect vectors $\langle \mathbf{p}^{[r]} |$ are easy to define using the following algorithm:

$$\langle \mathbf{p}^{[-r]} | = \left(..., p^{-r}, ..., 5^{-r}, 3^{-r}, 2^{-r} \right) , \quad (74)$$

where one can use as a source the inward inverse of the vector $\langle \mathbf{p} |$ as defined in the equation :

$$\langle \mathbf{p}^{[-1]} | = \left(..., p^{-1}, ..., 5^{-1}, 3^{-1}, 2^{-1} \right), \quad (75)$$

upon the procedure of the inward power construction, which in this case is written like this:

$$\langle \mathbf{p}^{[-r]} | = \mathop{*}_{K=1}^{r} \langle \mathbf{p}^{[-1]} | . \quad (76)$$

Of course, the infinite vector set: $\left\{ \langle \mathbf{p}^{[-r]} | \middle| \forall r \in \mathbb{N} \right\}$ no longer belongs to the vector space $\mathrm{V}_{\infty}(\mathbb{U})$ but to an extended one. To define this new vector space over all the powers described by the integer set, one can define the set $\mathbb{U}^{[-1]}$ simply using the following construct:

$$\forall p^{r} \in \mathbb{U} \Rightarrow p^{-r} \in \mathbb{U}^{[-1]} \quad (77)$$

in such a manner that:

$$\mathbb{W} = \mathbb{U} \bigcup \mathbb{U}^{[-1]} \subset \mathbb{Q} , \quad (78)$$

therefore, one can suppose that:

$$\forall p \in \mathbb{P} \wedge \forall s \in \mathbb{Z} : p^{s} \in \mathbb{W} . \quad (79)$$

Consequently, one can redefine the vector space of the prime powers as the vector space of the positive and negative prime powers $\mathrm{V}_{\infty}(\mathbb{W})$.

Of course, the vector pair: $\left\{ \langle \mathbf{p}^{[r]} | ; \langle \mathbf{p}^{[-r]} | \right\}$ can be considered the inverse one from the other, and one can see that their inward product produces the *unity* vector of the appropriate dimension:

$$\langle \mathbf{p}^{[r]} | * \langle \mathbf{p}^{[-r]} | = \langle \mathbf{p}^{[-r]} | * \langle \mathbf{p}^{[r]} | = \langle \mathbf{1}_{Card(\mathbb{P})} | . \quad (80)$$

Furthermore, the scalar product of any whole perfect [9,24] vector and its inward inverse is the space dimension, as one can write:

$$\left\langle \langle \mathbf{p}^{[r]} | * \langle \mathbf{p}^{[-r]} | \right\rangle = \left\langle \langle \mathbf{p}^{[-r]} | * \langle \mathbf{p}^{[r]} | \right\rangle = \left\langle \langle \mathbf{1}_{Card(\mathbb{P})} | \right\rangle = Card(\mathbb{P}) . \quad (81)$$

5.2.3. Splitting the Riemann Zeta Function

One can define the well-known Riemann zeta function $\zeta(z)$ as a complex variable function possessing a simple form:

$$\zeta(z)=\sum_{n=1}^{\infty}n^{-z}=\sum_{n\in\mathbb{N}}n^{-z}. \tag{82}$$

However, it can be split in two contributions, whenever one splits the natural number set into the prime set $\mathbb{P}$, which we have been dealing with in the present study, and the composite natural numbers, $\mathbb{K}$ say, in such a way that one can write:

$$\mathbb{N}=\mathbb{P}\cup\mathbb{K}, \tag{83}$$

therefore, one can rewrite the equation (82) like this:

$$\forall z\in\mathbb{C}:\ \zeta(z)=1+\pi(z)+\kappa(z)=$$
$$1+\sum_{p\in\mathbb{P}}p^{-z}+\sum_{k\in\mathbb{K}}k^{-z}=1+\left(2^{-z}+3^{-z}+5^{-z}+\ldots\right)+\left(4^{-z}+6^{-z}+8^{-z}\ldots\right). \tag{84}$$

Of course, the variable in the functions of the equation (84) is a complex number.

Still, one can also obtain some particular values of the Riemann function for variable values lying within the natural number set, as simply: $\mathbb{N}\subset\mathbb{C}$.

The Riemann function part possessing inverses of prime numbers has been described by Euler, constituting a previous construction leading Riemann to the function $\zeta(z)$ over the complex field variables.

One can find more information in references [2,27]. Interestingly enough, the Euler function $\pi(z)$, with the variable defined as $z\in\mathbb{N}$, diverges for $z=1$, but converges for $\forall z>1$.

5.2.4. The Euler Prime Function and Generalized Inward Powers of the Prime Vector.

Then, for natural numbers, the Euler prime function values: $\pi(s)$ are related to the set of prime vector powers: $\left\{\left\langle\mathbf{p}^{[s]}\right|\middle|\forall s\in\mathbb{Z}\right\}$.

However, one can easily extend the inward powers of a vector within the real and complex fields, just extending the definitions of equations (72), (75), and (76) with a direct inward power provided with the complex field elements:

$$\forall z\in\mathbb{C}:\left\langle\mathbf{p}^{[\pm z]}\right|=\left(\ldots,p^{\pm z},\ldots,5^{\pm z},3^{\pm z},2^{\pm z}\right). \tag{85}$$

One can connect this last expression with Hadamard's [1,2,2828] extended idea of a function over a diagonal matrix, acting over an isomorphic vector form.

Once the extended diagonal powers of the equation defined, one can quickly obtain the Euler prime function $\pi(z)$, employing the complete sum of a vector, defined and applied in many cases, see references [12,16,25] for more details, along our research path, and in this paper.

5.2.5. Complete sum of a vector, norms, and multiple scalar products.

Indeed, the complete sum of any vector like:

$$\forall z \in \mathbb{C}: \langle \mathbf{p}^{[-z]} | = \left(..., p^{-z}, ..., 5^{-z}, 3^{-z}, 2^{-z}\right), \tag{86}$$

say, can be particularly described by the sum:

$$\forall z \in \mathbb{C}: \langle\langle \mathbf{p}^{[-z]} |\rangle = 2^{-z} + 3^{-z} + 5^{-z} + ... + p^{-z} + ... = \sum_{\forall p \in \mathbb{P}} p^{-z} \tag{87}$$

which is coincident with the Euler prime function so that one can write:

$$\forall z \in \mathbb{C}: \pi(z) = \langle\langle \mathbf{p}^{[-z]} |\rangle \in \mathbb{C}. \tag{88}$$

One must remark now that the resultant vector constructed as in the equation shall be considered as belonging to a broader vector space: $\langle \mathbf{p}^{[-z]} | \in \mathrm{V}_{\infty}(\mathbb{C})$.

At the same time, the Euler prime function generally possesses values within the complex field, as the equation (88) indicates. One can extend such properties to the Riemann composite function $\kappa(z)$.

The complete sum of a vector, like in the equations (87) and (88), corresponds within the vector sets one has been working up to here to the sum leading to the Euler or Riemann functions. At the same time, one can associate a vector's complete sum with a first-order vector norm, a Minkowski norm, as described in the previous paragraph about the extension of Fermat's theorem.

More than this, the inward product of two or more vectors, being a vector, the result, if put under a complete sum operation, corresponds to a norm of the order of the number of vectors involved:

$$\left\langle \mathop{*}_{K=1}^{r} \langle \mathbf{p} | \right\rangle = \langle\langle \mathbf{p}^{[r]} |\rangle = \sum_{K=1}^{\infty} p_K^r . \tag{89}$$

The same can be said when the involved vectors are different. Suppose one has a vector set like: $\left\{ \langle \mathbf{p}_{I_K}^{\mathbb{N}} \| K = 1, N \right\}$, one can construct the scalar product involving N vectors using the simple algorithm:

$$\mathop{*}_{K=1}^{N} \langle \mathbf{p}_{I_K}^{\mathbb{N}} | = \langle \mathbf{q}^{\mathbb{N}} | \Rightarrow \left(\mathbf{p}_{I_1}^{\mathbb{N}}; \mathbf{p}_{I_2}^{\mathbb{N}}; \mathbf{p}_{I_3}^{\mathbb{N}}; ...\mathbf{p}_{I_N}^{\mathbb{N}} \right) = \langle\langle \mathbf{q}^{\mathbb{N}} |\rangle \in \mathbb{N}. \tag{90}$$

The elements of the inward products vector $\langle \mathbf{q}^{\mathbb{N}} |$ also present the products of the chosen N primes, power to every natural number.

VI. DISCUSSION AND RESULTS

Along this work, one has studied the powers of prime numbers as a source to construct vectors whose elements conveniently manipulated yield composite natural numbers.

Initially, one has described the sets of powers of prime numbers as a first step to building up a Euclidian vector space.

The basis set made of the canonical basis homothecies of the prime numbers, and their natural powers can be presented, with the ancillary use of a geometric norm, as the origin of the composite numbers.

Several characteristics of some natural numbers have been discussed, among others, the possibility of describing the twins of Mersenne numbers and extending such initial Mersenne pairs to the whole set of prime numbers.

Furthermore, one has studied two applications of the powers of natural numbers.

First, in Fermat's theorem, transforming the original problem into a zero norm of a tri-dimensional vector.

Then, one has studied the extension to higher dimensions of the Fermat theorem, employing the whole natural number set. One achieves this by defining a natural Minkowski metric space, where vectors with null norms correspond to Fermat vectors, whose components fulfill the extended version of the original theorem.

Second, one has discussed the Euler and Riemann functions. One has described from the point of view of this paper the Euler function involving primes, using the whole natural set, and the Riemann function.

While one can initially keep the vectors belonging to natural spaces, the nature of the Euler and Riemann functions needs, at least, the use of natural numbers powers chosen as integers, constructed similarly as in the previous sections of the present work.

Starting from previously defined perfect whole [9,24] natural vectors containing natural numbers powers, one can construct inward inverse natural perfect whole vectors. However, the inward natural vector inverses induce the need to transport the initial algebraic tools from vector spaces defined over the natural numbers into at least vector spaces defined over the field of rational numbers.

With the variable defined over the complex field, the Riemann function furthers the original natural space simplicity into even more extended vector spaces. Therefore, one has not studied this issue in depth, as the present work essentially studies the humble set of natural numbers.

To perform the mentioned goals, one has constructed a suggestive theoretical framework where the natural numbers can be studied from many points of view. Generally speaking, one has used a set of well-known simple operations over natural whole perfect vectors.

Such operations are (1) the inward product and power of a vector, (2) a Minkowski norm related to the total sum of the components of a vector, (3) a geometric norm related to the product of the total components of a vector, and (4) the inward inverse of a vector.

Finally, it can be said, as a resumé, that a simple analysis of the natural powers of prime numbers allows for presenting a picture where vector spaces, Euclidian and Minkowskian, and natural numbers can be easily connected.

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Compliance with ethical standards

Conflict of interest: The author state that this work has no conflict of interest.

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