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Necessary and sufficient conditions are obtained under which the martingale measure is unique. A significant number of examples of the discounted evolution of risky assets are presented for which the existence of a single martingale measure is established.

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Necessary and sufficient conditions are obtained under which the martingale measure is unique. A significant number of examples of the discounted evolution of risky assets are presented for which the existence of a single martingale measure is established. An explicit construction of a single martingale measure in these cases is given. Formulas for fair price of options contracts and investor hedging strategies are provided. A parametric model of evolution of risky asset is introduced so that the single martingale measure does not depend on the entered parameters. A complete description of the family of martingale measures is given for multinomial models of the evolution of risky asset. Each martingale measure is a finite sum of the introduced spot measures. The attractive side of such models is that the lower and upper price of the interval non arbitrage prices are, respectively, the minimum and maximum of the average values of the payment functions on a set of spot measures.

A class of parametric models is introduced that describe the multinomial evolution of risky asset such that the family of martingale measures does not depend on the entered parameters.

Keywords: random process; spot set of measures; parametric model of evolution; unique martingale measure; martingale; assessment of derivatives.

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I. INTRODUCTION

This paper continues the papers [1] - [5] and generalizes them to the case of different evolutions of risky assets. These examples of evolutions are quite realistic because they contain the memory of the past and describe the phenomenon of clustering and other effects. Our results concerning construction of risk neutral measures are quite general relative to volatility evolution and therefore they contain a wide class of evolutions of risky asset. The construction of the set of martingale measures for the above class of evolution of risky asset is based on the result of the work [4] (see Lemma 5) where, for a given random variable and a measure on an abstract

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probability space, the set of all measures equivalent to the original one and such that the average value over such measures of the considered random variable is equal to zero is described. The notion of consistency of a family of measures with filtration introduced in this paper and the proven Lemma 5 [4] made it possible to propose a new method for constructing a family of martingale measures equivalent to a given measure, which is different from the Escher transformation and generalizations of Girsanov's theorem. The ideas proposed in this paper [4] made it possible to propose a recursive method for constructing a set of risk neutral measures and to give a complete description of them for a certain class of evolutions of risky asset. It turned out that it is possible to introduce a set of spot martingale measures in a recurrent way and prove that any equivalent martingale measure to the original measure is an integral over the set of spot martingale measures. The latter made it possible to establish formulas for the boundaries of non-arbitrage prices for nonnegative contingent claims, as well as a formula for the fair price of a complete hedging of systematic risk. In the paper [3], formulas for the interval of non-arbitrage prices for put and call options are found for the evolution of a risky asset occurring in accordance with the geometric Brownian motion. The work [2] contains a general construction of building risk-neutral measures by the recursive method.

In the present paper, a significant generalization of the class of evolutions of risky assets is made, which contains ARCH and GARCH processes and their generalizations.

The study of non-arbitrage markets was begun for the first time in Bachelier's work [6]. Then, in the famous works of Black F. and Scholes M. [7] and Merton R. S. [8] the formula was found for the fair price of the standard call option of European type. The absence of arbitrage in the financial market has a very transparent economic sense, since it can be considered reasonably arranged. The concept of non arbitrage in financial market is associated with the fact that one cannot earn money without risking, that is, to make money you need to invest in risky or risk-free assets. The exact mathematical substantiation of the concept of non arbitrage was first made in the papers [9], [10] [11] for the finite probability space and in the general case in the paper [12]. In the continuous time evolution of risky asset, the proof of absent of arbitrage possibility see in [13]. The value of the established Theorems is that they make it possible to value assets. They got a special name "The First and The Second Fundamental Asset Pricing Theorems." Generalizations of these Theorems are contained in papers [14], [15], [16].

If the martingale measure is not the only one for a given evolution of a risky asset, then a rather difficult problem of describing all martingale measures arises in order to evaluate, for example, derivatives.

Assessment of risk in various systems was begun in papers [17], [18], [19], [20].

Statistical studies of the time series of the logarithm of the price ratio of risky assets contain heavy tails in distributions with strong elongation in the central region. The temporal behavior of these quantities exhibits the property of clustering and a strong dependence on the past. All this should be taken into account when building models for the evolution of risky assets.

In this paper, we generalize the results of the papers [1] - [5] and construct the evolution of risky assets for which we completely describe the set of equivalent martingale measures.

The aim of this study is to describe the family of martingale measures for a wide class of risky asset evolutions. The paper proposes the general concept for constructing the family of martingale measures equivalent to a given measure for a wide class of evolutions of risky assets. In particular, it also contains the description of the family of martingale measures for the evolution of risky assets given by the ARCH [21] and GARCH [22], [23] models. In section 2, we formulate the conditions relative to the evolution of risky assets and give the examples of risky asset evolution satisfying these conditions. Section 3 contains the construction of measures by recurrent relations. It is shown that under the conditions relative to the evolution of risky asset such construction is meaningful. It is proved that the constructed set of measures is equivalent to an initial measure. In theorem 1, we are proved that under certain integrability conditions of risky asset evolution the set of constructed measures is a set of martingale measures relative to this evolution of risky asset. In section 4, a family of spot martingale measures is introduced and a set of measures is constructed from it and a family of random variables, and it is shown in Theorem 2 that the constructed family of measures is absolutely continuous with respect to the original measure. And in Theorem 3, it is proved that the family of measures constructed in this way is a family of martingale measures which are equivalent to the original measure. A complete description of all martingale measures is found in Theorem 4. Theorem 7 establishes that the infimum and supremum of the mean value of payment functions all over martingale measures equals, correspondingly, infimum and supremum of the mean value of payment functions all over spot martingale measures. Theorem 8 establishes that the constructed class of martingale measures is invariant with respect to a certain class of evolutions of risky assets. This statement is important and makes it possible to build parametric models of financial markets. In Section 5, estimates for both the lower and upper limits of the interval of non-arbitrage prices are found for the constructed parametric model. The proposed parametric model based on the canonical model of the evolution of risky asset (9), which takes into account both memory and clustering, takes into account the fact that the price of a risky asset cannot fall to zero. As a consequence of these estimates, explicit formulas for the fair prices of a superhedge in the case of the payment functions of a standard call and put options are found in Theorems 11, 12. Analogous results are found in Theorems 13 and 14 for the payment functions of Asian-type call and put options.

Theorem 15 provides estimates for the parameters through realizations of the random parametric evolution of the risky asset. In Theorems 16 - 19 the formulas for interval of non arbitrage prices and the fair prices of superhedge are given through the obtained parameter estimates.

Another parametric model of the evolution of risky assets is considered in Section 6. It differs from the previous one in that it considers the discounted evolution of risky asset. Theorems 20 - 21 are proved, in which estimates are obtained both from above and from below and established. Theorems 22 - 23 derive formulas for the fair price of a superhedge for the payment functions of call and put options, respectively. A similar result is obtained in Theorems 24 - 25 for the payment functions of Asian-type put and call options. In Theorems 26 - 29, based on the sample for the evolution of the risky asset, the formulas for the fair price of the superhedge through parameter estimation are presented. Section 7 establishes Theorem 30, which gives the necessary and sufficient conditions for the unity of an equivalent martingale measure.

In Section 8, Proposition 2 proposes a model of the financial market with a single martingale measure that is invariant with respect to the evolution of each of the assets. In Theorems 32 and 33, various examples of discounted evolutions of risky assets are presented, conditions for the existence of a single martingale measure are found, and its explicit construction is given. Formulas for fair pricing options contracts and investor hedging strategies are provided. In proposition 3, a parametric model of the evolution of risky asset is proposed; the single martingale measure constructed for this evolution does not depend on these parameters. Estimates of the model parameters were built based on the realizations of the random evolution of asset.

Section 9 contains a description of all martingale measures for the multinomial evolution of risk assets. This result is obtained in Theorem 35.

In section 10, models of incomplete financial markets are proposed for which inequalities are established for the fair price of a superhedge for various models of the evolution of risky asset. Theorem 37 establishes that for a certain class of payment functions and for a wide class of evolutions of risky assets, the fair price of the superhedge is strictly less than the price of the underlying asset. Among such payment functions is the payment function of the standard call option of the European type. Theorems 39, 40 give various examples of discounted evolutions of risky assets that satisfy the conditions of the proved theorems 35 - 37, and find the conditions under which the family of martingale measures is nonempty. Formulas for a fair superhedge price have been found. Proposition 5 contains the construction of a parametric model of an incomplete financial market, a family of martingale measures of which does not depend on the considered parameters. Proposition 6 provides an estimates of the parameters of the constructed models of incomplete markets through realizations of the considered evolutions of risky asset.

II. GENERAL ASSUMPTIONS RELATIVE TO EVOLUTIONS OF RISKY ASSETS

Let $\{\Omega_N, \mathcal{F}_N, P_N\}$ be a direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, $i = \overline{1, N}$, $\Omega_N = \prod_{i=1}^{N} \Omega_i^0$, $P_N = \prod_{i=1}^{N} P_i^0$, $\mathcal{F}_N = \prod_{i=1}^{N} \mathcal{F}_i^0$, where the σ-algebra $\mathcal{F}_N$ is a minimal σ-algebra, generated by the sets $\prod_{i=1}^{N} G_i$, $G_i \in \mathcal{F}_i^0$. On the measurable space $\{\Omega_N, \mathcal{F}_N\}$, under the filtration $\mathcal{F}_n$, $n = \overline{1, N}$, we understand the minimal σ-algebra generated by the sets $\prod_{i=1}^{N} G_i$, $G_i \in \mathcal{F}_i^0$, where $G_i = \Omega_i^0$ for $i > n$. We also introduce the probability spaces $\{\Omega_n, \mathcal{F}_n, P_n\}, n = \overline{1, N}$, where $\Omega_n = \prod_{i=1}^{n} \Omega_i^0$, $\mathcal{F}_n = \prod_{i=1}^{n} \mathcal{F}_i^0$, $P_n = \prod_{i=1}^{n} P_i^0$. There is a one-to-one correspondence between the sets of the σ-algebra $\mathcal{F}_n$, belonging to the introduced filtration, and the sets of the σ-algebra $\mathcal{F}_n = \prod_{i=1}^{n} \mathcal{F}_i^0$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}, n = \overline{1, N}$. Therefore, we don't introduce new denotation for the σ-algebra $\mathcal{F}_n$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}$, since it always will be clear the difference between the above introduced σ-algebra $\mathcal{F}_n$ of filtration on the measurable space $\{\Omega_N, \mathcal{F}_N\}$ and the σ-algebra $\mathcal{F}_n$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}, n = \overline{1, N}$.

We assume that the evolution of risky asset $\{S_n\}_{n=0}^N$, given on the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, is consistent with the filtration $\mathcal{F}_n$, that is, S_n is a $\mathcal{F}_n$-measurable. Due to the above one-to-one correspondence between the sets of the σ-algebra $\mathcal{F}_n$, belonging to the introduced filtration, and the sets of the σ-algebra $\mathcal{F}_n$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}, n = \overline{1, N}$, we give the evolution of risky assets in the form

$$\{S_n(\omega_1, \ldots, \omega_n)\}_{n=0}^N, \qquad (1)$$

where $S_n(\omega_1, \dots, \omega_n)$ is an $\mathcal{F}_n$-measurable random variable, given on the measurable space $\{\Omega_n, \mathcal{F}_n\}$. It is evident that such evolution is consistent with the filtration $\mathcal{F}_n$ on the measurable space $\{\Omega_N, \mathcal{F}_N, P_N\}$.

Further, we assume that

$$P_n((\omega_1, \dots, \omega_n) \in \Omega_n,\ \Delta S_n > 0) > 0,$$

$$P_n((\omega_1, \dots, \omega_n) \in \Omega_n,\ \Delta S_n < 0) > 0, \quad n = \overline{1, N}, \tag{2}$$

where $\Delta S_n = S_n(\omega_1, \dots, \omega_n) - S_{n-1}(\omega_1, \dots, \omega_{n-1}),\ n = \overline{1, N}$.

Let us introduce the denotations

$$\Omega_n^- = \{(\omega_1, \dots, \omega_n) \in \Omega_n,\ \Delta S_n \leq 0\}, \quad \Omega_n^+ = \{(\omega_1, \dots, \omega_n) \in \Omega_n,\ \Delta S_n > 0\}, \tag{3}$$

$$\Delta S_n^- = -\Delta S_n \chi_{\Omega_n^-}(\omega_1, \dots, \omega_n), \quad \Delta S_n^+ = \Delta S_n \chi_{\Omega_n^+}(\omega_1, \dots, \omega_n), \tag{4}$$

$$V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2) = \Delta S_n^-(\omega_1, \dots, \omega_{n-1}, \omega_n^1) + \Delta S_n^+(\omega_1, \dots, \omega_{n-1}, \omega_n^2),$$

$$(\omega_1, \dots, \omega_{n-1}, \omega_n^1) \in \Omega_n^-, \quad (\omega_1, \dots, \omega_{n-1}, \omega_n^2) \in \Omega_n^+. \tag{5}$$

Our assumptions relative to Ω_n^- and Ω_n^+ are the following

$$\Omega_n^- = \Omega_{n-1} \times \Omega_n^{0-}, \quad \Omega_n^+ = \Omega_{n-1} \times \Omega_n^{0+}, \quad \Omega_n^{0-},\ \Omega_n^{0+} \in \Omega_n^0, \quad n = \overline{1, N}, \tag{6}$$

where

$$\Omega_n^{0-} \cup \Omega_n^{0+} = \Omega_n^0, \quad n = \overline{1, N}, \tag{7}$$

$$P_n^0(\Omega_n^{0+}) > 0, \quad P_n^0(\Omega_n^{0-}) > 0, \quad n = \overline{1, N}. \tag{8}$$

Below, we give the examples of evolutions $\{S_n(\omega_1, \dots, \omega_n)\}_{n=1}^N$, for which the conditions (6) - (8) are true. Let us consider the evolution of risky asset given by the law

$$S_n(\omega_1, \dots, \omega_{n-1}, \omega_n) = S_0 \prod_{i=1}^{n} e^{\sigma_i(\omega_1, \dots, \omega_{i-1})\varepsilon_i(\omega_i)}, \quad n = \overline{1, N}, \quad S_0 > 0, \tag{9}$$

relative to which we assume that the conditions

$$\sigma_i(\omega_1, \dots, \omega_{i-1}) \geq \sigma_i^0 > 0, \quad P_i^0(\varepsilon_i(\omega_i) > 0) > 0, \quad P_i^0(\varepsilon_i(\omega_i) < 0) > 0, \quad i = \overline{1, N},$$

are true. For the evolution of risky asset (9), we have

$$\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$S_{n-1}(\omega_1,\dots,\omega_{n-1})(e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n)}-1) = \qquad (10)$$

$$d_n(\omega_1,\dots,\omega_{n-1},\omega_n)(e^{\sigma_n^0\varepsilon_n(\omega_n)}-1),$$

where

$$d_n(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$S_{n-1}(\omega_1,\dots,\omega_{n-1})\frac{(e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n)}-1)}{(e^{\sigma_n^0\varepsilon_n(\omega_n)}-1)}. \qquad (11)$$

It is evident that $d_n(\omega_1,\dots,\omega_{n-1},\omega_n) > 0$ and for Ω_n^-, Ω_n^+ the representations (6) are true with

$$\Omega_n^{0-} = \{\omega_n \in \Omega_n^0, \varepsilon_n(\omega_n) \le 0\}, \quad \Omega_n^{0+} = \{\omega_n \in \Omega_n^0, \varepsilon_n(\omega_n) > 0\}.$$

The more general example of risky asset evolution, satisfying the conditions (6) - (8), is given by the formula

$$S_n(\omega_1,\dots,\omega_n) =$$

$$S_0\prod_{i=1}^{n}(1+a_i(\omega_1,\dots,\omega_i)\eta_i(\omega_i)), \ \{\omega_1,\dots,\omega_{n-1},\omega_n\}\in\Omega_n, \ n=\overline{1,N}, \quad S_0>0, \ (12)$$

where the random values $a_n(\omega_1,\dots,\omega_{n-1},\omega_n)$, $\eta_n(\omega_n)$, $n=\overline{1,N}$, given on the probability space $\{\Omega_n,\mathcal{F}_n,P_n\}$, satisfy the conditions

$$a_n(\omega_1,\dots,\omega_{n-1},\omega_n) > 0, \qquad \sup_{\omega_n\in\Omega_n^0,\eta_n^-(\omega_n)>0}\eta_n^-(\omega_n) < \infty,$$

$$\sup_{\{\omega_1,\dots,\omega_n\}\in\Omega_n} a_n(\omega_1,\dots,\omega_{n-1},\omega_n) < \frac{1}{\sup\limits_{\omega_n\in\Omega_n^0,\eta_n^-(\omega_n)>0}\eta_n^-(\omega_n)}. \qquad (13)$$

So, for $\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)$, $n=\overline{1,N}$, the representation

$$\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$S_{n-1}(\omega_1,\dots,\omega_{n-1})a_n(\omega_1,\dots,\omega_{n-1},\omega_n)\eta_n(\omega_n) =$$

$$d_n(\omega_1,\dots,\omega_{n-1},\omega_n)\eta_n(\omega_n), \quad n=\overline{1,N}, \qquad (14)$$

is true, where $d_n(\omega_1,\dots,\omega_{n-1},\omega_n) > 0$. From the representation (14) we obtain $\Omega_n^- = \Omega_{n-1}\times\Omega_n^{0-}$, $\Omega_n^+ = \Omega_{n-1}\times\Omega_n^{0+}$, where $\Omega_n^{0-} = \{\omega_n\in\Omega_n^0, \eta_n(\omega_n)\le 0\}$, $\Omega_n^{0+} = \{\omega_n\in\Omega_n^0, \eta_n(\omega_n) > 0\}$.

Further, we assume that $P_n^0(\Omega_n^{0-}) > 0$, $P_n^0(\Omega_n^{0+}) > 0$. The measure P_n^{0-} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0-} = \Omega_n^{0-} \cap \mathcal{F}_n^0$, P_n^{0+} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0+} = \Omega_n^{0+} \cap \mathcal{F}_n^0$.

Below we give an example of discount evolution having the representation (12). Suppose that risky asset evolution is given by the formula (9) and an evolution of non risky asset is given by the law

$$B_n = \prod_{i=1}^{n} e^{r_i}, \quad 0 < r_n < \infty, \quad n = \overline{1, N}. \tag{15}$$

Let us assume that

$$P_i^0(\{\omega_i \in \Omega_i^0,\ \sigma_i^0 \varepsilon_i(\omega_i) - r_i < 0\}) > 0,$$

$$P_i^0(\{\omega_i \in \Omega_i^0, \quad \sigma_i^0 \varepsilon_i(\omega_i) - r_i > 0\}) > 0, \quad i = \overline{1, N}. \tag{16}$$

Then for the discount evolution

$$S_n^1(\omega_1, \ldots, \omega_{n-1}, \omega_n) = \frac{S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)}{B_n}, \quad n = \overline{1, N}, \tag{17}$$

the representation (12) is true, where

$$a_i(\omega_1, \ldots, \omega_{i-1}, \omega_i) = \frac{e^{\sigma_i(\omega_1, \ldots, \omega_{i-1})\varepsilon_i(\omega_i) - r_i} - 1}{e^{\sigma_i^0 \varepsilon_i(\omega_i) - r_i} - 1} \geq 1, \quad \eta_i(\omega_i) = e^{\sigma_i^0 \varepsilon_i(\omega_i) - r_i} - 1.$$

In this case,

$$\Omega_i^{0-} = \{\omega_i \in \Omega_i^0,\ \varepsilon_i(\omega_i) \leq \frac{r_i}{\sigma_i^0}\}, \quad \Omega_i^{0+} = \{\omega_i \in \Omega_i^0, \varepsilon_i(\omega_i) > \frac{r_i}{\sigma_i^0}\}, \tag{18}$$

$$\Omega_i^- = \Omega_{i-1} \times \Omega_i^{0-}, \quad \Omega_i^+ = \Omega_{i-1} \times \Omega_i^{0+}. \tag{19}$$

The evolution of risky asset, given by the formula (9), includes a wide class of evolutions of risky assets, used in economics. For example, under an appropriate choice of probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$ and a choice of sequence of independent random values $\varepsilon_i(\omega_i)$ with the normal distribution $N(0,1)$, we obtain ARCH model (Autoregressive Conditional Heteroskedastic Model) introduced by Engle in [21] and GARCH model (Generalized Autoregressive Conditional Heteroskedastic Model) introduced later by Bollerslev in [22]. In these models, the random variables $\sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $i = \overline{1, N}$, are called the volatilities which satisfy the nonlinear set of equations.

Further, we do not restrict ourselves only the above considered case of evolutions of risky assets. We assume that the random variables $\sigma_i(\omega_1, \ldots, \omega_{i-1})$ entering in the formulas (9) satisfy only the inequalities $\sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $i = \overline{1, N}$, and the random values $\varepsilon_i(\omega_i)$, $i = \overline{1, N}$, are non correlated between themselves. For example, they may be independent random values having the normal distribution with zero mean value and not only.

III. RECURSIVE CONSTRUCTION OF THE SET OF MARTINGALE MEASURES

In this section, we present the construction of the set of measures on the basis of evolution of risky asset, given by the formula (1), satisfying the conditions (6) - (8). For this purpose, we use the set of nonnegative random values $\alpha_n(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\})$, given on the probability space $\{\Omega_n^- \times \Omega_n^+, \mathcal{F}_n^- \times \mathcal{F}_n^+, P_n^- \times P_n^+\}$, $n = \overline{1, N}$, where $\mathcal{F}_n^- = \mathcal{F}_n \cap \Omega_n^-$, $\mathcal{F}_n^+ = \mathcal{F}_n \cap \Omega_n^+$. The measure P_n^- is a contraction of the measure P_n on the σ-algebra $\mathcal{F}_n^-$ and the measure P_n^+ is a contraction of the measure P_n on the σ-algebra $\mathcal{F}_n^+$. After that, we prove that this set of measures is equivalent to the measure P_N. At last, Theorem 1 gives the sufficient conditions under which the constructed set of measures is a set of martingale measures for the considered evolution of risky asset. Sometimes, we use the abbreviated denotations $\{\omega_1^1, \ldots, \omega_n^1\} = \{\omega\}_n^1$, $\{\omega_1^2, \ldots, \omega_n^2\} = \{\omega\}_n^2$.

We assume that the set of random values $\alpha_n(\{\omega_1^1, \ldots, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_n^2\}) = \alpha_n(\{\omega\}_n^1; \{\omega\}_n^2)$, $(\{\omega\}_n^1; \{\omega\}_n^2) \in \Omega_n^- \times \Omega_n^+$, $n = \overline{1, N}$, satisfies the following conditions:

$$P_n^- \times P_n^+((\{\omega\}_n^1; \{\omega\}_n^2) \in \Omega_n^- \times \Omega_n^+, \alpha_n(\{\omega\}_n^1; \{\omega\}_n^2) > 0) =$$

$$P_n(\Omega_n^-) \times P_n(\Omega_n^+), \quad n = \overline{1, N}; \tag{20}$$

$$\int\limits_{\Omega_n^0 \times \Omega_n^0} \chi_{\Omega_n^-}(\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1) \chi_{\Omega_n^+}(\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2) \times$$

$$\alpha_n(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\}) \times$$

$$\frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} dP_n^0(\omega_n^1) dP_n^0(\omega_n^2) < \infty,$$

$$(\{\omega_1^1, \ldots, \omega_{n-1}^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2\}) \in \Omega_{n-1} \times \Omega_{n-1},$$

$$(\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1}, \quad n = \overline{1, N}; \tag{21}$$

$$\int\limits_{\Omega_n^0 \times \Omega_n^0} \chi_{\Omega_n^-}(\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1) \chi_{\Omega_n^+}(\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2) \times$$

$$\alpha_n(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\}) dP_n^0(\omega_n^1) dP_n^0(\omega_n^2) = 1,$$

$$(\{\omega_1^1, \ldots, \omega_{n-1}^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2\}) \in \Omega_{n-1} \times \Omega_{n-1}, \quad n = \overline{1, N}. \tag{22}$$

In the next Lemma 1, we give the sufficient conditions under which the conditions (20) - (22) are valid.

Lemma 1. *Suppose that the evolution of risky asset, given by the formula (1), satisfies the conditions (6) - (8). If the inequalities*

$$\int\limits_{\Omega_N} \Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n) dP_N < \infty, \quad n = \overline{1, N}, \tag{23}$$

are true, then the set of bounded random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)$, satisfying the conditions (20) - (22), is a nonempty set.

Proof. If the random values

$$0 < c_1 \leq \alpha_n^1(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\}) \leq c_2 < \infty,\ n=\overline{1,N}, \tag{24}$$

are bounded as from below and above, then the random values

$$\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\}) =$$

$$\frac{\alpha_n^1(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})}{T(\{\omega_1^1,\dots,\omega_{n-1}^1\};\{\omega_1^2,\dots,\omega_{n-1}^2\})},\ n=\overline{1,N}, \tag{25}$$

where

$$T(\{\omega_1^1,\dots,\omega_{n-1}^1\};\{\omega_1^2,\dots,\omega_{n-1}^2\}) =$$

$$\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0}\chi_{\Omega_n^-}(\omega_1^1,\dots,\omega_n^1)\chi_{\Omega_n^+}(\omega_1^2,\dots,\omega_n^2)\alpha_n^1(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})\times$$

$$dP_n^0(\omega_n^1)dP_n^0(\omega_n^2),$$

is also bounded as from below and above. Really,

$$\frac{c_1}{c_2 P_n^0(\Omega_n^-)P_n^0(\Omega_n^+)} \leq \alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\}) \leq$$

$$\frac{c_2}{c_1 P_n^0(\Omega_n^-)P_n^0(\Omega_n^+)} = C_n < \infty, \quad (\{\omega\}_n^1;\{\omega\}_n^2) \in \Omega_n^- \times \Omega_n^+, \quad n=\overline{1,N}. \tag{26}$$

It is evident that the random values (25) satisfy the condition (20) - (21). Really, due to the inequalities (26), the random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2\})$, $n=\overline{1,N}$, are strictly positive. Therefore, the conditions (20) are true.

Owing to the boundedness of $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2\}) \leq C_n,\ n=\overline{1,N}$, the inequalities

$$\int\limits_{\Omega_n^0\times\Omega_n^0}\chi_{\Omega_n^-}(\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1)\chi_{\Omega_n^+}(\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2)\times$$

$$\alpha_n(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1)dP_n^0(\omega_n^2) \leq$$

$$C_n\int\limits_{\Omega_n^0}\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)dP_n^0(\omega_n^1) < \infty, \quad n=\overline{1,N}, \tag{27}$$

are true for almost everywhere $(\omega_1,\dots,\omega_{n-1}) \in \Omega_{n-1}$, $n=\overline{1,N}$, relative to the measure P_{n-1}, owing to the inequalities (23) and Foubini Theorem. This proves the inequality (21). The equality (22) is also satisfied, due to the construction of $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)$. Lemma 1 is proved.

On the basis of the set of random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)$, $n=\overline{1,N}$, constructed in Lemma 1, let us introduce into consideration the family of measure $\mu_0(A)$ on the measurable space $\{\Omega_N,\mathcal{F}_N\}$ by the recurrent relations

$$\mu_N^{(\omega_1,\dots,\omega_{N-1})}(A)=\int\limits_{\Omega_N^0\times\Omega_N^0}\chi_{\Omega_N^-}(\omega_1,\dots,\omega_{N-1},\omega_N^1)\chi_{\Omega_N^+}(\omega_1,\dots,\omega_{N-1},\omega_N^2)\times$$
$$\alpha_N(\{\omega_1,\dots,\omega_{N-1},\omega_N^1\};\{\omega_1,\dots,\omega_{N-1},\omega_N^2\})\times$$
$$\left[\frac{\Delta S_N^+(\omega_1,\dots,\omega_{N-1},\omega_N^2)}{V_N(\omega_1,\dots,\omega_{N-1},\omega_N^1,\omega_N^2)}\mu_N^{(\omega_1,\dots,\omega_{N-1},\omega_N^1)}(A)+\right.$$
$$\left.\frac{\Delta S_N^-(\omega_1,\dots,\omega_{N-1},\omega_N^1)}{V_N(\omega_1,\dots,\omega_{N-1},\omega_N^1,\omega_N^2)}\mu_N^{(\omega_1,\dots,\omega_{N-1},\omega_N^2)}(A)\right]dP_N^0(\omega_N^1)dP_N^0(\omega_N^2), \tag{28}$$

$$\mu_{n-1}^{(\omega_1,\dots,\omega_{n-1})}(A)=\int\limits_{\Omega_n^0\times\Omega_n^0}\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\times$$
$$\alpha_n(\{\omega_1,\dots,\omega_{n-1},\omega_n^1\};\{\omega_1,\dots,\omega_{n-1},\omega_n^2\})\times$$
$$\left[\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{(\omega_1,\dots,\omega_{n-1},\omega_n^1)}(A)+\right.$$
$$\left.\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{(\omega_1,\dots,\omega_{n-1},\omega_n^2)}(A)\right]dP_n^0(\omega_n^1)dP_n^0(\omega_n^2),\quad n=\overline{2,N}, \tag{29}$$

$$\mu_0(A)=\int\limits_{\Omega_1^0\times\Omega_1^0}\chi_{\Omega_1^-}(\omega_1^1)\chi_{\Omega_1^+}(\omega_1^2)\alpha_1(\omega_1^1;\omega_1^2)\times$$
$$\left[\frac{\Delta S_1^+(\omega_1^2)}{V_1(\omega_1^1,\omega_1^2)}\mu_1^{(\omega_1^1)}(A)+\frac{\Delta S_1^-(\omega_1^1)}{V_1(\omega_1^1,\omega_1^2)}\mu_1^{(\omega_1^2)}(A)\right]dP_1^0(\omega_1^1)dP_1^0(\omega_1^2), \tag{30}$$

where we put

$$\mu_N^{(\omega_1,\dots,\omega_{N-1},\omega_N)}(A)=\chi_A(\omega_1,\dots,\omega_{N-1},\omega_N),\quad A\in\mathcal{F}_N. \tag{31}$$

Lemma 2. *Suppose that the conditions of Lemma 1 are true. For the measure $\mu_0(A)$, $A\in\mathcal{F}_N$, constructed by the recurrent relations (28) - (30), the representation*

$$\mu_0(A)=\int\limits_{\Omega_N}\prod_{n=1}^N\psi_n(\omega_1,\dots,\omega_n)\chi_A(\omega_1,\dots,\omega_N)\prod_{i=1}^N dP_i^0(\omega_i) \tag{32}$$

is true and $\mu_0(\Omega_N) = 1$, *that is, the measure* $\mu_0(A)$ *is a probability measure, being equivalent to the measure* P_N, *where we put*

$$\psi_n(\omega_1, \ldots, \omega_n) = \chi_{\Omega_n^-}(\omega_1, \ldots, \omega_{n-1}, \omega_n)\psi_n^1(\omega_1, \ldots, \omega_n) +$$

$$\chi_{\Omega_n^+}(\omega_1, \ldots, \omega_{n-1}, \omega_n)\psi_n^2(\omega_1, \ldots, \omega_n), \tag{33}$$

$$\psi_n^1(\omega_1, \ldots, \omega_{n-1}, \omega_n) =$$

$$\int\limits_{\Omega_n^0} \chi_{\Omega_n^+}(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)\alpha_n(\{\omega_1, \ldots, \omega_{n-1}, \omega_n\}; \{\omega_1, \ldots, \omega_{n-1}, \omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} dP_n^0(\omega_n^2), \quad (\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1}, \tag{34}$$

$$\psi_n^2(\omega_1, \ldots, \omega_{n-1}, \omega_n) =$$

$$\int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)\alpha_n(\{\omega_1, \ldots, \omega_{n-1}, \omega_n^1\}; \{\omega_1, \ldots, \omega_{n-1}, \omega_n\})\times$$

$$\frac{\Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} dP_n^0(\omega_n^1), \quad (\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1}. \tag{35}$$

Proof. Due to Lemma 1 conditions, the set of strictly positive bounded random values $\alpha_n(\{\omega\}_n^1; \{\omega\}_n^2)$, $n = \overline{1, N}$, satisfying the conditions (20) - (22), is a non empty set. The proof of formula (32) see in [2]. To prove Lemma 2, we need to prove that $\psi_n(\omega_1, \ldots, \omega_n) > 0, \ n = \overline{1, N}$. Really,

$$\psi_n^1(\omega_1, \ldots, \omega_{n-1}, \omega_n) \geq$$

$$\frac{c_1}{c_2} \int\limits_{\Omega_n^{0+}} \frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} dP_n^0(\omega_n^2) > 0, \quad (\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1}, \tag{36}$$

$$\psi_n^2(\omega_1, \ldots, \omega_{n-1}, \omega_n) \geq$$

$$\frac{c_1}{c_2} \int\limits_{\Omega_n^{0-}} \frac{\Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} dP_n^0(\omega_n^1) > 0, \quad (\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1}. \tag{37}$$

From the inequalities (36), (37) we have what we need. To prove that $\mu_0(\Omega_N) = 1$, let us prove the equality

$$\int\limits_{\Omega_n^0} \psi_n(\omega_1, \dots, \omega_n) dP_n^0(\omega_n) = 1, \quad (\omega_1, \dots, \omega_{n-1}) \in \Omega_{n-1}, \qquad n = \overline{1, N}. \tag{38}$$

We have

$$\int\limits_{\Omega_n^0} \psi_n(\omega_1, \dots, \omega_n) dP_n^0(\omega_n) =$$

$$\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1, \dots, \omega_{n-1}, \omega_n^1)\chi_{\Omega_n^+}(\omega_1, \dots, \omega_{n-1}, \omega_n^2)\times$$
$$\alpha_n(\{\omega_1, \dots, \omega_{n-1}, \omega_n^1\}; \{\omega_1, \dots, \omega_{n-1}, \omega_n^2\})\times$$
$$\left[\frac{\Delta S_n^+(\omega_1, \dots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2)} + \right.$$
$$\left.\frac{\Delta S_n^-(\omega_1, \dots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2)}\right] dP_n^0(\omega_n^1) dP_n^0(\omega_n^2) =$$
$$\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1, \dots, \omega_{n-1}, \omega_n^1)\chi_{\Omega_n^+}(\omega_1, \dots, \omega_{n-1}, \omega_n^2)\times$$
$$\alpha_n(\{\omega_1, \dots, \omega_{n-1}, \omega_n^1\}; \{\omega_1, \dots, \omega_{n-1}, \omega_n^2\}) dP_n^0(\omega_n^1) dP_n^0(\omega_n^2) = 1. \tag{39}$$

The last equality follows from the fact that the set of random values $\alpha_n(\{\omega_1\}_n^1; \{\omega_1\}_n^2)$, $n = \overline{1, N}$, satisfies the condition (22). The equalities (38) proves that every measure (32), defined by the set of random values $\alpha_n(\{\omega_1^1, \dots, \omega_n^1\}; \{\omega_1^2, \dots, \omega_n^2\})$, $n = \overline{1, N}$, satisfying the conditions (20) - (22), is a probability measure, being equivalent to the measure P_N.

This proves Lemma 2.

Note 1. *Assume that for* $\alpha_n(\{\omega_1^1, \dots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \dots, \omega_{n-1}^2, \omega_n^2\})$, *constructed in Lemma 1, the inequalities*

$$0 < c_n \le \alpha_n(\{\omega_1^1, \dots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \dots, \omega_{n-1}^2, \omega_n^2\}) \le C_n < \infty,$$

are true. Suppose that the conditions

$$\Delta S_n^-(\omega_1, \dots, \omega_{n-1}, \omega_n) \le B_n < \infty, \quad n = \overline{1, N}, \tag{40}$$

are valid, where c_n, C_n, B_n *are constant, then the set of equivalent measures to the measure* P_N, *described in Lemma 2, is nonempty one.*

Proof. Due to Lemma 2 conditions, the equality (20) is true. Further,

$$\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1)\chi_{\Omega_n^+}(\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2)\times$$

$$\alpha_n(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)\leq B_n,$$

$$(\{\omega_1^1,\dots,\omega_{n-1}^1\};\{\omega_1^2,\dots,\omega_{n-1}^2\})\in\Omega_{n-1}\times\Omega_{n-1},\quad (\omega_1,\dots,\omega_{n-1})\in\Omega_{n-1},$$

$$\int\limits_{\Omega_n^0\times\Omega_n^0} \chi_{\Omega_n^-}(\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1)\chi_{\Omega_n^+}(\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2)\times$$

$$\alpha_n(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)=1,$$

$$(\{\omega_1^1,\dots,\omega_{n-1}^1\};\{\omega_1^2,\dots,\omega_{n-1}^2\})\in\Omega_{n-1}\times\Omega_{n-1}. \tag{41}$$

The last inequality and the equality (41) means that the conditions (20) - (22) are satisfied. Note 1 is proved.

For a nonnegative random value $f_N(\omega_1,\dots,\omega_N)$ let us define the integral relative to the measure $\mu_0(A)$, given by the formula

$$E^{\mu_0}f_N=\int\limits_{\Omega_N}\prod_{n=1}^N\psi_n(\omega_1,\dots,\omega_n)f_N(\omega_1,\dots,\omega_{N-1},\omega_N)\prod_{i=1}^N dP_i^0(\omega_i). \tag{42}$$

Theorem 1. *Suppose that the conditions of Lemma 1 are true. Then, the set of nonnegative random values* $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2), n=\overline{1,N}$, *satisfying the conditions*

$$E^{\mu_0}|\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)|=$$

$$\int\limits_{\Omega_N}\prod_{i=1}^N\psi_i(\omega_1,\dots,\omega_i)|\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)|\prod_{i=1}^N dP_i^0(\omega_i)<\infty,\quad n=\overline{1,N}, \tag{43}$$

is a nonempty one and the convex linear span of the set of measures (32), defined by the random values $\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})$, $n=\overline{1,N}$, *and satisfying the conditions (43), is a set of martingale measures, being equivalent to the measure* P_N.

Proof. Taking into account the equality (38), the right hand side of equality (43) can be written in the form

$$\int\limits_{\Omega_N} \prod_{i=1}^{N} \psi_i(\omega_1,\ldots,\omega_i)|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)| \prod_{i=1}^{N} dP_i^0(\omega_i) =$$

$$\int\limits_{\Omega_n} \prod_{i=1}^{n} \psi_i(\omega_1,\ldots,\omega_i)|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)| \prod_{i=1}^{n} dP_i^0(\omega_i) =$$

$$2\int\limits_{\Omega_{n-1}} \prod_{i=1}^{n-1} \psi_i(\omega_1,\ldots,\omega_i) \int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\alpha_n(\{\omega_1,\ldots,\omega_{n-1},\omega_n^1\};\{\omega_1,\ldots,\omega_{n-1},\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\times$$

$$dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)\prod_{i=1}^{n-1} dP_i^0(\omega_i), \quad n=\overline{1,N}. \tag{44}$$

Since the conditions of Lemma 1 are true, then the set of bounded random values $\alpha_n(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\})$, $n=\overline{1,N}$, satisfying the conditions (20) - (22), is nonempty one. From the equality (44) for the set of bounded random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)$, $n=\overline{1,N}$, figuring in Lemma 1, we obtain the inequality

$$\int\limits_{\Omega_N} \prod_{i=1}^{N} \psi_i(\omega_1,\ldots,\omega_i)|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)| \prod_{i=1}^{N} dP_i^0(\omega_i) \leq$$

$$\prod_{n=1}^{N} \frac{2c_2}{c_1 P_n^0(\Omega_n^-)P_n^0(\Omega_n^+)} \int\limits_{\Omega_N} \Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)dP_N < \infty, \quad n=\overline{1,N}. \tag{45}$$

This proves that the set of nonnegative random values $\alpha_n(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\})$, $n=\overline{1,N}$, satisfying the conditions (43), is a non empty set.

Let us prove that

$$\int\limits_{\Omega_n^0} \psi_n(\omega_1,\ldots,\omega_n)\Delta S_n(\omega_1,\ldots,\omega_n)dP_n^0(\omega_n)=0,$$

$$(\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}, \quad n=\overline{1,N}. \tag{46}$$

Really,

$$\int\limits_{\Omega_n^0} \psi_n(\omega_1,\ldots,\omega_n)\Delta S_n(\omega_1,\ldots,\omega_n)dP_n^0(\omega_n) =$$

$$\int_{\Omega_n^0}\int_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\times$$

$$\alpha_n(\{\omega_1,\dots,\omega_{n-1},\omega_n^1\};\{\omega_1,\dots,\omega_{n-1},\omega_n^2\})\times$$

$$\Big[-\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)+$$

$$\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)\Big]\,dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)=0, \qquad (47)$$

due to the condition (21).

To complete the proof of Theorem 1, let A belong to the filtration $\mathcal{F}_{n-1}$, then $A = B\times\prod_{i=n}^{N}\Omega_i^0$, where B belongs to the σ-algebra $\mathcal{F}_{n-1}$ of the measurable space $\{\Omega_{n-1},\mathcal{F}_{n-1}\}$. Taking into account the equality (39), (47), we have, due to Foubini theorem,

$$\int_{\Omega_N}\prod_{i=1}^{N}\psi_i(\omega_1,\dots,\omega_i)\chi_A(\omega_1,\dots,\omega_N)\Delta S_n(\omega_1,\dots,\omega_n)\prod_{i=1}^{N}dP_i^0(\omega_i)=$$

$$\int_{\Omega_n}\prod_{i=1}^{n}\psi_i(\omega_1,\dots,\omega_i)\chi_B(\omega_1,\dots,\omega_{n-1})\Delta S_n(\omega_1,\dots,\omega_n)\prod_{i=1}^{n}dP_i^0(\omega_i)=$$

$$\int_{\Omega_{n-1}}\prod_{i=1}^{n-1}\psi_i(\omega_1,\dots,\omega_i)\chi_B(\omega_1,\dots,\omega_{n-1})\prod_{i=1}^{n-1}dP_i^0(\omega_i)\times$$

$$\int_0\psi_n(\omega_1,\dots,\omega_n)\Delta S_n(\omega_1,\dots,\omega_n)dP_n^0(\omega_n)=0. \qquad (48)$$

The last means that $E^{\mu_0}\{S_n(\omega_1,\dots,\omega_n)|\mathcal{F}_{n-1}\}=S_{n-1}(\omega_1,\dots,\omega_{n-1})$. Since every measure, belonging to the convex linear span of the measures considered above, is a finite sum of such measures, then it is a martingale measure, being equivalent to the measure P_N. Theorem 1 is proved.

Our aim is to describe this convex span of martingale measures.

IV. INTEGRAL REPRESENTATION FOR MARTINGALE MEASURES

In this section we consider the spot measures $\mu_{\{\omega_n^1,\omega_n^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$, introduced in [2]. Let us consider the random values

$$\psi_n(\omega_1,\dots,\omega_n)=\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n)\psi_n^1(\omega_1,\dots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n)\psi_n^2(\omega_1,\dots,\omega_n), \qquad (49)$$

where

$$\psi_n^1(\omega_1,\dots,\omega_{n-1},\omega_n)=\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\times$$

$$\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)},\quad (\omega_1,\dots,\omega_{n-1})\in\Omega_{n-1}, \tag{50}$$

$$\psi_n^2(\omega_1,\dots,\omega_{n-1},\omega_n)=\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\times$$

$$\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)},\quad (\omega_1,\dots,\omega_{n-1})\in\Omega_{n-1}. \tag{51}$$

Definition 1. *Let the evolution of risky asset, given by the formula (1), satisfies the conditions (6) - (8). On the measurable space* $\{\prod_{i=1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}],\prod_{i=1}^{N}[\mathcal{F}_i^{0-}\times\mathcal{F}_i^{0+}]\}$, *being the direct product of the measurable spaces* $\{\Omega_i^{0-}\times\Omega_i^{0+},\mathcal{F}_i^{0-}\times\mathcal{F}_i^{0+}\}$, *for every point* $\{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}\}\in\prod_{i=1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]$ *let us introduce the set of spot measures (see also [2])*

$$\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)=$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\chi_A(\omega_1^{i_1},\dots,\omega_N^{i_N}),\quad A\in\mathcal{F}_N, \tag{52}$$

where $\psi_n(\omega_1,\dots,\omega_n)$ *is determined by the formulas (49) - (51).*

Let us define the integral for the random value $f_N\ (\omega_1,\dots,\omega_{N-1},\omega_N)$ relative to the measure $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ by the formula

$$\int_{\Omega_N} f_N(\omega_1,\dots,\omega_{N-1},\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}=$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})f_N(\omega_1^{i_1},\dots,\omega_N^{i_N}). \tag{53}$$

To describe the convex set of equivalent martingale measures, we introduce the family of α-spot measures, depending on the point $(\{\omega_1^1,\{\omega_1^2\},\dots,\{\omega_N^1,\{\omega_N^2\})$ belonging to $\prod_{i=1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]$ and the set of strictly positive random values

$$\alpha_n(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\}),\quad n=\overline{1,N}, \tag{54}$$

at points $W_n=(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})$, being constructed by the point $(\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\})$.

Let us determine the random values

$$\psi_n^\alpha(\omega_1,\dots,\omega_n)=\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n)\psi_n^{1,\alpha}(\omega_1,\dots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n)\psi_n^{2,\alpha}(\omega_1,\ldots,\omega_n), \quad (55)$$

$$\psi_n^{1,\alpha}(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\int\limits_{\Omega_n^0} \chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^2), \quad (\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}, \quad (56)$$

$$\psi_n^{2,\alpha}(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})\times$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1), \quad (\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}. \quad (57)$$

Let us define the set of the measures on the σ-algebra $\mathcal{F}_N$ by the formula

$$\mu_0(A) = \int\limits_{\prod_{i=1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]} \prod_{i=1}^{N} \alpha_i(\{\omega_1^1,\ldots,\omega_i^1\};\{\omega_1^2,\ldots,\omega_i^2\})\times$$

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)\prod_{i=1}^{N} d[P_i^0 \times P_i^0], \ A \in \mathcal{F}_N. \quad (58)$$

Theorem 2. *Suppose that the strictly positive random value*

$$\alpha_n(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\}), \ n = \overline{1,N}, \quad (59)$$

given on the measurable space $\{\prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}], \prod_{i=1}^{N}[\mathcal{F}_i^{0-} \times \mathcal{F}_i^{0+}]\}$, *satisfies the conditions of Lemma 1, then for the measure* $\mu_0(A)$, *given by the formula (58), the representation*

$$\mu_0(A) =$$

$$\int\limits_{\Omega_N} \prod_{i=1}^{N} \psi_i^{\alpha}(\omega_1,\ldots,\omega_i)\chi_A(\omega_1,\ldots,\omega_N)dP_N \quad (60)$$

is true.

Proof. Due to Lemma 1, the set of random values $\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})$, $n=\overline{1,N}$, satisfying the conditions (20) - (22), is a non empty set. Introduce into consideration the sequence of measures

$$\mu_{n-1}^{\omega_1,\dots\omega_{n-1}}(A)=\int\limits_{\prod_{i=n}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]}\prod_{i=n}^{N}\alpha_i(\{\omega_1^1,\dots,\omega_i^1\};\{\omega_1^2,\dots,\omega_i^2\})\times$$

$$\sum_{i_n=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{j=n}^{N}\psi_j(\omega_1,\dots\omega_{n-1},\omega_n^{i_n},\dots,\omega_j^{i_j})\chi_A(\omega_1,\dots\omega_{n-1},\omega_n^{i_n},\dots,\omega_N^{i_N})\times$$

$$dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)\dots dP_N^0(\omega_N^1)dP_N^0(\omega_N^2),\quad n=\overline{1,N}, \tag{61}$$

and find the recurrent relations between them. Using Fubini Theorem, we have

$$\mu_{n-1}^{\omega_1,\dots\omega_{n-1}}(A)=$$

$$\int\limits_{\Omega_n^{0-}}\int\limits_{\Omega_n^{0+}}dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})\sum_{i_n=1}^{2}\psi_n(\omega_1,\dots\omega_{n-1},\omega_n^{i_n})$$

$$\int\limits_{\prod_{i=n+1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]}\prod_{i=n+1}^{N}\alpha_i(\{\omega_1^1,\dots,\omega_i^1\};\{\omega_1^2,\dots,\omega_i^2\})\times$$

$$\sum_{i_{n+1}=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{j=n+1}^{N}\psi_j(\omega_1,\dots\omega_{n-1},\omega_n^{i_n},\dots,\omega_j^{i_j})\chi_A(\omega_1,\dots\omega_{n-1},\omega_n^{i_n},\dots,\omega_N^{i_N})\times$$

$$dP_{n+1}^0(\omega_{n+1}^1)dP_{n+1}^0(\omega_{n+1}^2)\dots dP_N^0(\omega_N^1)dP_N^0(\omega_N^2)=$$

$$\int\limits_{\Omega_n^{0-}}\int\limits_{\Omega_n^{0+}}\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})\sum_{i_n=1}^{2}\psi_n(\omega_1,\dots\omega_{n-1},\omega_n^{i_n})\times$$

$$\mu_n^{\omega_1,\dots\omega_{n-1},\omega_n^{i_n}}(A)dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)=$$

$$\int\limits_{\Omega_n^{0-}}\int\limits_{\Omega_n^{0+}}\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})\psi_n(\omega_1,\dots\omega_{n-1},\omega_n^1)\times$$

$$\mu_n^{\omega_1,\dots\omega_{n-1},\omega_n^1}(A)dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)+$$

$$\int\limits_{\Omega_n^{0-}}\int\limits_{\Omega_n^{0+}}\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})\psi_n(\omega_1,\dots\omega_{n-1},\omega_n^2)\times$$

$$\mu_n^{\omega_1,\dots\omega_{n-1},\omega_n^2}(A)dP_n^0(\omega_n^1)dP_n^0(\omega_n^2). \tag{62}$$

In accordance with the formulas (49) - (51), for $\psi_n(\omega_1,\dots,\omega_n)$ we have

$$\psi_n(\omega_1,\dots,\omega_n^1)=\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\psi_n^1(\omega_1,\dots,\omega_n^1)+$$

$$\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\psi_n^2(\omega_1,\dots,\omega_n^1)=$$

$$\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\times$$

$$\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}+$$

$$\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\times$$

$$\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}=$$

$$\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\times$$

$$\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}. \tag{63}$$

Further,

$$\psi_n(\omega_1,\dots,\omega_n^2)=\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\psi_n^1(\omega_1,\dots,\omega_n^2)+$$

$$\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\psi_n^2(\omega_1,\dots,\omega_n^2)=$$

$$\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\times$$

$$\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}+$$

$$\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\times$$

$$\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}=$$

$$\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\times$$

$$\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}. \tag{64}$$

Substituting (63), (64) into (62), we obtain the recurrent relations

$$\mu_{n-1}^{\omega_1,\dots\omega_{n-1}}(A)=$$

$$\int\limits_{\Omega_n^{0-}}\int\limits_{\Omega_n^{0+}}\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\times$$

$$\left[\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{\omega_1,\dots\omega_{n-1},\omega_n^1}(A)+\right.$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{\omega_1,\ldots\omega_{n-1},\omega_n^2}(A)\Big] dP_n^0(\omega_n^1)dP_n^0(\omega_n^2). \quad (65)$$

To prove Theorem 2, we need to prove that the recurrent relations for

$$\mu_n^{\omega_1\ldots\omega_n}(A) =$$

$$\int\limits_{\prod_{k=n+1}^{N}\Omega_k^0} \prod_{i=n+1}^{N} \psi_i^\alpha(\omega_1,\ldots,\omega_i)\chi_A(\omega_1,\ldots,\omega_N) \prod_{i=n+1}^{N} dP_i^0(\omega_i^0)$$

are the same as (65). Really,

$$\mu_n^{\omega_1\ldots\omega_n}(A) =$$

$$\int\limits_{\Omega_{n+1}^0} dP_{n+1}^0(\omega_{n+1})\psi_{n+1}^\alpha(\omega_1,\ldots,\omega_n,\omega_{n+1})\times$$

$$\int\limits_{\prod_{k=n+2}^{N}\Omega_k^0} \prod_{i=n+2}^{N} \psi_i^\alpha(\omega_1,\ldots,\omega_i)\chi_A(\omega_1,\ldots,\omega_N) \prod_{i=n+2}^{N} dP_i^0(\omega_i^0)dP_{n+1}^0(\omega_n^0) =$$

$$\int\limits_{\Omega_{n+1}^0} \psi_{n+1}^\alpha(\omega_1,\ldots,\omega_n,\omega_{n+1})\mu_{n+1}^{\omega_1\ldots\omega_n,\omega_{n+1}}(A)dP_{n+1}^0(\omega_{n+1}). \quad (66)$$

Substituting (55) - (57) into (66), we obtain

$$\mu_n^{\omega_1\ldots\omega_n}(A) =$$

$$\int\limits_{\Omega_{n+1}^{0-}} \chi_{\Omega_{n+1}^-}(\omega_1,\ldots,\omega_n,\omega_{n+1}^1)\psi_{n+1}^{1,\alpha}(\omega_1,\ldots,\omega_n,\omega_{n+1}^1)\mu_{n+1}^{\omega_1\ldots\omega_n,\omega_{n+1}^1}(A)dP_{n+1}^0(\omega_{n+1}^1)+$$

$$\int\limits_{\Omega_{n+1}^{0+}} \chi_{\Omega_{n+1}^+}(\omega_1,\ldots,\omega_n,\omega_{n+1}^2)\psi_{n+1}^{2,\alpha}(\omega_1,\ldots,\omega_n,\omega_{n+1}^2)\mu_{n+1}^{\omega_1\ldots\omega_n,\omega_{n+1}^2}(A)dP_{n+1}^0(\omega_{n+1}^2) =$$

$$\int\limits_{\Omega_{n+1}^{0-}} \chi_{\Omega_{n+1}^-}(\omega_1,\ldots,\omega_n,\omega_{n+1}^1) \int\limits_{\Omega_{n+1}^{0+}} \chi_{\Omega_{n+1}^+}(\omega_1,\ldots,\omega_n,\omega_{n+1}^2)\times$$

$$\alpha_{n+1}(\{\omega_1^1,\ldots,\omega_{n+1}^1\};\{\omega_1^2,\ldots,\omega_{n+1}^2\})\frac{\Delta S_{n+1}^+(\omega_1,\ldots,\omega_n,\omega_{n+1}^2)}{V_{n+1}(\omega_1,\ldots,\omega_n,\omega_{n+1}^1,\omega_{n+1}^2)}\times$$

$$\mu_{n+1}^{\omega_1\ldots\omega_n,\omega_{n+1}^1}(A)dP_{n+1}^0(\omega_{n+1}^1)dP_{n+1}^0(\omega_{n+1}^2)+$$

$$\int\limits_{\Omega_{n+1}^{0+}} \chi_{\Omega_{n+1}^+}(\omega_1,\ldots,\omega_n,\omega_{n+1}^2) \int\limits_{\Omega_{n+1}^{0-}} \chi_{\Omega_{n+1}^-}(\omega_1,\ldots,\omega_n,\omega_{n+1}^1)\times$$

$$\alpha_{n+1}(\{\omega_1^1,\dots,\omega_{n+1}^1\};\{\omega_1^2,\dots,\omega_{n+1}^2\})\frac{\Delta S_{n+1}^-(\omega_1,\dots,\omega_n,\omega_{n+1}^1)}{V_{n+1}(\omega_1,\dots,\omega_n,\omega_{n+1}^1,\omega_{n+1}^2)}\times$$

$$\mu_{n+1}^{\omega_1\dots\omega_n,\omega_{n+1}^2}(A)dP_{n+1}^0(\omega_{n+1}^1)dP_{n+1}^0(\omega_{n+1}^2).$$

So, we obtained the recurrent relations

$$\mu_n^{\omega_1\dots\omega_n}(A)=$$

$$\int\limits_{\Omega_{n+1}^{0-}}\int\limits_{\Omega_{n+1}^{0+}}\chi_{\Omega_{n+1}^-}(\omega_1,\dots,\omega_n,\omega_{n+1}^1)\chi_{\Omega_{n+1}^+}(\omega_1,\dots,\omega_n,\omega_{n+1}^2)\times$$

$$\alpha_{n+1}(\{\omega_1^1,\dots,\omega_{n+1}^1\};\{\omega_1^2,\dots,\omega_{n+1}^2\})\left[\frac{\Delta S_{n+1}^+(\omega_1,\dots,\omega_n,\omega_{n+1}^2)}{V_{n+1}(\omega_1,\dots,\omega_n,\omega_{n+1}^1,\omega_{n+1}^2)}\mu_{n+1}^{\omega_1\dots\omega_n,\omega_{n+1}^1}(A)+\right.$$

$$\left.\frac{\Delta S_{n+1}^-(\omega_1,\dots,\omega_n,\omega_{n+1}^1)}{V_{n+1}(\omega_1,\dots,\omega_n,\omega_{n+1}^1,\omega_{n+1}^2)}\mu_{n+1}^{\omega_1\dots\omega_n,\omega_{n+1}^2}(A)\right]dP_{n+1}^0(\omega_{n+1}^1)dP_{n+1}^0(\omega_{n+1}^2), \qquad (67)$$

which are the same as (65). Theorem 2 is proved.

Theorem 3. *Suppose that the conditions of Lemma 1 are true. Then, the set of strictly positive random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2), n=\overline{1,N}$, satisfying the conditions*

$$E^{\mu_0}|\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)|=$$

$$\int\limits_{\Omega_N}\prod_{i=1}^N\psi_i^\alpha(\omega_1,\dots,\omega_i)|\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)|\prod_{i=1}^N dP_i^0(\omega_i)<\infty, \quad n=\overline{1,N}, \qquad (68)$$

is a non empty set for the measures $\mu_0(A)$, given by the formula (60). The measure $\mu_0(A)$, constructed by the strictly positive random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2), n=\overline{1,N}$, satisfying the conditions (68) is a martingale measure for the evolution of risky asset, given by the formula (1). Every measure, belonging to the convex linear span of such measures, is also martingale measure for the evolution of risky asset, given by the formula (1). They are equivalent to the measure P_N. The set of spot measures $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ is a set of martingale measures for the evolution of risky asset, given by the formula (1).

Proof. The first fact, that the set of random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2), n=\overline{1,N}$, satisfying the conditions (68), is a non empty one, follows from Lemma 1. From the representation for the set of measures $\mu_0(A)$, given by the formula (60), as in the proof of Theorem 1, it is proved that this set of measures is a set of martingale measures, being equivalent to the measure P_N.

Let us prove the last statement of Theorem 3. Since for the spot measure $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ the representation

$$\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)=$$

$$\sum_{i_1=1}^{2}\cdots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\chi_A(\omega_1^{i_1},\ldots,\omega_N^{i_N}),\quad A\in\mathcal{F}_N, \tag{69}$$

is true, let us calculate $\sum_{i_j=1}^{2}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})$. We have

$$\sum_{i_j=1}^{2}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})=\psi_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)+\psi_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)=$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\psi_j^1(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^1)+$$
$$\chi_{\Omega_n^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\psi_j^2(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^1)+$$
$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\psi_j^1(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^2)+$$
$$\chi_{\Omega_n^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\psi_j^2(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^2)=$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)}+$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)}=$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)}=$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)=1,$$

if $\omega_j^1\in\Omega_j^{0-},\omega_j^2\in\Omega_j^{0+},\ j=\overline{1,N}$.

So, for every point $\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_2^2\}\} \in \prod_{j=1}^{N} [\Omega_j^{0-} \times \Omega_j^{0+}]$ the spot measure (69) is nonzero probability measure on the σ-algebra $\mathcal{F}_N$. Further,

$$\sum_{i_j=1}^{2} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \Delta S_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) =$$

$$\psi_j(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^1) \Delta S_j(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^1) +$$
$$\psi_j(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^2) \Delta S_j(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^2) =$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^1) \chi_{\Omega_j^+}(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^2) \times$$

$$\left[-\frac{\Delta S_j^+(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^2)}{V_j(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^1, \omega_j^2)} \Delta S_j^-(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^1) + \right.$$

$$\left. \frac{\Delta S_j^-(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^1)}{V_j(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^1, \omega_j^1)} \Delta S_j^+(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^2) \right] = 0, \quad j = \overline{1, N}. \qquad (70)$$

Let us prove that the set of measures $\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}(A)$ is a set of martingale measures. Really, for A, belonging to the σ-algebra $\mathcal{F}_{n-1}$ of the filtration, we have $A = B \times \prod_{i=n}^{N} \Omega_i^0$, where B belongs to σ-algebra $\mathcal{F}_{n-1}$ of the measurable space $\{\Omega_{n-1}, \mathcal{F}_{n-1}\}$. Then,

$$\int_A \Delta S_n(\omega_1, \ldots, \omega_n) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} =$$

$$\sum_{i_1=1}^{2} \cdots \sum_{i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \chi_B(\omega_1^{i_1}, \ldots, \omega_{n-1}^{i_{n-1}}) \Delta S_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) =$$

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{j=1}^{n} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \chi_B(\omega_1^{i_1}, \ldots, \omega_{n-1}^{i_{n-1}}) \Delta S_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) =$$

$$\sum_{i_1=1}^{2} \cdots \sum_{i_{n-1}=1}^{2} \prod_{j=1}^{n-1} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \chi_B(\omega_1^{i_1}, \ldots, \omega_{n-1}^{i_{n-1}}) \times$$

$$\sum_{i_n=1}^{2} \psi_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) \Delta S_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) = 0, \quad A \in \mathcal{F}_{n-1}. \qquad (71)$$

The last means the needed statement. Theorem 3 is proved.

Below, in Theorem 4, we present the consequence of Theorems 2, 3.
Let us introduce the denotations

$$\gamma_n(\{\omega\}_N^1; \{\omega\}_N^2) = \prod_{i=n}^{N} \chi_{\Omega_i^-}(\{\omega_1^1, \dots, \omega_i^1\})\chi_{\Omega_i^+}(\{\omega_1^2, \dots, \omega_i^2\}), \quad n = \overline{1, N}. \tag{72}$$

From the assumptions (6) - (8), it follows that

$$\gamma_n(\{\omega\}_N^1; \{\omega\}_N^2) = \prod_{i=n}^{N} \chi_{\Omega_i^{0-}}(\omega_i^1)\chi_{\Omega_i^{0+}}(\omega_i^2), \quad n = \overline{1, N}. \tag{73}$$

We also use the denotations

$$\Gamma_N = \{\{\{\omega\}_N^1, \{\omega\}_N^2\} \in \prod_{i=1}^{N}[\Omega_i^0 \times \Omega_i^0], \gamma_1(\{\omega\}_N^1, \{\omega\}_N^2) = 1\}, \tag{74}$$

$$\mu_N = \{\{\{\omega\}_N^1, \{\omega\}_N^2\} \in \prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}],\ \mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(\Omega_N) = 1\}. \tag{75}$$

. From the construction of spot measures $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ and assumptions (6) - (8), it follows that these sets (74), (75) coincide.

Theorem 4. *Let the evolution of risky asset, given by the formula (1), satisfy the conditions (6) - (8). Suppose that the random value* $\alpha_N(\{\omega\}_N^1; \{\omega\}_N^2)$*, given on the measurable space* $\{\prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}], \prod_{i=1}^{N}[\mathcal{F}_i^{0-} \times \mathcal{F}_i^{0+}]\}$*, satisfies the conditions*

$$0 < c_N \leq \alpha_N(\{\omega_1^1, \dots, \omega_N^1\}; \{\omega_1^2, \dots, \omega_N^2\}) \leq C_N < \infty. \tag{76}$$

If

$$\int\limits_{\prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N(\{\omega_1^1, \dots, \omega_N^1\}; \{\omega_1^2, \dots, \omega_N^2\}) \times$$

$$\prod_{i=1}^{N} dP_i^0(\omega_i^1) dP_i^0(\omega_i^2) = 1, \tag{77}$$

then the measure $\mu_0(A)$*, given by the formula (78)*

$$\mu_0(A) = \int\limits_{\prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N(\{\omega_1^1, \dots, \omega_N^1\}; \{\omega_1^2, \dots, \omega_N^2\}) \times$$

$$\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A) \prod_{i=1}^{N} d[P_i^0(\omega_i^1) \times P_i^0(\omega_i^2)], \tag{78}$$

is a martingale measure, being equivalent to the measure P_N.

Proof. It is evident that

$$\int\limits_{\Omega_n^0 \times \Omega_n^0} \chi_{\Omega_n^-}(\{\omega_1^1, \ldots, \omega_n^1\})\chi_{\Omega_n^+}(\{\omega_1^2, \ldots, \omega_n^2\})\times$$

$$\alpha_n^1(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\})dP_n^0(\omega_n^1)dP_n^0(\omega_n^2) = 1,$$

where

$$\alpha_N^1(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\}) =$$

$$\frac{\alpha_N(\{\omega_1^1, \ldots, \omega_N^1\}; \{\omega_1^2, \ldots, \omega_N^2\})}{\int\limits_{\prod_{i=N}^{N}[\Omega_i^0 \times \Omega_i^0]} \gamma_N(\{\omega\}_N^1; \{\omega\}_N^2)\alpha_N(\{\omega_1^1, \ldots, \omega_N^1\}; \{\omega_1^2, \ldots, \omega_N^2\}) \prod_{i=N}^{N} dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)},$$

$$\alpha_n^1(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\}) =$$

$$\frac{\int\limits_{\prod_{i=n+1}^{N}[\Omega_i^0 \times \Omega_i^0]} \gamma_{n+1}(\{\omega\}_N^1; \{\omega\}_N^2)\alpha_N(\{\omega_1^1, \ldots, \omega_N^1\}; \{\omega_1^2, \ldots, \omega_N^2\}) \prod_{i=n+1}^{N} dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)}{\int\limits_{\prod_{i=n}^{N}[\Omega_i^0 \times \Omega_i^0]} \gamma_n(\{\omega\}_N^1; \{\omega\}_N^2)\alpha_N(\{\omega_1^1, \ldots, \omega_N^1\}; \{\omega_1^2, \ldots, \omega_N^2\}) \prod_{i=n}^{N} dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)},$$

$$n = \overline{1, N-1}. \tag{79}$$

The set of positive random values $\alpha_n^1(\{\omega\}_n^1; \{\omega\}_n^2), n = \overline{1, N}$, given by the formula (79), are bounded as from below and above. Really,

$$\alpha_n^1(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\}) \leq$$

$$\frac{C_N}{c_N} \frac{\int\limits_{\prod_{i=n+1}^{N}[\Omega_i^0 \times \Omega_i^0]} \gamma_{n+1}(\{\omega\}_N^1; \{\omega\}_N^2) \prod_{i=n+1}^{N} dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)}{\int\limits_{\prod_{i=n}^{N}[\Omega_i^0 \times \Omega_i^0]} \gamma_n(\{\omega\}_N^1; \{\omega\}_N^2) \prod_{i=n}^{N} dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)} =$$

$$\frac{C_N}{c_N P_n^0(\Omega_n^{0-})P_n^0(\Omega_n^{0+})} < \infty.$$

Further,

$$\alpha_n^1(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\}) \geq$$

$$\frac{c_N}{C_N}\frac{\int\limits_{\prod\limits_{i=n+1}^{N}[\Omega_i^0\times\Omega_i^0]}\gamma_{n+1}(\{\omega\}_N^1;\{\omega\}_N^2)\prod\limits_{i=n+1}^{N}dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)}{\int\limits_{\prod\limits_{i=n}^{N}[\Omega_i^0\times\Omega_i^0]}\gamma_n(\{\omega\}_N^1;\{\omega\}_N^2)\prod\limits_{i=n}^{N}dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)}=$$

$$\frac{c_N}{P_n^0(\Omega_n^{0-})P_n^0(\Omega_n^{0+})C_N}>0.$$

Therefore, they satisfy the conditions

$$E^{\mu_0}|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)|=$$

$$\int\limits_{\Omega_N}\prod_{i=1}^{N}\psi_i(\omega_1,\ldots,\omega_i)|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)|\prod_{i=1}^{N}dP_i^0(\omega_i)<\infty,\quad n=\overline{1,N}.\qquad(80)$$

The boundedness of random values $\alpha_n^1(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})$, $n=\overline{1,N}$, means that they satisfy the conditions (20) - (22). It is evident that

$$\alpha_N(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\})=\prod_{n=1}^{N}\alpha_n^1(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\}).\qquad(81)$$

Owing to Theorem 3, $\mu_0(A)$, given by the formula (78), is a martingale measure, being equivalent to the measure P_N. Theorem 4 is proved.

Theorem 5. *Let the conditions of Theorem 4 be true. If the contingent claim $f_N=f_N(\omega_1,\ldots,\omega_N)$ satisfies the condition*

$$\sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}}\int\limits_{\Omega_N}f_N(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}<\infty,$$

then the equalities

$$\inf_{P\in M_b}E^Pf_N=\inf_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}}\int\limits_{\Omega_N}f_N(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}},\qquad(82)$$

$$\sup_{P\in M_b}E^Pf_N=\sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}}\int\limits_{\Omega_N}f_N(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1\omega_N^2\}},\qquad(83)$$

are true, where M_b is the set of all martingale measures, figuring in Theorem 4, with $\alpha_N(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\})$ running all nonnegative bounded as from below and above the random values.

Proof. The inequality

$$\inf_{P\in M_b} E^P f_N \leq (1-\alpha)E^Q f_N + \alpha \int\limits_{\Omega_N} f_N(\omega_1,\dots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}, \quad 0<\alpha<1,$$

is valid for all $0<\alpha<1$, since $(1-\alpha)Q+\alpha\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}$ is a martingale measure, being equivalent to P_N. Tending α to one, we have

$$\inf_{P\in M_b} E^P f_N \leq \int\limits_{\Omega_N} f_N(\omega_1,\dots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}},$$

or

$$\inf_{P\in M_b} E^P f_N \leq \inf_{\{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\dots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}.$$

To prove the inverse inequality, we use the representation

$$E^Q f_N =$$

$$\int\limits_{\prod_{i=1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]} \alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\})\times$$

$$\int\limits_{\Omega_N} f_N(\omega_1,\dots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}} \prod_{i=1}^{N} d[P_i^0(\omega_i^1)\times P_i^0(\omega_i^2)]. \tag{84}$$

Using the representation (84), we obtain the inequality

$$E^Q f_N \geq \inf_{\{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\dots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}. \tag{85}$$

Taking into account the inequality (85), we obtain the inequality

$$\inf_{Q\in M_b} E^Q f_N \geq \inf_{\{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\dots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}. \tag{86}$$

This proves the equality (82). As before, $(1-\alpha)Q+\alpha\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}$ is a martingale measure, being equivalent to P_N, therefore the inequality

$$\sup_{P\in M_b} E^P f_N \geq (1-\alpha)E^Q f_N + \alpha \int\limits_{\Omega_N} f_N(\omega_1,\dots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}, \quad 0<\alpha<1,$$

is true. Tending α to one, we have

$$\sup_{P\in M_b} E^P f_N \geq \int\limits_{\Omega_N} f_N(\omega_1,\dots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}},$$

or

$$\sup_{P \in M_b} E^P f_N \geq \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\} \in \mu_N\}} \int_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}.$$

To prove the inverse inequality, we use the representation

$$E^Q f_N =$$

$$\int_{\prod_{i=1}^N [\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\}) \times$$

$$\int_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \prod_{i=1}^N d[P_i^0(\omega_i^1) \times P_i^0(\omega_i^2)]. \quad (87)$$

From (87) we have

$$E^Q f_N \leq \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\} \in \mu_N\}} \int_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}. \quad (88)$$

Taking into account the inequality (88), we obtain the inequality

$$\sup_{Q \in M_b} E^Q f_N \leq \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\} \in \mu_N\}} \int_{\Omega_N} f_N(\omega_1,\ldots,\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}. \quad (89)$$

This proves the equality (83). Theorem 5 is proved.

Let us introduce into the set of measure M_b the norm. If $P_1, P_2 \in M_b$, where

$$P_1(A) = \int_{\prod_{i=1}^N [\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N^1(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\}) \times$$

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A) \prod_{i=1}^N d[P_i^0(\omega_i^1) \times P_i^0(\omega_i^2)], \quad (90)$$

$$P_2(A) = \int_{\prod_{i=1}^N [\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N^2(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\}) \times$$

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A) \prod_{i=1}^N d[P_i^0(\omega_i^1) \times P_i^0(\omega_i^2)], \quad (91)$$

then we put

$$||P_1 - P_2|| = \int\limits_{\prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}]} |\alpha_N^1(\{\omega\}_N^1; \{\omega\}_N^2) - \alpha_N^2(\{\omega\}_N^1; \{\omega\}_N^2)| \times$$

$$\prod_{i=1}^{N} d[P_i^0(\omega_i^1) \times P_i^0(\omega_i^2)]. \qquad (92)$$

Denote M_0 the completion of the set M_b in the introduced metrics.

Theorem 6. *Let the conditions of Theorem 5 be true. Then, the equalities*

$$\inf_{P \in M_0} E^P f_N = \inf_{\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\} \in \mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}, \qquad (93)$$

$$\sup_{P \in M_0} E^P f_N = \sup_{\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\} \in \mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} \qquad (94)$$

are valid.

Proof. For arbitrary small $\varepsilon > 0$ there exists a measure $P_0 \in M_0$ such that $|\inf_{P \in M_0} E^P f_N - E^{P_0} f_N| < \varepsilon$. Since $|E^{P_1} f_N - E^{P_2} f_N| \leq ||P_1 - P_2||$, then there exists a measure $P_n \in M_b$ such that $|E^{P_n} f_N - E^{P_0} f_N| \leq ||P_n - P_0|| < \varepsilon$. Due to the above inequalities, we have

$$\inf_{\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\} \in \mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} = \inf_{P \in M_b} E^P f_N \geq$$

$$\inf_{P \in M_0} E^P f_N \geq -\varepsilon + E^{P_0} f_N \geq -2\varepsilon + E^{P_n} f_N \geq$$

$$-2\varepsilon + \inf_{\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\} \in \mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}.$$

Since $\varepsilon > 0$ is arbitrary small we have the proof of (93).

Analogously, for arbitrary small $\varepsilon > 0$ there exists a measure $P_0 \in M_0$ such that $|\sup_{P \in M_0} E^P f_N - E^{P_0} f_N| < \varepsilon$. Since $|E^{P_1} f_N - E^{P_2} f_N| \leq ||P_1 - P_2||$, then there exists a measure $P_n \in M_b$ such that $|E^{P_n} f_N - E^{P_0} f_N| \leq ||P_n - P_0|| < \varepsilon$. Due to the above inequalities, we have

$$\sup_{\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\} \in \mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} = \sup_{P \in M_b} E^P f_N \leq$$

$$\sup_{P \in M_0} E^P f_N \leq \varepsilon + E^{P_0} f_N \leq 2\varepsilon + E^{P_n} f_N \leq$$

$$2\varepsilon + \sup_{\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\} \in \mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}.$$

Since $\varepsilon > 0$ is arbitrary small we have the proof of (94).

Denote $M \subseteq M_0$ the subset of all martingale measures from the set M_0, which are equivalent to P_N. As a consequence of Theorem 6, we obtain

Theorem 7. *Let the conditions of Theorem 5 be valid. Then, the equalities*

$$\inf_{P\in M} E^P f_N = \inf_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}, \quad (95)$$

$$\sup_{P\in M} E^P f_N = \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \quad (96)$$

are true.

Proof. The proof of Theorem 7 follows from the inclusions $M_b \subseteq M \subseteq M_0$ and Theorems 5, 6.

Theorem 8. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_n^0, \mathcal{F}_n^0, P_n^0\}$, let the evolution of risky asset be given by the formula (12), with $a_n(\omega_1,\ldots,\omega_n\ ) = b_n(\omega_1,\ldots,\omega_{n-1}),\ n=\overline{1,N}$,* where *the random variables $f_n(\omega_1,\ldots,\omega_n),\ b_n(\omega_1,\ldots,\omega_{n-1}),\ \eta_n(\omega_n)$ satisfy the inequalities*

$$b_n(\omega_1,\ldots,\omega_{i-1},\omega_n) > 0, \quad f_n(\omega_1,\ldots,\omega_{n-1},\omega_n) > 0, \quad \sup_{\omega_n\in\Omega_n^0,\eta_n^-(\omega_n)>0} \eta_n^-(\omega_n) < \infty,$$

$$\sup_{\{\omega_1,\quad \ldots,\omega_n\}\in\Omega_n} b_n(\omega_1,\quad \ldots,\omega_{n-1},\omega_n) <$$

$$\frac{1}{\sup\limits_{\{\omega_1,\quad\ldots,\omega_n\}\in\Omega_n} f_n(\omega_1,\ldots,\omega_{n-1},\omega_n) \sup\limits_{\omega_n\in\Omega_n^0,\eta_n^-(\omega_i)>0} \eta_n^-(\omega_n)}, \quad n=\overline{1,N}. \quad (97)$$

For such an evolution, the family of martingale measures (78) described in Theorem 4 does not depend on the random variables $b_n(\omega_1,\ldots,\omega_{n-1}),\ n=\overline{1,N}$.

Proof. Due to the representation (78) for the measure $\mu_0(A)$ in Theorem 4, to prove Theorem 8, it needs to prove that all spot measures $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ do not depend on the random variables $b_n(\omega_1,\ldots,\omega_{n-1}),\ n=\overline{1,N}$. For this purpose, it is need to prove that $\psi_n(\omega_1,\ldots,\omega_n)$ do not depend on the random variables $b_n(\omega_1,\ldots,\omega_{n-1}),\ n=\overline{1,N}$, where

$$\psi_n(\omega_1,\ldots,\omega_n) = \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n)\,\psi_n^1(\omega_1,\ldots,\omega_n) +$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n)\,\psi_n^2(\omega_1,\ldots,\omega_n), \quad (98)$$

$$\psi_n^1(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}, \quad (99)$$

$$\psi_n^2(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\dots,\omega_{n-1}) \in \Omega_{n-1}. \quad (100)$$

It is evident that $\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n)$ and $\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n)$ do not depend on the random variables $b_n(\omega_1,\dots,\omega_{n-1}),\ n=\overline{1,N}$, where
Since,

$$\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2) =$$

$$S_{n-1}(\omega_1,\dots,\omega_{n-1})b_n(\omega_1,\dots,\omega_{n-1})f_n(\omega_1,\dots,\omega_n^2)\eta_n^+(\omega_n^2), \quad (101)$$

$$\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1) =$$

$$S_{n-1}(\omega_1,\dots,\omega_{n-1})b_n(\omega_1,\dots,\omega_{n-1})f_n(\omega_1,\dots,\omega_n^1)\eta_n^-(\omega_n^1), \quad (102)$$

we have

$$\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)} =$$

$$\frac{f_n(\omega_1,\dots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)}{f_n(\omega_1,\dots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)+f_n(\omega_1,\dots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}, \quad (103)$$

$$\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)} =$$

$$\frac{f_n(\omega_1,\dots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}{f_n(\omega_1,\dots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)+f_n(\omega_1,\dots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}, \quad (104)$$

$$(\omega_1,\dots,\omega_{n-1}) \in \Omega_{n-1}.$$

The equalities (103), (104) prove Theorem 8.

V. ASSESSMENT OF CONTINGENT CLAIM

In this section, we prove Theorems, giving us the formula for the fair price of super-hedge for the evolution of risky asset, given by the formula

$$S_n(\omega_1,\dots,\omega_{n-1},\omega_n) = S_0\prod_{i=1}^{n}\left(1+a_i\left(e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)}-1\right)\right), \quad n=\overline{1,N}, \quad (105)$$

where the random value $\varepsilon_i(\omega_i),\ \omega_i \in \Omega_i^0, i=\overline{1,N}$, takes all real values from R^1, $S_0>0$. The random values $\sigma_i(\omega_1,\dots,\omega_{i-1})$ satisfy the inequalities $\sigma_i(\omega_1,\dots,\omega_{i-1}) \geq \sigma_i^0>0,\ 0<a_i\leq 1, i=\overline{1,N}$. Due to Theorem 8, the set of equivalent martingale measures constructed by the evolution of risky asset, given by the formula (105), do not depend on parameters $0<a_i\leq 1, i=\overline{1,N}$. The proposed parametric model based on the canonical model of the evolution of risky asset (9), which takes into account both memory and clustering, takes into account the fact that the price of a risky asset cannot fall to zero.

Theorem 9. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105) and the conditions of Theorem 7 are true. Suppose that $0 \leq a_i \leq 1$, $\infty > \sigma_i > \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $i = \overline{1, N}$. If the nonnegative continuous payoff function $f(x)$, $x \in [0, \infty)$, satisfies the conditions:*
1) $f(0) = 0$, $f(x) \leq ax$, $\lim\limits_{x \to \infty} \frac{f(x)}{x} = a$, $a > 0$, then the inequalities

$$f\left(S_0 \prod_{i=1}^{N}(1 - a_i)\right) + aS_0 \left(1 - \prod_{i=1}^{N}(1 - a_i)\right) \leq \sup_{P \in M} E^P f(S_N) \leq aS_0 \tag{106}$$

are true. If, in addition, the nonnegative payoff function $f(x)$ is a convex down one, then

$$\inf_{P \in M} E^P f(S_N) = f(S_0), \tag{107}$$

where M is a set of equivalent martingale measures for the evolution of risky asset, given by the formula (105).

Proof. Due to Theorem 7,

$$\sup_{P \in M} E^P f(S_N) = \sup_{\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\} \in \mu_N\}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}.$$

So, we have

$$aS_0 \geq \sup_{\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\} \in \mu_N\}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} =$$

$$\sup_{\{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, \ i = \overline{1, N}\}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} =$$

$$\sup_{\{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, \ i = \overline{1, N}\}} \sum_{i_1 = 1, \ldots, i_N = 1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N} \left(1 + a_s \left(e^{\sigma_s(\omega_1^{i_1}, \ldots, \omega_{s-1}^{i_{s-1}}) \varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right). \tag{108}$$

Further,

$$\sup_{\{\omega_N^1 \in \Omega_N^{0-}, \omega_N^2 \in \Omega_N^{0+}\}} \sum_{i_N=1}^{2} \psi_N(\omega_1^{i_1}, \ldots, \omega_N^{i_N}) \times$$

$$f\left(S_0 \prod_{s=1}^{N} \left(1 + a_s \left(e^{\sigma_s(\omega_1^{i_1}, \ldots, \omega_{s-1}^{i_{s-1}}) \varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right) =$$

$$\sup_{\{\omega_N^1 \in \Omega_N^{0-}, \omega_N^2 \in \Omega_N^{0+}\}} \left[\frac{\Delta S_N^+(\omega_1^{i_1}, \ldots, \omega_{N-1}^{i_{N-1}}, \omega_N^2)}{V_N(\omega_1^{i_1}, \ldots, \omega_{N-1}^{i_{N-1}}, \omega_N^1, \omega_N^2)} \times \right.$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)+$$

$$\left.\frac{\Delta S_N^-(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)\right]\geq$$

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\left[\frac{\Delta S_N^+(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^2)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}\times\right.$$
$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)+$$

$$\left.\frac{\Delta S_N^-(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)\right]=$$

$$f(S_{N-1}(1-a_N))+aa_NS_{N-1}, \tag{109}$$

where we put

$$S_{N-1}=S_0\prod_{s=1}^{N-1}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right). \tag{110}$$

Really,

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\Delta S_N^+(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^2)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}\times$$
$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)=$$

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)}{\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}\right)}\times$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)=f(S_{N-1}(1-a_N)).$$

Further,

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\Delta S_N^-(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}\times$$
$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)=$$

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\left(1-e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}\right)}{\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}\right)}\times$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)=aa_NS_{N-1}.$$

Substituting the inequality (109) into (108), we obtain the inequality

$$\sup_{\{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},\ i=\overline{1,N}\}}\sum_{i_1=1,\dots,i_N=1}^{2}\sum_{i_1=1,\dots,i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)\geq$$

$$\sup_{\{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},\ i=\overline{1,N-1}\}}\sum_{i_1=1,\dots,i_{N-1}=1}^{2}\prod_{j=1}^{N-1}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\times$$

$$f\left(S_0(1-a_N)\prod_{s=1}^{N-1}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)+aa_NS_0. \quad (111)$$

Applying $(N-1)$ times the inequality (111), we obtain the inequality

$$\sup_{Q\in M}\int_{\Omega}f(S_N)dQ\geq f(S_0\prod_{i=1}^{N}(1-a_i))+aS_0\sum_{i=1}^{N}a_i\prod_{s=i+1}^{N}(1-a_s)=$$

$$f\left(S_0\prod_{i=1}^{N}(1-a_i)\right)+aS_0\left(1-\prod_{i=1}^{N}(1-a_i)\right). \quad (112)$$

Let us prove the equality (107). Using the Jensen inequality, we obtain

$$\inf_{P\in M}E^Pf(S_N)\geq f(S_0). \quad (113)$$

Let us prove the inverse inequality. The inequality

$$\sum_{i_1=1,\dots,i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)\geq\inf_{P\in M}E^Pf(S_N) \quad (114)$$

is true. If to put $\varepsilon_s(\omega_s^1)=0,\ s=\overline{1,N}$, then the inequality (114) turns into the inequality

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^2,\dots,\omega_{s-1}^2)\varepsilon_s(\omega_s^2)}-1\right)\right)\right)\geq\inf_{P\in M}E^Pf(S_N). \quad (115)$$

In the considered case $\Omega_i^{0-}=\{\omega_i\in\Omega_i^0,\varepsilon_i(\omega_i)\leq 0\}$, $\Omega_i^{0+}=\{\omega_i\in\Omega_i^0,\varepsilon_i(\omega_i)>0\}$. Since the value $\varepsilon_s(\omega_s^2)>0$ can be made as small as it needs for $\omega_s^2\in\Omega_s^{0+}$, then we can do the left side of the inequality (115) as close to $f(S_0)$ as it needs, since $\sigma_s(\omega_1^2,\dots,\omega_{s-1}^2)$ is bounded and $f(x)$ is a continuous one. The last proves the needed inequality. Theorem 9 is proved.

Theorem 10. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105) and the conditions of Theorem 7 are true. Suppose that $0 \leq a_i \leq 1$, $\infty > \sigma_i > \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $\quad i = \overline{1,N}$. If the nonnegative continuous payoff function $f(x)$, $x \in [0, \infty)$, satisfies the conditions:*
1) $f(0) = K$, $f(x) \leq K$, then

$$f\left(S_0 \prod_{i=1}^{N}(1-a_i)\right) \leq \sup_{P \in M} E^P f(S_N) \leq K. \tag{116}$$

If, in addition, the nonnegative payoff function $f(x)$ is a convex down one, then

$$\inf_{P \in M} E^P f(S_N) = f(S_0), \tag{117}$$

where M is the set of equivalent martingale measures for the evolution of risky asset, given by the formula (105).

Proof. Let us obtain the estimate from below. Due to Theorem 7,

$$\sup_{P \in M} E^P f_N = \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\} \in \mu_N\}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}.$$

So, we have

$$K \geq \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\} \in \mu_N\}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} =$$

$$\sup_{\{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+},\ i=\overline{1,N}\}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} =$$

$$\sup_{\{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+},\ i=\overline{1,N}\}} \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N}\left(1 + a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right). \tag{118}$$

Further,

$$\sup_{\{\omega_N^1 \in \Omega_N^{0-}, \omega_N^2 \in \Omega_N^{0+}\}} \sum_{i_N=1}^{2} \psi_N(\omega_1^{i_1}, \ldots, \omega_N^{i_N}) \times$$

$$f\left(S_0 \prod_{s=1}^{N}\left(1 + a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right) =$$

$$\sup_{\{\omega_N^1 \in \Omega_N^{0-}, \omega_N^2 \in \Omega_N^{0+}\}} \left[\frac{\Delta S_N^+(\omega_1^{i_1}, \ldots, \omega_{N-1}^{i_{N-1}}, \omega_N^2)}{V_N(\omega_1^{i_1}, \ldots, \omega_{N-1}^{i_{N-1}}, \omega_N^1, \omega_N^2)} \times\right.$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)+$$

$$\left.\frac{\Delta S_N^-(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)\right]\geq$$

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\left[\frac{\Delta S_N^+(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^2)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}\times\right.$$
$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)+$$

$$\left.\frac{\Delta S_N^-(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)\right]=$$

$$f(S_{N-1}(1-a_N)), \tag{119}$$

where we put

$$S_{N-1}=S_0\prod_{s=1}^{N-1}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right). \tag{120}$$

Really,

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\Delta S_N^+(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^2)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}\times$$
$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)=$$

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)}{\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}\right)}\times$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)=f(S_{N-1}(1-a_N)).$$

Further,

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\Delta S_N^-(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}\times$$
$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)=$$

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\frac{\left(1-e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}\right)}{\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}\right)}\times$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)=0.$$

Substituting the inequality (119) into (118), we obtain the inequality

$$\sup_{\{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},\ i=\overline{1,N}\}}\sum_{i_1=1,\dots,i_N=1}^{2}\sum_{i_1=1,\dots,i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)\geq$$

$$\sup_{\{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},\ i=\overline{1,N-1}\}}\sum_{i_1=1,\dots,i_{N-1}=1}^{2}\prod_{j=1}^{N-1}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\times$$

$$f\left(S_0(1-a_N)\prod_{s=1}^{N-1}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right). \tag{121}$$

Applying $(N-1)$ times the inequality (121), we obtain the inequality

$$\sup_{Q\in M}\int_{\Omega}f(S_N)dQ\geq f(S_0\prod_{i=1}^{N}(1-a_i)). \tag{122}$$

Let us prove the equality (117). Using the Jensen inequality, we obtain

$$\inf_{P\in M}E^Pf(S_N)\geq f(S_0). \tag{123}$$

Let us prove the inverse inequality. It is evident that the inequality

$$\sum_{i_1=1,\dots,i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)\geq\inf_{P\in M}E^Pf(S_N) \tag{124}$$

is valid. If to put $\varepsilon_s(\omega_s^1)=0,\ s=\overline{1,N}$, then the inequality (124) turns into the inequality

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{2},\dots,\omega_{s-1}^{2})\varepsilon_s(\omega_s^{2})}-1\right)\right)\right)\geq\inf_{P\in M}E^Pf(S_N). \tag{125}$$

In the considered case $\Omega_i^{0-}=\{\omega_i\in\Omega_i^0,\varepsilon_i(\omega_i)\leq 0\}$, $\Omega_i^{0+}=\{\omega_i\in\Omega_i^0,\varepsilon_i(\omega_i)>0\}$. Since the value $\varepsilon_s(\omega_s^2)>0$ can be made as small as it needs for $\omega_s^2\in\Omega_s^{0+}$, we can do the left side of the inequality (125) as close to $f(S_0)$ as it needs, since $\sigma_s(\omega_1^2,\dots,\omega_{s-1}^2)$ is bounded and $f(x)$ is a continuous one. The last proves the needed inequality. Theorem 10 is proved.

Theorem 11. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105) and the conditions of Theorem 7 are true. Suppose that* $0 \leq a_i \leq 1$, $\infty > \sigma_i > \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $\quad i = \overline{1, N}$. *For the payoff function* $f(x) = (x - K)^+$, $x \in (0, \infty)$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if} \quad S_0 \prod_{i=1}^{N} (1 - a_i)) \geq K, \\ S_0 \left(1 - \prod_{i=1}^{N} (1 - a_i)\right), & \text{if} \quad S_0 \prod_{i=1}^{N} (1 - a_i) < K. \end{cases} \tag{126}$$

For $S_0 \prod_{i=1}^{N} (1 - a_i)) \geq K$, *the set of non arbitrage prices coincides with the point* $(S_0 - K)^+$, *in case if* $S_0 \prod_{i=1}^{N} (1 - a_i) < K$ *the set of non arbitrage prices coincides with the set* $\left((S_0 - K)^+, S_0 \left(1 - \prod_{i=1}^{N} (1 - a_i)\right)\right)$.

Proof. Let us introduce the denotations

$$I_N = \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N} \left(1 + a_s \left(e^{\sigma_s(\omega_1^{i_1}, \ldots, \omega_{s-1}^{i_{s-1}}) \varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right), \tag{127}$$

$$I_N^1 = \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f_1\left(S_0 \prod_{s=1}^{N} \left(1 + a_s \left(e^{\sigma_s(\omega_1^{i_1}, \ldots, \omega_{s-1}^{i_{s-1}}) \varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right), \tag{128}$$

$$I_N^0 = \sup_{\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\} \in \mu_N\}} \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N} \left(1 + a_s \left(e^{\sigma_s(\omega_1^{i_1}, \ldots, \omega_{s-1}^{i_{s-1}}) \varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right), \tag{129}$$

where we put $f_1(x) = (K - x)^+$. Let us estimate from above the value I_N. For this, we use the equality

$$I_N = I_N^1 + S_0 - K, \tag{130}$$

which follows from the identity: $f(x) = f_1(x) + x - K,\ x \geq 0$. Since

$$f_1\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right) \leq f_1\left(S_0\prod_{s=1}^{N}(1-a_s)\right), \quad (131)$$

we obtain the inequality

$$I_N \leq S_0 - K + f_1\left(S_0\prod_{s=1}^{N}(1-a_s)\right). \quad (132)$$

From the inequality (132), we have

$$I_N^0 \leq S_0 - K + f_1\left(S_0\prod_{s=1}^{N}(1-a_s))\right) =$$

$$\begin{cases} (S_0-K)^+, & \text{if} \quad S_0\prod_{i=1}^{N}(1-a_i)) \geq K, \\ S_0\left(1-\prod_{i=1}^{N}(1-a_i)\right), & \text{if} \quad S_0\prod_{i=1}^{N}(1-a_i) < K. \end{cases} \quad (133)$$

From the inequality (106) of Theorem 9

$$I_N^0 \geq f\left(S_0\prod_{i=1}^{N}(1-a_i)\right) + S_0\left(1-\prod_{i=1}^{N}(1-a_i)\right) \quad (134)$$

and the inequality

$$I_N^0 \geq (S_0-K)^+, \quad (135)$$

which follows from the Jensen inequality, we have

$$I_N^0 \geq \max\left\{S_0-K)^+, f\left(S_0\prod_{i=1}^{N}(1-a_i)\right) + S_0\left(1-\prod_{i=1}^{N}(1-a_i)\right)\right\} =$$

$$\begin{cases} (S_0-K)^+, & \text{if} \quad S_0\prod_{i=1}^{N}(1-a_i)) \geq K, \\ S_0\left(1-\prod_{i=1}^{N}(1-a_i)\right), & \text{if} \quad S_0\prod_{i=1}^{N}(1-a_i) < K. \end{cases} \quad (136)$$

This proves Theorem 11.

Theorem 12. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105) and the conditions of Theorem 7 are true. Suppose that* $0 \leq a_i \leq 1$, $\infty > \sigma_i > \sigma_i(\omega_1,\dots,\omega_{i-1}) \geq \sigma_i^0 > 0$, $\quad i = \overline{1,N}$. *For the payoff* *function* $f_1(x) = (K-x)^+, \ x \in (0,\infty), \ K>0,$ *the fair price of super-hedge is given by the formula*

$$\sup_{Q\in M} E^Q f_1(S_N) = f_1\left(S_0 \prod_{i=1}^{N}(1-a_i)\right). \tag{137}$$

The set of non arbitrage prices coincides with the interval $\left((K-S_0)^+, f_1\left(S_0\prod_{i=1}^{N}(1-a_i)\right)\right).$

Proof. The inequality

$$I_N^1 = \sum_{i_1=1,\dots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\times$$

$$f_1\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right) \leq f_1\left(S_0\prod_{i=1}^{N}(1-a_i)\right) \tag{138}$$

is true. Taking into account the inequality (116) of Theorem 10, we prove Theorem 12.

Theorem 13. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (105) and the conditions of Theorem 7 are true. Suppose that* $0 \leq a_i \leq 1, \ \infty > \sigma_i > \sigma_i(\omega_1,\dots,\omega_{i-1}) \geq \sigma_i^0 > 0, \quad i=\overline{1,N}$. *For the payoff function* $f_1(S_0,S_1,\dots,S_N) = \left(K - \frac{\sum_{i=0}^{N} S_i}{N+1}\right)^+, \ K>0,$ *the fair price of super-hedge is given by the formula*

$$\sup_{Q\in M} E^Q f_1(S_0,S_1,\dots,S_N) = \left(K - \frac{S_0\sum_{i=0}^{N}\prod_{s=1}^{i}(1-a_s)}{N+1}\right)^+. \tag{139}$$

The set of non arbitrage prices coincides with the interval

$$\left((K-S_0)^+, \left(K - \frac{S_0\sum_{i=0}^{N}\prod_{s=1}^{i}(1-a_s)}{N+1}\right)^+\right).$$

Proof. Let us denote

$$S_n(\omega_1^1,\dots,\omega_n^1) = S_0\prod_{s=1}^{n}\left(1+a_s\left(e^{\sigma_s(\omega_1^1,\dots,\omega_{s-1}^1)\varepsilon_s(\omega_s^1)}-1\right)\right), \quad n=\overline{1,N},$$

$$t_N(\omega_1^1,\dots,\omega_N^1) = \prod_{s=1}^{N}\frac{\left(e^{\sigma_s(\omega_1^1,\dots,\omega_{s-1}^1)\varepsilon_s(\omega_s^2)}-1\right)}{\left(e^{\sigma_s(\omega_1^1,\dots,\omega_{s-1}^1)\varepsilon_s(\omega_s^2)}-e^{\sigma_s(\omega_1^1,\dots,\omega_{s-1}^1)\varepsilon_s(\omega_s^1)}\right)}. \tag{140}$$

It is evident that

$$I_N^2 = \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\times$$

$$f_1\left(S_0, S_1(\omega_1^{i_1}),\ldots,S_N(\omega_1^{i_1},\ldots,\omega_N^{i_N})\right) \geq \tag{141}$$

$$\lim_{\varepsilon_s(\omega_s^1)=-\infty,\ \varepsilon_s(\omega_s^2)\to\infty, s=\overline{1,N}} f_1\left(S_0, S_1(\omega_1^1),\ldots,S_N(\omega_1^1,\ldots,\omega_N^1)\right)\times$$

$$t_N(\omega_1^1,\ldots,\omega_N^1) = f_1\left(S_0, S_0(1-a_1),\ldots,S_0\prod_{s=1}^{N}(1-a_s)\right).$$

So, we obtain the inequality

$$I_N^2 \geq f_1\left(S_0, S_0(1-a_1),\ldots,S_0\prod_{s=1}^{N}(1-a_s)\right) = \left(K - \frac{S_0\sum_{i=0}^{N}\prod_{s=1}^{i}(1-a_s)}{N+1}\right)^+. \tag{142}$$

Let us prove the inverse inequality. We have

$$I_N^2 \leq \sup_{\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\in\mu_N\}} \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\times$$

$$f_1\left(S_0, S_0(1-a_1),\ldots,S_0\prod_{s=1}^{N}(1-a_s)\right) =$$

$$f_1\left(S_0, S_0(1-a_1),\ldots,S_0\prod_{s=1}^{N}(1-a_s)\right) = \left(K - \frac{S_0\sum_{i=0}^{N}\prod_{s=1}^{N}(1-a_s)}{N+1}\right)^+. \tag{143}$$

Therefore,

$$I_N^2 \leq \left(K - \frac{S_0\sum_{i=0}^{N}\prod_{s=1}^{i}(1-a_s)}{N+1}\right)^+. \tag{144}$$

The inequalities (142), (144) prove Theorem 13.

Theorem 14. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105) and the conditions of Theorem 7 are true. Suppose that $0 \leq a_i \leq 1$, $\infty > \sigma_i > \sigma_i(\omega_1,\ldots,\omega_{i-1}) \geq \sigma_i^0 > 0$, $\quad i = \overline{1,N}$. For the payoff* *function* $f(S_0, S_1, \ldots, S_N) = \left(\frac{\sum_{i=0}^{N} S_i}{N+1} - K\right)^+$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_0, S_1, \ldots, S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if } \frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i} (1-a_i)}{N+1} \geq K, \\ S_0 \left(1 - \frac{\sum_{i=0}^{N} \prod_{s=1}^{i} (1-a_s)}{N+1}\right), & \text{if } S_0 \frac{\sum_{i=0}^{N} \prod_{s=1}^{i} (1-a_s)}{N+1} < K. \end{cases} \tag{145}$$

The set of non arbitrage prices coincides with the point $(S_0 - K)^+$ *for* $\frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i} (1-a_i)}{N+1} \geq K$, *in case if* $S_0 \frac{\sum_{i=0}^{N} \prod_{s=1}^{i} (1-a_s)}{N+1} < K$ *the set of non arbitrage prices coincides with the interval* $\left((S_0 - K)^+, S_0 \left(1 - \frac{\sum_{i=0}^{N} \prod_{s=1}^{i} (1-a_s)}{N+1}\right)\right)$.

Proof. Let us introduce the denotation

$$V_N = \sup_{\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\} \in \mu_N\}} \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f\left(S_0, S_1(\omega_1^{i_1}), \ldots, S_N(\omega_1^{i_1}, \ldots, \omega_N^{i_N})\right). \tag{146}$$

Then, we have

$$V_N = \sup_{\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\} \in \mu_N\}} \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f_1\left(S_0, S_1(\omega_1^{i_1}), \ldots, S_N(\omega_1^{i_1}, \ldots, \omega_N^{i_N})\right) + S_0 - K. \tag{147}$$

Due to Theorem 13,

$$V_N = (S_0 - K) + \left(K - \frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i} (1 - a_s)}{N + 1}\right)^+ =$$

$$\begin{cases} (S_0 - K)^+, & \text{if } \frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i} (1-a_i)}{N+1} \geq K, \\ S_0 \left(1 - \frac{\sum_{i=0}^{N} \prod_{s=1}^{i} (1-a_s)}{N+1}\right), & \text{if } S_0 \frac{\sum_{i=0}^{N} \prod_{s=1}^{i} (1-a_s)}{N+1} < K. \end{cases} \tag{148}$$

In the formula (147) we introduced the denotation

$$f_1(S_0, S_1, \dots, S_N) = \left(K - \frac{\sum\limits_{i=0}^{N} S_i}{N+1} \right)^+ . \tag{149}$$

Theorem 14 is proved.

If $S_0, \dots, S_N$ is a sample of the process (105), let us denote the order statistic $S_{(0)}, \dots, S_{(N)}$ of this sample.

Theorem 15. *Suppose that $S_0, \dots, S_N$ is a sample of the random process (105). Then, for the parameters $a_1, \dots, a_N$ the estimations*

$$a_i = 1 - \frac{S_{(N-i)}}{S_{(N-i+1)}}, \quad i = \overline{1, N}, \tag{150}$$

are valid. Under such estimations $0 < a_i < 1$, $i = \overline{1, N}$, the equalities

$$\prod_{i=1}^{N-k} (1 - a_i) = \frac{S_{(k)}}{S_{(N)}}, \quad i = \overline{0, N-1}, \tag{151}$$

are true.

Proof. The estimation of the parameters $a_1, \dots, a_N$ we do using the representation of random process S_n, $n = \overline{1, N}$. The smallest value of the random variable S_n is equal $S_0 \prod\limits_{i=1}^{n}(1 - a_i)$, $n = \overline{1, N}$. Let us determine the parameters a_i from the relations

$$S_0 \prod_{i=1}^{N} (1 - a_i) = \tau S_{(0)}, \dots, S_0 \prod_{i=1}^{N-k} (1 - a_i) = \tau S_{(k)},$$

$$S_0 \prod_{i=1}^{N-k-1} (1 - a_i) = \tau S_{(k+1)}, \dots, S_0(1 - a_1) = \tau S_{(N-1)}, \tag{152}$$

where $\tau > 0$. Taking into account the relations (152), we obtain

$$S_0(1 - a_1) = \tau S_{(N-1)},$$

$$(1 - a_{N-k}) = \frac{S_{(k)}}{S_{(k+1)}} \quad k = \overline{0, N-1}. \tag{153}$$

Solving the relations (153), we have

$$a_1 = 1 - \frac{\tau}{S_0} S_{(N-1)}, \quad a_{N-k} = 1 - \frac{S_{(k)}}{S_{(k+1)}}, \quad k = \overline{1, N-2}. \tag{154}$$

It is evident that if to put $\tau = \frac{S_0}{S_{(N)}}$, then $1 - a_1 = \frac{S_{(N-1)}}{S_{(N)}}$. Therefore, $\prod\limits_{i=1}^{N-k}(1 - a_i) = \frac{S_{(k)}}{S_{(N)}}$, $k = \overline{0, N-1}$. Theorem 15 is proved.

Theorem 16. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105), with parameters a_i, $i = \overline{1, N}$, given by the formula (150). For the payoff function $f(x) = (x-K)^+$, $x \in (0, \infty)$, $K > 0$, the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if} \quad S_0 \frac{S_{(0)}}{S_{(N)}} \geq K, \\ S_0 \left(1 - \frac{S_{(0)}}{S_{(N)}}\right), & \text{if} \quad S_0 \frac{S_{(0)}}{S_{(N)}} < K. \end{cases} \tag{155}$$

If $S_0 \frac{S_{(0)}}{S_{(N)}} \geq K$, then the set of non arbitrage prices coincides with the point $(S_0 - K)^+$, in case if $S_0 \frac{S_{(0)}}{S_{(N)}} < K$ the set of non arbitrage prices coincides with the set $\left((S_0 - K)^+, S_0 \left(1 - \frac{S_{(0)}}{S_{(N)}}\right)\right)$.

Corollary 1. *Suppose that the strike price $K = S_0 \frac{S_{(0)}}{S_{(N)}}$, then the set of non arbitrage prices consists of one point $S_0 \left(1 - \frac{S_{(0)}}{S_{(N)}}\right)$. It is a fair price of a standard call option of European type with the payoff function $(S_N - K)^+$.*

This corollary is very important for practical application. The fair price of a standard call option of European type is proportional to the initial spot price of the underlying asset multiplied by the value of the relative swing of the market in the given horizon.

Theorem 17. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105) with the parameters a_i, $i = \overline{1, N}$, given by the formula (150). For the payoff function $f_1(x) = (K - x)^+$, $x \in (0, \infty)$, $K > 0$, the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_N) = f_1 \left(S_0 \frac{S_{(0)}}{S_{(N)}}\right). \tag{156}$$

The set of non arbitrage prices coincides with the interval

$\left((K - S_0)^+, f_1 \left(S_0 \frac{S_{(0)}}{S_{(N)}}\right)\right)$.

Theorem 18. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105) with the parameters a_i, $i = \overline{1, N}$, given by the formula (150). For the payoff function $f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{\sum_{i=0}^{N} S_i}{N+1}\right)^+$, $K > 0$, the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{S_0 + S_0 \sum_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)}\right)^+. \tag{157}$$

The set of non arbitrage prices coincides with the interval

$$\left((K-S_0)^+, \left(K - \frac{S_0 + S_0\sum\limits_{i=0}^{N-1}\frac{S_{(i)}}{S_{(N)}}}{(N+1)}\right)^+\right).$$

Theorem 19. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (105) with the parameters a_i, $i=\overline{1,N}$, given by the formula (150). For the payoff function $f(S_0, S_1, \ldots, S_N) = \left(\frac{\sum\limits_{i=0}^{N} S_i}{N+1} - K\right)^+$, $K>0$, the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_0, S_1, \ldots, S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if } \frac{S_0 + S_0\sum\limits_{i=0}^{N-1}\frac{S_{(i)}}{S_{(N)}}}{(N+1)} \geq K, \\ \left(S_0 - \frac{S_0 + S_0\sum\limits_{i=0}^{N-1}\frac{S_{(i)}}{S_{(N)}}}{(N+1)}\right), & \text{if } \frac{S_0 + S_0\sum\limits_{i=0}^{N-1}\frac{S_{(i)}}{S_{(N)}}}{(N+1)} < K. \end{cases} \quad (158)$$

If $\frac{S_0 + S_0\sum\limits_{i=0}^{N-1}\frac{S_{(i)}}{S_{(N)}}}{(N+1)} \geq K$, then the set of non arbitrage prices coincides with the point $(S_0 - K)^+$, in case if $\frac{S_0 + S_0\sum\limits_{i=0}^{N-1}\frac{S_{(i)}}{S_{(N)}}}{(N+1)} < K$ the set of non arbitrage prices coincides with the interval $\left((S_0 - K)^+, \left(S_0 - \frac{S_0 + S_0\sum\limits_{i=0}^{N-1}\frac{S_{(i)}}{S_{(N)}}}{(N+1)}\right)\right)$.

VI. DISCOUNTED EVOLUTION OF RISKY ASSET

In this section, we formulate Theorems, giving us the formula for the fair price of super-hedge for the evolution of risky asset, given by the formula

$$S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) = S_0 \prod_{i=1}^{n}\left(1 + a_i\left(\frac{e^{\sigma_i(\omega_1, \ldots, \omega_{i-1})\varepsilon_i(\omega_i)}}{e^{r_i}} - 1\right)\right), \quad n = \overline{1,N}. \ (159)$$

where the random value $\varepsilon_i(\omega_i)$, $\omega_i \in \Omega_i^0, i = \overline{1,N}$, takes all real values from R^1, $S_0 > 0$. The random values $\sigma_i(\omega_1, \ldots, \omega_{i-1})$ satisfy the inequalities $\sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$ and $0 \leq a_i \leq 1$, $0 < r_i < \infty$. This parametric evolution (159) is built on the discounted evolution of risky asset (17) for which the representation (12) is valid. From this representation, it follows that for such a discounted evolution, all proven Theorems regarding the existence of a family of martingale measures are valid, since the representations (18), (19) is true. Due to Theorem 8, the set of martingale measures do not depend on parameters $0 \leq a_i \leq 1$. The proof of Theorems formulated below is analogous to the proof of Theorems 9 - 14.

Theorem 20. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (159). Suppose that* $0 \leq a_i \leq 1$, $\infty > \sigma_i \geq \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $0 < r_i < \infty$, $i = \overline{1, N}$. *If the nonnegative continuous payoff function* $f(x)$, $x \in [0, \infty)$, *satisfies the conditions:*

1) $f(0) = 0$, $f(x) \leq ax$, $\lim\limits_{x \to \infty} \frac{f(x)}{x} = a$, $a > 0$, *then the inequalities*

$$f\left(S_0 \prod_{i=1}^{N}(1 - a_i)\right) + aS_0\left(1 - \prod_{i=1}^{N}(1 - a_i)\right) \leq \sup_{P \in M} E^P f(S_N) \leq aS_0 \tag{160}$$

are true. If, in addition, the nonnegative payoff function $f(x)$ is a convex down one, then

$$\inf_{P \in M} E^P f(S_N) = f(S_0), \tag{161}$$

where M is the set of equivalent martingale measures for the evolution of risky asset, given by the formula (159).

Theorem 21. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (159). Suppose that* $0 \leq a_i \leq 1$, $\infty > \sigma_i \geq \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $0 < r_i < \infty$, $i = \overline{1, N}$. *If the nonnegative continuous payoff function* $f(x)$, $x \in [0, \infty)$, *satisfies the conditions:*
1) $f(0) = K$, $f(x) \leq K$, *then*

$$f\left(S_0 \prod_{i=1}^{N}(1 - a_i)\right) \leq \sup_{P \in M} E^P f(S_N) \leq K. \tag{162}$$

If, in addition, the nonnegative payoff function $f(x)$ is a convex down one, then

$$\inf_{P \in M} E^P f(S_N) = f(S_0), \tag{163}$$

where M is the set of equivalent martingale measures for the evolution of risky asset, given by the formula (159).

Theorem 22. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (159). Suppose that* $0 \leq a_i \leq 1$, $\infty > \sigma_i \geq \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $0 < r_i < \infty$, $i = \overline{1, N}$. *For the payoff function* $f(x) = (x - K)^+$, $x \in (0, \infty)$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if} \quad S_0 \prod_{i=1}^{N}(1 - a_i)) \geq K, \\ S_0\left(1 - \prod_{i=1}^{N}(1 - a_i)\right), & \text{if} \quad S_0 \prod_{i=1}^{N}(1 - a_i) < K. \end{cases} \tag{164}$$

For $S_0 \prod_{i=1}^{N}(1 - a_i)) \geq K$, *the set of non arbitrage prices coincides with the point* $(S_0 - K)^+$, *in case if* $S_0 \prod_{i=1}^{N}(1 - a_i) < K$ *the set of non arbitrage prices coincides with the set* $\left((S_0 - K)^+, S_0\left(1 - \prod_{i=1}^{N}(1 - a_i)\right)\right)$.

Theorem 23. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (159). Suppose that* $0 \leq a_i \leq 1$, $\infty > \sigma_i \geq \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $0 < r_i < \infty$, $i = \overline{1, N}$. *For the payoff function* $f_1(x) = (K - x)^+$, $x \in (0, \infty)$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_N) = f_1\left(S_0 \prod_{i=1}^{N}(1 - a_i)\right). \tag{165}$$

The set of non arbitrage prices coincides with the interval

$$\left((K - S_0)^+, f_1\left(S_0 \prod_{i=1}^{N}(1 - a_i)\right)\right).$$

Theorem 24. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (159). Suppose that* $0 \leq a_i \leq 1$, $\infty > \sigma_i \geq \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $0 < r_i < \infty$, $i = \overline{1, N}$. *For the payoff function* $f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{\sum_{i=0}^{N} S_i}{N+1}\right)^+$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{S_0 + S_0 \sum_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)}\right)^+. \tag{166}$$

The set of non arbitrage prices coincides with the interval

$$\left((K - S_0)^+, \left(K - \frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i}(1 - a_s)}{N+1}\right)^+\right).$$

Theorem 25. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (159). Suppose that* $0 < a_i \leq 1$, $\infty > \sigma_i \geq \sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0$, $0 < r_i < \infty$, $i = \overline{1, N}$. *For the payoff function* $f(S_0, S_1, \ldots, S_N) = \left(\frac{\sum_{i=0}^{N} S_i}{N+1} - K\right)^+$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_0, S_1, \ldots, S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if } \frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i}(1 - a_i)}{N+1} \geq K, \\ S_0\left(1 - \frac{\sum_{i=0}^{N} \prod_{s=1}^{i}(1 - a_s)}{N+1}\right), & \text{if } S_0 \frac{\sum_{i=0}^{N} \prod_{s=1}^{i}(1 - a_s)}{N+1} < K. \end{cases} \tag{167}$$

The set of non arbitrage prices coincides with the point $(S_0 - K)^+$ *for* $\frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i}(1 - a_i)}{N+1} \geq$

K, *in case if* $S_0 \frac{\sum_{i=0}^{N}\prod_{s=1}^{i}(1-a_s)}{N+1} < K$ *the set of non arbitrage prices coincides with the interval* $\left((S_0 - K)^+, S_0\left(1 - \frac{\sum_{i=0}^{N}\prod_{s=1}^{i}(1-a_s)}{N+1}\right)\right)$.

Theorem 26. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (159), with parameters* a_i, $i = \overline{1,N}$, *given by the formula (150). For the payoff function* $f(x) = (x-K)^+$, $x \in (0,\infty)$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q\in M} E^Q f(S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if} \quad S_0\frac{S_{(0)}}{S_{(N)}} \geq K, \\ S_0\left(1 - \frac{S_{(0)}}{S_{(N)}}\right), & \text{if} \quad S_0\frac{S_{(0)}}{S_{(N)}} < K. \end{cases} \tag{168}$$

If $S_0\frac{S_{(0)}}{S_{(N)}} \geq K$, *then the set of non arbitrage prices coincides with the point* $(S_0 - K)^+$, *in case if* $S_0\frac{S_{(0)}}{S_{(N)}} < K$ *the set of non arbitrage prices coincides with the set* $\left((S_0 - K)^+, S_0\left(1 - \frac{S_{(0)}}{S_{(N)}}\right)\right)$.

Corollary 2. *Suppose that the strike price* $K = S_0\frac{S_{(0)}}{S_{(N)}}$, *then the set of non arbitrage prices consists of one point* $S_0\left(1 - \frac{S_{(0)}}{S_{(N)}}\right)$. *It is a fair price of a standard call option of European type with the payoff function* $(S_N - K)^+$.

This corollary is very important for practical application. The fair price of a standard call option of European type is proportional to the initial spot price of the underlying asset multiplied by the value of the relative swing of the market in the given horizon.

Theorem 27. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (159) with the parameters* a_i, $i = \overline{1,N}$, *given by the formula (150). For the payoff function* $f_1(x) = (K-x)^+$, $x \in (0,\infty)$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q\in M} E^Q f_1(S_N) = f_1\left(S_0\frac{S_{(0)}}{S_{(N)}}\right). \tag{169}$$

The set of non arbitrage prices coincides with the interval $\left((K - S_0)^+, f_1\left(S_0\frac{S_{(0)}}{S_{(N)}}\right)\right)$.

Theorem 28. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (159) with the parameters* a_i, $i = \overline{1,N}$, *given by the formula (150). For the payoff function* $f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{\sum_{i=0}^{N} S_i}{N+1}\right)^+$, $K > 0$, *the fair* *price of super-hedge is given by the formula*

$$\sup_{Q\in M} E^Q f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{S_0 + S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)}\right)^+. \tag{170}$$

The set of non arbitrage prices coincides with the interval

$$\left((K-S_0)^+, \left(K - \frac{S_0+S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)}\right)^+\right).$$

Theorem 29. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (159) with the parameters a_i, $i = \overline{1,N}$, given by the formula (150). For the payoff function $f(S_0, S_1, \ldots, S_N) = \left(\frac{\sum\limits_{i=0}^{N} S_i}{N+1} - K\right)^+$, $K > 0$, the fair price of super-hedge is given by the formula*

$$\sup_{Q\in M} E^Q f(S_0, S_1, \ldots, S_N) =$$

$$\begin{cases} (S_0-K)^+, & \text{if } \frac{S_0+S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)} \geq K, \\ \left(S_0 - \frac{S_0+S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)}\right), & \text{if } \frac{S_0+S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)} < K. \end{cases} \tag{171}$$

If $\frac{S_0+S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)} \geq K$, then the set of non arbitrage prices coincides with the point $(S_0-K)^+$, in case if $\frac{S_0+S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)} < K$ the set of non arbitrage prices coincides with the interval $\left((S_0-K)^+, \left(S_0 - \frac{S_0+S_0 \sum\limits_{i=0}^{N-1} \frac{S_{(i)}}{S_{(N)}}}{(N+1)}\right)\right)$.

VII. UNIQUENESS OF THE MARTINGALE MEASURE

In this section, the necessary and sufficient conditions of the uniqueness of martingale measure in terms of the evolution of risky assets are obtained. Under the fairly wide assumptions about the evolution of risky assets, an expression for a single martingale measure is found. Based on the explicit construction of the martingale measure and its invariance with respect to a certain type of evolutions, it is possible to construct the models of non arbitrage markets, both complete and incomplete.

In this and section 8, we put that $\Omega_i^0 = \{\omega_i^1, \omega_i^2\}$. Denote by $\mathcal{F}_i^0$ the σ-algebra of all subsets of the set Ω_i^0. Let P_i^0 be a probability measure on $\mathcal{F}_i^0$. We assume that $P_i^0(\omega_i^s) > 0$, $i = \overline{1,N}$, $s = \overline{1,2}$. As before, we put that the probability space

$\{\Omega_N, \mathcal{F}_N, P_N\}$ is a direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, $i = \overline{1,N}$, and we put $N < \infty$. We also consider the probability spaces $\{\Omega_n, \mathcal{F}_n, P_n\}, n = \overline{1,N}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, $i = \overline{1,n}$. We assume that the evolution of a risky asset is given by the formula

$$S_n(\omega_1, \ldots, \omega_n) =$$

$$S_0 \prod_{i=1}^{n} (1 + a_i(\omega_1, \ldots, \omega_i)\eta_i(\omega_i)), \ \{\omega_1, \ldots, \omega_{n-1}, \omega_n\} \in \Omega_n, \ n = \overline{1,N}, \quad S_0 > 0, \tag{172}$$

where the random values $a_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)$, $\eta_n(\omega_n)$, $n = \overline{1,N}$, given on the probability space $\{\Omega_n, \mathcal{F}_n, P_n\}$, satisfy the conditions

$$a_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) > 0, \qquad \max_{\{\omega_1, \ldots, \omega_{n-1}\} \in \Omega_{n-1}} a_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) < \frac{1}{\eta_n^-(\omega_n^1)},$$

$$\eta_n(\omega_n^2) > 0, \quad \eta_n(\omega_n^1) < 0. \tag{173}$$

So, for $\Delta S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)$, $n = \overline{1,N}$, the representation

$$\Delta S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) =$$

$$S_{n-1}(\omega_1, \ldots, \omega_{n-1}) a_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)\eta_n(\omega_n) =$$

$$d_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)\eta_n(\omega_n), \quad d_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) > 0, \quad n = \overline{1,N}, \tag{174}$$

is true, From these conditions, we obtain $\Omega_n^- = \Omega_{n-1} \times \Omega_n^{0-}$, $\Omega_n^+ = \Omega_{n-1} \times \Omega_n^{0+}$, where $\Omega_n^{0-} = \{\omega_n \in \Omega_n^0, \eta_n(\omega_n) < 0\}$, $\Omega_n^{0+} = \{\omega_n \in \Omega_n^0, \eta_n(\omega_n) > 0\}$.

Further, we assume that $P_n^0(\Omega_n^{0-}) > 0$, $P_n^0(\Omega_n^{0+}) > 0$. The measure P_n^{0-} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0-} = \Omega_n^{0-} \cap \mathcal{F}_n^0$, P_n^{0+} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0+} = \Omega_n^{0+} \cap \mathcal{F}_n^0$.

Let us introduce the following denotations. For every point $\{\omega_1, \ldots, \omega_{n-1}, \omega_n\} \in \Omega_n$, we introduce the set $A(\omega_1, \ldots, \omega_{n-1}, \omega_n) \in \Omega_N$, where

$$A(\omega_1, \ldots, \omega_{n-1}, \omega_n) = \bigcup_{i_{n+1}=1, \ldots, i_N=1}^{2} \{\omega_1, \ldots, \omega_{n-1}, \omega_n, \omega_{n+1}^{i_{n+1}}, \ldots, \omega_N^{i_N}\}.$$

Sometimes, for fixed indexes $i_1, \ldots, i_n$ we also use the denotation

$$A(\omega_1^{i_1}, \ldots, \omega_{n-1}^{i_{n-1}}, \omega_n^{i_n}) = A^{i_1, \ldots, i_n}.$$

It is evident that every set $A^{i_1, \ldots, i_n}$ has the form

$$A^{i_1, \ldots, i_n} = \bigcup_{i_{n+1}=1, \ldots, i_N=1}^{2} \{\omega_1^{i_1}, \ldots, \omega_n^{i_n}, \omega_{n+1}^{i_{n+1}}, \ldots, \omega_N^{i_N}\},$$

where indexes i_s, $s = \overline{1,N}$, take values from the set $\{1,2\}$. Then, $A^{i_1,\dots,i_{n-1}} = A^{i_1,\dots,i_{n-1},1} \cup A^{i_1,\dots,i_{n-1},2} \in \mathcal{F}_{n-1}$, where

$$A^{i_1,\dots,i_{n-1},1} = \bigcup_{i_{n+1}=1,\dots,i_N=1}^{2} \{\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^1,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\} \in \mathcal{F}_n,$$

$$A^{i_1,\dots,i_{n-1},2} = \bigcup_{i_{n+1}=1,\dots,i_N=1}^{2} \{\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^2,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\}\} \in \mathcal{F}_n.$$

If P_N is a measure on $\mathcal{F}_N$, then

$$P_N(A(\omega_1,\dots,\omega_{n-1},\omega_n)) = \sum_{i_{n+1}=1,\dots,i_N=1}^{2} P_N(\{\omega_1,\dots,\omega_{n-1},\omega_n,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\}).$$

We give an evident construction of martingale measure for risky asset evolution, given by the formula (172). Let us put $P_n^0(\omega_n^1) = p_n$, $P_n^0(\omega_n^2) = 1 - p_n$, where $0 < p_n < 1$. Then, to satisfy the conditions (14) - (16), (see [2]) we need to put that

$$\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1) < \infty, \quad (\omega_1,\dots,\omega_{n-1},\omega_n^1) \in \Omega_n^-,$$

$$\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2) < \infty, \quad (\omega_1,\dots,\omega_{n-1},\omega_n^2) \in \Omega_n^+. \qquad (175)$$

The next Lemma 3 is a consequence of results in [2].

Lemma 3. *On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, for the evolution of risky asset given by the formula (172) only one spot measure $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ exists, where $\{\omega_i^1,\omega_i^2\} \in \Omega_i^0$, $i = \overline{1,N}$. For it the representation*

$$\mu_0(A) = \mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A) =$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\chi_A(\omega_1^{i_1},\dots,\omega_N^{i_N}), \quad A \in \mathcal{F}_N, \qquad (176)$$

is true. This measure is a martingale one for the considered evolution of risky asset, where

$$\psi_n(\omega_1,\dots,\omega_n) = \chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n)\psi_n^1(\omega_1,\dots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n)\psi_n^2(\omega_1,\dots,\omega_n), \qquad (177)$$

$$\psi_n^1(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$\chi_{\Omega_n^{0+}}(\omega_n^2)\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\dots,\omega_{n-1}) \in \Omega_{n-1}, \quad \omega_n^2 \in \Omega_n^{0+}, \qquad (178)$$

$$V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2) = \Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1) + \Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2),$$

$$\psi_n^2(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$\chi_{\Omega_n^{0-}}(\omega_n^1)\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\dots,\omega_{n-1}) \in \Omega_{n-1}, \quad \omega_n^1 \in \Omega_n^{0-}. \qquad (179)$$

Next Theorem 30 appeared first in [24] (Theorem 1.4.1), where it was proved under the less general conditions.

Theorem 30. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, suppose that the evolution of risky asset $\{S_n(\omega_1, \dots, \omega_n)\}_{n=1}^N$ is given by the formula (172). The necessary and sufficient conditions of the uniqueness of martingale measure $\mu_0(A)$, $A \in \mathcal{F}_N$, are the inequalities*

$$S_n(\omega_1^{i_1}, \dots, \omega_{n-1}^{i_{n-1}}, \omega_n^1) \neq S_n(\omega_1^{i_1}, \dots, \omega_{n-1}^{i_{n-1}}, \omega_n^2), \quad n = \overline{1, N}, \tag{180}$$

for every set of indexes $i_1, \dots, i_{n-1}$. For any martingale $\{m_n(\omega_1, \dots, \omega_{n-1}, \omega_n)\}_{n=0}^N$ relative to the unique measure $\mu_0(A)$ the representation

$$m_n(\omega_1, \dots, \omega_{n-1}, \omega_n) =$$

$$\sum_{k=1}^{n} C_k(\omega_1, \dots, \omega_{k-1})[S_k(\omega_1, \dots, \omega_{k-1}, \omega_i) - S_{k-1}(\omega_1, \dots, \omega_{k-1})] +$$

$$m_0, \quad n = \overline{1, N}, \tag{181}$$

is true, where

$$C_k(\omega_1, \dots, \omega_{k-1}) = \sum_{i_1=1,\dots,i_{k-1}=1}^{2} d_{i_1,\dots,i_{k-1}} \chi_{A^{i_1,\dots,i_{k-1}}}(\omega_1, \dots, \omega_{k-1}), \tag{182}$$

$$d_{i_1,\dots,i_{k-1}} =$$

$$\frac{m_k(\omega_1^{i_1}, \dots, \omega_{k-1}^{i_{k-1}}, \omega_k^1) - m_k(\omega_1^{i_1}, \dots, \omega_{k-1}^{i_{k-1}}, \omega_k^2)}{S_k(\omega_1^{i_1}, \dots, \omega_{k-1}^{i_{k-1}}, \omega_k^1) - S_k(\omega_1^{i_1}, \dots, \omega_{k-1}^{i_{k-1}}, \omega_k^2)}, \quad k = \overline{1, N}. \tag{183}$$

Proof. The necessity. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution $\{S_n(\omega_1, \dots, \omega_n)\}_{n=1}^N$ of risky asset be such that the martingale measure $\mu_0(A)$, $A \in \mathcal{F}_N$, being equivalent to the measure P_N, is unique. Then, for every contingent liability $m_N(\omega_1, \dots, \omega_N)$ the representation (181) is true [13] for some $\mathcal{F}_{k-1}$-measurable finite valued random value $C_k(\omega_1, \dots, \omega_{k-1})$, $k = \overline{1, N}$, where $m_n(\omega_1, \dots, \omega_{n-1}, \omega_n) = E^{\mu_0}\{m_N(\omega_1, \dots, \omega_N)|\mathcal{F}_n\}$. For $m_n(\omega_1, \dots, \omega_{n-1}, \omega_n)$ and $S_n(\omega_1, \dots, \omega_{n-1}, \omega_n)$ the representations

$$m_n(\omega_1, \dots, \omega_{n-1}, \omega_n) =$$

$$\sum_{i_1=1,\dots,i_n=1}^{2} \frac{\chi_{A^{i_1,\dots,i_{n-1},i_n}}(\omega_1, \dots, \omega_n)}{\mu_0(A^{i_1,\dots,i_{n-1},i_n})} \int_{A^{i_1,\dots,i_{n-1},i_n}} m_N(\omega_1, \dots, \omega_N) d\mu_0, \ n = \overline{1, N}, \tag{184}$$

$$S_n(\omega_1, \dots, \omega_{n-1}, \omega_n) =$$

$$\sum_{i_1=1,\dots,i_n=1}^{2} \frac{\chi_{A^{i_1,\dots,i_{n-1},i_n}}(\omega_1, \dots, \omega_n)}{\mu_0(A^{i_1,\dots,i_{n-1},i_n})} \int_{A^{i_1,\dots,i_{n-1},i_n}} S_N(\omega_1, \dots, \omega_N) d\mu_0, \ n = \overline{1, N}, \tag{185}$$

are true. From the representation (181) and the equality (182) for $\{\omega_1, \ldots, \omega_{n-1}\} \in A^{i_1,\ldots,i_{n-1}}$ we obtain the equality

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1},1}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},1})} \int\limits_{A^{i_1,\ldots,i_{n-1},1}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 +$$

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1},2}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},2})} \int\limits_{A^{i_1,\ldots,i_{n-1},2}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 -$$

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1}}}(\omega_1,\ldots,\omega_{n-1})}{\mu_0(A^{i_1,\ldots,i_{n-1}})} \int\limits_{A^{i_1,\ldots,i_{n-1}}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 =$$

$$d_{i_1,\ldots,i_{n-1}}\chi_{A^{i_1,\ldots,i_{n-1}}}(\omega_1,\ldots,\omega_{n-1})\times$$

$$\left[\frac{\chi_{A^{i_1,\ldots,i_{n-1},1}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},1})} \int\limits_{A^{i_1,\ldots,i_{n-1},1}} S_N(\omega_1,\ldots,\omega_N)d\mu_0 +\right.$$

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1},2}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},2})} \int\limits_{A^{i_1,\ldots,i_{n-1},2}} S_N(\omega_1,\ldots,\omega_N)d\mu_0 -$$

$$\left.\frac{\chi_{A^{i_1,\ldots,i_{n-1}}}(\omega_1,\ldots,\omega_{n-1})}{\mu_0(A^{i_1,\ldots,i_{n-1}})} \int\limits_{A^{i_1,\ldots,i_{n-1}}} S_N(\omega_1,\ldots,\omega_N)d\mu_0\right], \tag{186}$$

where $d_{i_1,\ldots,i_{n-1}}$ is finite. Since

$$\int\limits_{A^{i_1,\ldots,i_{n-1}}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 =$$

$$\int\limits_{A^{i_1,\ldots,i_{n-1},1}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 + \int\limits_{A^{i_1,\ldots,i_{n-1},2}} m_N(\omega_1,\ldots,\omega_N)d\mu_0, \tag{187}$$

we have

$$\mu_0(A^{i_1,\ldots,i_{n-1}}) \int\limits_{A^{i_1,\ldots,i_{n-1},1}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\ldots,i_{n-1},1}) \int\limits_{A^{i_1,\ldots,i_{n-1}}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 =$$

$$[\mu_0(A^{i_1,\ldots,i_{n-1},1}) + \mu_0(A^{i_1,\ldots,i_{n-1},2})] \int\limits_{A^{i_1,\ldots,i_{n-1},1}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\dots,i_{n-1},1})\left[\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0 + \int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0\right] =$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2})\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\dots,i_{n-1},1})\int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0. \tag{188}$$

Further,

$$\mu_0(A^{i_1,\dots,i_{n-1}})\int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2})\int\limits_{A^{i_1,\dots,i_{n-1}}} m_N(\omega_1,\dots,\omega_N)d\mu_0 =$$

$$[\mu_0(A^{i_1,\dots,i_{n-1},1}) + \mu_0(A^{i_1,\dots,i_{n-1},2})]\int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2})\left[\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0 + \int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0\right] =$$

$$-\left[\mu_0(A^{i_1,\dots,i_{n-1},2})\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0 -\right.$$

$$\left.\mu_0(A^{i_1,\dots,i_{n-1},1})\int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0\right]. \tag{189}$$

If to put

$$R_1^m(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}}) = \mu_0(A^{i_1,\dots,i_{n-1},1})\int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2})\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0, \tag{190}$$

$$R_1^{S_N}(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}}) = \mu_0(A^{i_1,\dots,i_{n-1},1})\int\limits_{A^{i_1,\dots,i_{n-1},2}} S_N(\omega_1,\dots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2}) \int\limits_{A^{i_1,\dots,i_{n-1},1}} S_N(\omega_1,\dots,\omega_N)d\mu_0, \tag{191}$$

then the equality (186) is transformed into the equality

$$R_1^m(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}}) = d_{i_1,\dots,i_{n-1}} R_1^{S_N}(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}}). \tag{192}$$

Due to that $S_n(\omega_1,\dots,\omega_n)$ and $m_n(\omega_1,\dots,\omega_n)$ are martingales relative to the measure μ_0 and $A^{i_1,\dots,i_{n-1},1}$, $A^{i_1,\dots,i_{n-1},2} \in \mathcal{F}_n$ we have

$$\int\limits_{A^{i_1,\dots,i_{n-1},1}} S_N(\omega_1,\dots,\omega_N)d\mu_0 = \int\limits_{A^{i_1,\dots,i_{n-1},1}} S_n(\omega_1,\dots,\omega_n)d\mu_0 =$$

$$\mu_0(A^{i_1,\dots,i_{n-1},1})S_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^1), \tag{193}$$

$$\int\limits_{A^{i_1,\dots,i_{n-1},2}} S_N(\omega_1,\dots,\omega_N)d\mu_0 = \int\limits_{A^{i_1,\dots,i_{n-1},2}} S_n(\omega_1,\dots,\omega_n)d\mu_0 =$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2})S_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^2), \tag{194}$$

$$\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0 = \int\limits_{A^{i_1,\dots,i_{n-1},1}} m_n(\omega_1,\dots,\omega_n)d\mu_0 =$$

$$\mu_0(A^{i_1,\dots,i_{n-1},1})m_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^1), \tag{195}$$

$$\int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0 = \int\limits_{A^{i_1,\dots,i_{n-1},2}} m_n(\omega_1,\dots,\omega_n)d\mu_0 =$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2})m_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^2). \tag{196}$$

Since $d_{i_1,\dots,i_{n-1}}$ is finite and $m_N(\omega_1,\dots,\omega_N)$ is arbitrary, then $R_1^{S_N}(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}}) \neq 0$. The last means that inequality (180) takes place. This proves the equality

$$d_{i_1,\dots,i_{n-1}} = \tag{197}$$

$$\frac{m_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^1) - m_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^2)}{S_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^1) - S_n(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^2)},$$

$$n = \overline{1, N},$$

which means that (183) is true, where we introduced the denotations

$$m_n(\omega_1,\dots,\omega_n) = E^{\mu_0}\{m_N(\omega_1,\dots,\omega_N)|\mathcal{F}_n\} =$$

$$\frac{\sum_{i_{n+1}=1,\dots,i_N=1}^{2} m(\omega_1,\dots,\omega_n,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N})\mu_0(\{\omega_1,\dots,\omega_n,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\})}{\sum_{i_{n+1}=1,\dots,i_N=1}^{2} \mu_0(\{\omega_1,\dots,\omega_n,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\})}, \quad (198)$$

$$S_n(\omega_1,\dots,\omega_n) = E^{\mu_0}\{S_N(\omega_1,\dots,\omega_N)|\mathcal{F}_n\} =$$

$$\frac{\sum_{i_{n+1}=1,\dots,i_N=1}^{2} S_N(\omega_1,\dots,\omega_n,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N})\mu_0(\{\omega_1,\dots,\omega_n,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\})}{\sum_{i_{n+1}=1,\dots,i_N=1}^{2} \mu_0(\{\omega_1,\dots,\omega_n,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\})}. \quad (199)$$

This proves the necessity.

Proof of the sufficiency. Suppose that the inequalities (180) are true. Let us prove that the martingale measure μ_0 is unique. For this purpose, we prove that for every martingale the representation (181) is true with validity of equalities (182), (183).

Let us note that the equality (186) is true if for $d_{i_1,\dots,i_{n-1}}$ to choose the right hand side of the equality (197), since the equalities

$$\left[\frac{\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1},1})} - \frac{\int\limits_{A^{i_1,\dots,i_{n-1}}} m_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1}})}\right] \times$$

$$\left[\frac{\int\limits_{A^{i_1,\dots,i_{n-1},1}} S_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1},1})} - \frac{\int\limits_{A^{i_1,\dots,i_{n-1}}} S_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1}})}\right]^{-1} =$$

$$\left[\frac{\int\limits_{A^{i_1,\dots,i_{n-1},2}} m_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1},2})} - \frac{\int\limits_{A^{i_1,\dots,i_{n-1}}} m_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1}})}\right] \times$$

$$\left[\frac{\int\limits_{A^{i_1,\dots,i_{n-1},2}} S_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1},2})} - \frac{\int\limits_{A^{i_1,\dots,i_{n-1}}} S_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1}})}\right]^{-1} =$$

$$d_{i_1,\dots,i_{n-1}} \quad (200)$$

are valid. Taking into account the equality (186) and the equalities

$$d_{i_1,\dots,i_{n-1}}\chi_{A^{i_1,\dots,i_{n-1}}}(\omega_1,\dots,\omega_{n-1})\times$$

$$\left[\frac{\chi_{A^{i_1,\dots,i_{n-1},1}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{i_1,\dots,i_{n-1},1})}\int\limits_{A^{i_1,\dots,i_{n-1},1}} S_N(\omega_1,\dots,\omega_N)d\mu_0+\right.$$

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1},2}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},2})} \int\limits_{A^{i_1,\ldots,i_{n-1},2}} S_N(\omega_1,\ldots,\omega_N)d\mu_0 -$$

$$\left.\frac{\chi_{A^{i_1,\ldots,i_{n-1}}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1}})} \int\limits_{A^{i_1,\ldots,i_{n-1}}} S_N(\omega_1,\ldots,\omega_N)d\mu_0 \right] =$$

$$d_{i_1,\ldots,i_{n-1}}\chi_{A^{i_1,\ldots,i_{n-1}}}(\omega_1,\ldots,\omega_{n-1})\times$$

$$\sum_{j_1=1,\ldots j_{n-1}=1}^{2} \left[\frac{\chi_{A^{j_1,\ldots,j_{n-1},1}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{j_1,\ldots,j_{n-1},1})} \int\limits_{A^{j_1,\ldots,j_{n-1},1}} S_N(\omega_1,\ldots,\omega_N)d\mu_0 + \right.$$

$$\frac{\chi_{A^{j_1,\ldots,j_{n-1},2}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{j_1,\ldots,j_{n-1},2})} \int\limits_{A^{j_1,\ldots,j_{n-1},2}} S_N(\omega_1,\ldots,\omega_N)d\mu_0 -$$

$$\left.\frac{\chi_{A^{j_1,\ldots,j_{n-1}}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{j_1,\ldots,j_{n-1}})} \int\limits_{A^{j_1,\ldots,j_{n-1}}} S_N(\omega_1,\ldots,\omega_N)d\mu_0 \right] = \tag{201}$$

$$d_{i_1,\ldots,i_{n-1}}\chi_{A^{i_1,\ldots,i_{n-1}}}(\omega_1,\ldots,\omega_n)[S_n(\omega_1,\ldots,\omega_{n-1},\omega_n) - S_{n-1}(\omega_1,\ldots,\omega_{n-1})],$$

we have

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1},1}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},1})} \int\limits_{A^{i_1,\ldots,i_{n-1},1}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 +$$

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1},2}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},2})} \int\limits_{A^{i_1,\ldots,i_{n-1},2}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 -$$

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1}}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1}})} \int\limits_{A^{i_1,\ldots,i_{n-1}}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 =$$

$$d_{i_1,\ldots,i_{n-1}}\chi_{A^{i_1,\ldots,i_{n-1}}}(\omega_1,\ldots,\omega_n)[S_n(\omega_1,\ldots,\omega_{n-1},\omega_n) - S_{n-1}(\omega_1,\ldots,\omega_{n-1})]. \tag{202}$$

Summing over all indexes $i_1,\ldots,i_{n-1}$ the left and right hand sides of the equality (202), we obtain the equalities

$$m_n(\omega_1,\ldots,\omega_n) - m_{n-1}(\omega_1,\ldots,\omega_{n-1}) =$$

$$C_n(\omega_1,\ldots,\omega_{n-1})[S_n(\omega_1,\ldots,\omega_{n-1},\omega_n) - S_{n-1}(\omega_1,\ldots,\omega_{n-1})], \tag{203}$$

$$C_n(\omega_1,\ldots,\omega_{n-1}) = \sum_{i_1=1,\ldots,i_{n-1}=1}^{2} d_{i_1,\ldots,i_{n-1}}\chi_{A^{i_1,\ldots,i_{n-1}}}(\omega_1,\ldots,\omega_{n-1}). \tag{204}$$

We proved that for every martingale $\{m_n(\omega_1,\ldots,\omega_n)\}_{n=0}^N$ relative to measure μ_0 the representation (181) is true, due to the conditions (180). Let us prove that the martingale measure is unique. Suppose that there are at most two martingale measures μ_0^1 and μ_0^2. If to put $m(\omega_1,\ldots,\omega_N)=\chi_A(\omega_1,\ldots,\omega_N)$, then

$$\chi_A(\omega_1,\ldots,\omega_N)=$$

$$\sum_{n=1}^{N} C_n(\omega_1,\ldots,\omega_{n-1})[S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)-S_{n-1}(\omega_1,\ldots,\omega_{n-1})]+c_0. \quad (205)$$

From this representation, we obtain the equalities $\mu_0^1(A)=\mu_0^2(A)=c_0,\ A\in\mathcal{F}_N$. Contradiction. The last proves Theorem 30.

Next Theorem is concerned the case as the set of martingale measures consists of one measure.

Theorem 31. *On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, suppose that the evolution of risky asset is given by the formula (172), then the set of martingale measures, being equivalent to the measure P_N, consists of one point*

$$\mu_0(A)=$$

$$\sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\ldots,\omega_n^{i_n})\chi_A(\omega_1^{i_1},\ldots,\omega_N^{i_N}),\quad A\in\mathcal{F}_N. \quad (206)$$

The fair price φ_0 of European type option with the payoff function $\varphi(\omega_1,\ldots,\omega_N)$ is given by the formula

$$\varphi_0=\sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\ldots,\omega_n^{i_n})\varphi(\omega_1^{i_1},\ldots,\omega_N^{i_N}), \quad (207)$$

where the number of shares is determined by the formula (208) and the number of bonds is determined by the formula (209)

$$\gamma_k(\omega_1,\ldots,\omega_{k-1})=$$

$$\frac{m_k(\omega_1,\ldots,\omega_{k-1},\omega_k^1)-m_k(\omega_1,\ldots,\omega_{k-1},\omega_k^2)}{S_k(\omega_1,\ldots,\omega_{k-1},\omega_k^1)-S_k(\omega_1,\ldots,\omega_{k-1},\omega_k^2)},\quad k=\overline{1,N}, \quad (208)$$

$$\beta_k(\omega_1,\ldots,\omega_{k-1})=$$

$$m_{k-1}(\omega_1,\ldots,\omega_{k-1})-\gamma_k(\omega_1,\ldots,\omega_{k-1})S_{k-1}(\omega_1,\ldots,\omega_{k-1}),\quad k=\overline{1,N}, \quad (209)$$

where

$$m_k(\omega_1,\ldots,\omega_k)=E^{\mu_0}\{\varphi(\omega_1,\ldots,\omega_N)|\mathcal{F}_k\}=$$

$$\frac{\sum\limits_{i_{k+1}=1,\ldots,i_N=1}^{2}\varphi(\omega_1,\ldots,\omega_k,\omega_{k+1}^{i_{k+1}},\ldots,\omega_N^{i_N})\mu_0(\{\omega_1,\ldots,\omega_k,\omega_{k+1}^{i_{n+1}},\ldots,\omega_N^{i_N}\})}{\sum\limits_{i_{k+1}=1,\ldots,i_N=1}^{2}\mu_0(\{\omega_1,\ldots,\omega_k,\omega_{k+1}^{i_{k+1}},\ldots,\omega_N^{i_N}\})},$$

$$\psi_n(\omega_1,\dots,\omega_n) = \chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n)\,\psi_n^1(\omega_1,\dots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n)\,\psi_n^2(\omega_1,\dots,\omega_n), \tag{210}$$

$$\psi_n^1(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$\chi_{\Omega_n^{0+}}(\omega_n^2)\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\dots,\omega_{n-1}) \in \Omega_{n-1}, \quad \omega_n^2 \in \Omega_n^{0+}, \tag{211}$$

$$\psi_n^2(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$\chi_{\Omega_n^{0-}}(\omega_n^1)\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\dots,\omega_{n-1}) \in \Omega_{n-1}, \quad \omega_n^1 \in \Omega_n^{0-}. \tag{212}$$

Proof. Since

$$\psi_n^1(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)} > 0, \quad (\omega_1,\dots,\omega_{n-1}) \in \Omega_{n-1}, \quad \omega_n^2 \in \Omega_n^{0+}, \tag{213}$$

$$\psi_n^2(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)} > 0, \quad (\omega_1,\dots,\omega_{n-1}) \in \Omega_{n-1}, \quad \omega_n^1 \in \Omega_n^{0-}, \tag{214}$$

we have

$$\psi_n(\omega_1,\dots,\omega_n) = \chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n)\,\psi_n^1(\omega_1,\dots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n)\,\psi_n^2(\omega_1,\dots,\omega_n) > 0, \quad (\omega_1,\dots,\omega_n) \in \Omega_n. \tag{215}$$

From this, it follows that $\mu_0(A) > 0$ for every $A \in \mathcal{F}_N$. It means that $\mu_0(A)$ is equivalent to P_N. The inequality

$$S_n(\omega_1,\dots,\omega_{n-1},\omega_n^1) = \prod_{i=1}^{n-1}(1+a_i(\omega_1,\dots,\omega_i)\eta_i(\omega_i))(1+a_n(\omega_1,\dots,\omega_n^1)\eta_i(\omega_n^1)) \neq$$

$$S_n(\omega_1,\dots,\omega_{n-1},\omega_n^2) =$$

$$\prod_{i=1}^{n-1}(1+a_i(\omega_1,\dots,\omega_i)\eta_i(\omega_i))(1+a_n(\omega_1,\dots,\omega_n^2)\eta_i(\omega_n^2)), \quad n=\overline{1,N}, \tag{216}$$

is true, since

$$(1+a_n(\omega_1,\dots,\omega_n^1)\eta_i(\omega_n^1)) \neq$$

$$(1 + a_n(\omega_1, \dots, \omega_n^2)\eta_i(\omega_n^2)), \quad n = \overline{1, N}, \tag{217}$$

due to the suppositions relative to the evolutions of risky asset, given by the formula (172). Thanks to Theorem 30, the martingale measure μ_0 is unique.

To prove the rest statement of Theorem 31, we need to construct the self-financing strategy π such that the capital corresponding this strategy on (B, S) market satisfies the condition

$$X_0^\pi = E^{\mu_0}\varphi(\omega_1, \dots, \omega_{n-1}, \omega_N), \quad X_N^\pi = \varphi(\omega_1, \dots, \omega_{n-1}, \omega_N).$$

Let us consider the martingale

$$m_n(\omega_1, \dots, \omega_{n-1}, \omega_n) = E^{\mu_0}\{\varphi(\omega_1, \dots, \omega_{n-1}, \omega_N)|\mathcal{F}_n\}.$$

Due to Theorem 30, for the finite martingale $\{m_n(\omega_1, \dots, \omega_{n-1}, \omega_n)\}_{n=0}^N$ relative to the the measure $\mu_0(A)$ the representation

$$m_n(\omega_1, \dots, \omega_{n-1}, \omega_n) =$$

$$\sum_{i=1}^{n} C_i(\omega_1, \dots, \omega_{i-1})[S_i(\omega_1, \dots, \omega_{i-1}, \omega_i) - S_{i-1}(\omega_1, \dots, \omega_{i-1})] +$$

$$m_0, \quad n = \overline{1, N}, \tag{218}$$

is true, where $C_i(\omega_1, \dots, \omega_{i-1})$ is $\mathcal{F}_{i-1}$ measurable random value, and $m_0 = E^{\mu_0}\varphi(\omega_1, \dots, \omega_{n-1}, \omega_N)$. If to put $\pi = \{\beta_n, \gamma_n\}_{n=0}^N$, where

$$\gamma_n = C_n(\omega_1, \dots, \omega_{n-1}), \quad \beta_n = m_{n-1}(\omega_1, \dots, \omega_{n-1}) - \gamma_n S_{n-1}(\omega_1, \dots, \omega_{n-1}),$$

then it easy to see that π is self-financed strategy. Really, since $B_n = 1, \ n = \overline{0, N}$, we have

$$\Delta\beta_n B_{n-1} + \Delta\gamma_n S_{n-1} = \Delta\beta_n + \Delta\gamma_n S_{n-1} =$$
$$m_{n-1} - \gamma_n S_{n-1} - m_{n-2} + \gamma_{n-1}S_{n-2} + (\gamma_n - \gamma_{n-1})S_{n-1} =$$
$$m_{n-1} - m_{n-2} - \gamma_{n-1}(S_{n-1} - S_{n-2}) = 0.$$

$\mathcal{F}_{n-1}$-measurability of (β_n, γ_n) is evident. It is easy to show that

$$X_n^\pi(\omega_1, \dots, \omega_n) = \beta_n B_n + \gamma_n S_n = m_n(\omega_1, \dots, \omega_n).$$

Therefore,

$$X_0^\pi = m_0 = E^{\mu_0}\varphi(\omega_1, \dots, \omega_{n-1}, \omega_N), \quad X_N = \varphi(\omega_1, \dots, \omega_{n-1}, \omega_N).$$

VIII. COMPLETE MARKET HEDGING

In this section, the securities market is constructed, the evolution of which occurs in accordance with formula (172). Possible for this was the observation that with respect to a certain class of evolutions of risky assets, the family of martingale measures is invariant. This fact turned out to be crucial for the construction of models of non-arbitrage markets. In papers [11], [13], such a possibility of the existence of non-arbitrage markets is established on the basis of the Hahn-Banach Theorem. This beautiful result has the disadvantage that it does not provide an algorithm for constructing models of non-arbitrage markets. How to build them having the evolution of risky assets is practically a difficult problem.

In Proposition 1, we establish the form of measurable transformations relative to which the only measure is invariant. Using that, a model of the securities market is built, which is complete. This result is constructive in contrast to the existence theorem from [11], [13]. Our denotations in this section are the same as in the previous section. We consider the evolution of risky asset, given by the formula (172), on the same probability space.

Proposition 1. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset be given by the formula (172), with $a_i(\omega_1, \ldots, \omega_i) = b_i(\omega_1, \ldots, \omega_{i-1}) f_i(\omega_1, \ldots, \omega_i)$, where the random variables $f_i(\omega_1, \ldots, \omega_i)$, $b_i(\omega_1, \ldots, \omega_{i-1})$, satisfy the inequalities*

$$f_i(\omega_1, \ldots, \omega_i) > 0, \quad b_i(\omega_1, \ldots, \omega_{i-1}) > 0, \quad \max_{\{\omega_1, \ldots, \omega_{i-1}\} \in \Omega_{i-1}} b_i(\omega_1, \ldots, \omega_{i-1}) <$$

$$\frac{1}{\max\limits_{\{\omega_1, \ldots, \omega_{i-1}\} \in \Omega_{i-1}} f_i(\omega_1, \ldots, \omega_{i-1}, \omega_i^1) \eta_i^-(\omega_i^1)}, \quad i = \overline{1, N}. \tag{219}$$

For such an evolution, the unique martingale measure μ_0 does not depend on the random variables $b_i(\omega_1, \ldots, \omega_{i-1})$, $i = \overline{1, N}$, and it is given by the formula

$$\mu_0(A) = \mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}(A) =$$

$$\sum_{i_1=1}^{2} \cdots \sum_{i_N=1}^{2} \prod_{n=1}^{N} \psi_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) \chi_A(\omega_1^{i_1}, \ldots, \omega_N^{i_N}), \quad A \in \mathcal{F}_N, \tag{220}$$

where

$$\psi_n(\omega_1, \ldots, \omega_n) = \chi_{\Omega_n^-}(\omega_1, \ldots, \omega_{n-1}, \omega_n)\, \psi_n^1(\omega_1, \ldots, \omega_n) +$$

$$\chi_{\Omega_n^+}(\omega_1, \ldots, \omega_{n-1}, \omega_n) \psi_n^2(\omega_1, \ldots, \omega_n), \tag{221}$$

$$\psi_n^1(\omega_1, \ldots, \omega_{n-1}, \omega_n) = \chi_{\Omega_n^{0+}}(\omega_n^2) \frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} =$$

$$\frac{f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \eta_n^+(\omega_n^2)}{f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \eta_n^+(\omega_n^2) + f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) \eta_n^-(\omega_n^1)}, \quad \omega_n^2 \in \Omega_n^{0+}, \tag{222}$$

$$\psi_n^2(\omega_1, \ldots, \omega_{n-1}, \omega_n) = \chi_{\Omega_n^{0-}}(\omega_n^1) \frac{\Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} =$$

$$\frac{f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) \eta_n^-(\omega_n^1)}{f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \eta_n^+(\omega_n^2) + f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) \eta_n^-(\omega_n^1)}, \quad \omega_n^1 \in \Omega_n^{0-}. \tag{223}$$

Proof. The proof of Proposition 1 is the same as proof of Theorem 8.

Suppose that the market consists of d assets the evolutions of which are given by the law

$$S_n((\omega_1,\dots,\omega_n)=\{S_n^1((\omega_1,\dots,\omega_n),\dots,S_n^d((\omega_1,\dots,\omega_n)\},\quad n=\overline{1,N},\qquad(224)$$

where

$$S_n^k((\omega_1,\dots,\omega_n)=S_0^k\prod_{i=1}^{n}(1+b_i^k(\omega_1,\dots,\omega_{i-1})f_i(\omega_1,\dots,\omega_i)\eta_i(\omega_i)),\quad k=\overline{1,d},\quad(225)$$

and the random values $\eta_i(\omega_i),\ f_i(\omega_1,\dots,\omega_i),\ i=\overline{1,N}$, does not depend on k, and satisfy inequalities

$$f_i(\omega_1,\dots,\omega_i)>0,\quad b_i^k(\omega_1,\dots,\omega_{i-1})>0,\quad \max_{\{\omega_1,\dots,\omega_{i-1}\}\in\Omega_{i-1}} b_i^k(\omega_1,\dots,\omega_{i-1})<$$

$$\frac{1}{\max\limits_{\{\omega_1,\dots,\omega_{i-1}\}\in\Omega_{i-1}} f_i(\omega_1,\dots,\omega_{i-1},\omega_i^1)\eta_i^-(\omega_i^1)},\quad k=\overline{1,d},\quad i=\overline{1,N}.\qquad(226)$$

Proposition 2. *On the probability space* $\{\Omega_N,\mathcal{F}_N,P_N\}$, *being the direct product of the probability spaces* $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, *if the evolution of d risky assets is given by the formula (224), (225), then such a market is complete non arbitrage one. The unique martingale measure does not depend on the random variables* $b_i^k(\omega_1,\dots,\omega_{i-1}),k=\overline{1,d},\quad i=\overline{1,N}$, *and it is determined by the formula (220). For the contingent claims* $\varphi_i(\omega_1,\dots,\omega_N),i=\overline{1,d}$, *the fair prices* φ_0^i *are given by the formulas*

$$\varphi_0^i=\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\varphi_i(\omega_1^{i_1},\dots,\omega_N^{i_N}),\quad i=\overline{1,d},\qquad(227)$$

where the number of i-th shares is determined by the formula (228) and the number of i-th bonds is determined by the formula (229)

$$\gamma_k^i(\omega_1,\dots,\omega_{k-1})=$$

$$\frac{m_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^1)-m_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^2)}{S_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^1)-S_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^2)},\quad k=\overline{1,N},\qquad(228)$$

$$\beta_k^i(\omega_1,\dots,\omega_{k-1})=$$

$$m_{k-1}^i(\omega_1,\dots,\omega_{k-1})-\gamma_k^i(\omega_1,\dots,\omega_{k-1})S_{k-1}^i(\omega_1,\dots,\omega_{k-1}),\quad k=\overline{1,N},\qquad(229)$$

$$m_k^i(\omega_1,\dots,\omega_k)=E^{\mu_0}\{\varphi_i(\omega_1,\dots,\omega_N)|\mathcal{F}_k\}=$$

$$\frac{\sum\limits_{i_{k+1}=1,\dots,i_N=1}^{2}\varphi_i(\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{k+1}},\dots,\omega_N^{i_N})\mu_0(\{\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{n+1}},\dots,\omega_N^{i_N}\})}{\sum\limits_{i_{k+1}=1,\dots,i_N=1}^{2}\mu_0(\{\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{k+1}},\dots,\omega_N^{i_N}\})},$$

Corollary 3. *(Cox, Ross, Rubinstein, see [25]) On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset is given by the formula*

$$S_n^1((\omega_1, \dots, \omega_n) = S_0 \prod_{i=1}^{n}(1+\rho_i(\omega_i)), \quad n = \overline{1, N}, \tag{230}$$

where the random values $\rho_i(\omega_i)$, $i = \overline{1, N}$, are such that $\rho_i(\omega_i^1) = a$, $\rho_i(\omega_i^2) = b$, and let the bank account evolution be given by the formula

$$B_n = B_0(1+r)^n, \quad r > 0, \quad B_0 > 0, \quad n = \overline{1, N}. \tag{231}$$

Then, for the discount evolution of risky asset

$$S_n((\omega_1, \dots, \omega_n) = \frac{S_0 \prod\limits_{i=1}^{n}(1+\rho_i(\omega_i))}{B_0(1+r)^n}, \quad n = \overline{1, N}, \tag{232}$$

the martingale measure μ_0 is unique if $a < r < b$. It is a direct product of measures $\mu_0^i(A)$, $A \in \mathcal{F}_i^0$, $i = \overline{1, N}$, given on the measurable space $\{\Omega_i^0, \mathcal{F}_i^0\}$, where $\mu_0^i(\omega_i^1) = \frac{b-r}{b-a}$, $\mu_0^i(\omega_i^2) = \frac{r-a}{b-a}$. The fair prices φ_0 of the contingent liability $\varphi_N(\omega_1, \dots, \omega_N)$ is given by the formula

$$\varphi_0 = \int\limits_{\Omega_N} \varphi_N(\omega_1, \dots, \omega_N) d\mu_0 =$$

$$\sum_{i_1=1}^{2} \dots \sum_{i_N=1}^{2} \varphi_N(\omega_1^{i_1}, \dots, \omega_N^{i_N}) \prod_{k=1}^{N} \mu_0^k(\omega_k^{i_k}), \tag{233}$$

where the number of shares is determined by the formula (234) and the number of bonds is determined by the formula (235)

$$\gamma_k(\omega_1, \dots, \omega_{k-1}) =$$

$$\frac{m_k(\omega_1, \dots, \omega_{k-1}, \omega_k^1) - m_k(\omega_1, \dots, \omega_{k-1}, \omega_k^2)}{S_k(\omega_1, \dots, \omega_{k-1}, \omega_k^1) - S_k(\omega_1, \dots, \omega_{k-1}, \omega_k^2)}, \quad k = \overline{1, N}, \tag{234}$$

$$\beta_k(\omega_1, \dots, \omega_{k-1}) =$$

$$m_{k-1}(\omega_1, \dots, \omega_{k-1}) - \gamma_k(\omega_1, \dots, \omega_{k-1}) S_{k-1}(\omega_1, \dots, \omega_{k-1}), \tag{235}$$

$$m_k(\omega_1, \dots, \omega_k) = E^{\mu_0}\{\varphi_N(\omega_1, \dots, \omega_N) | \mathcal{F}_k\} =$$

$$\frac{\sum\limits_{i_{k+1}=1, \dots, i_N=1}^{2} \varphi_N(\omega_1, \dots, \omega_k, \omega_{k+1}^{i_{k+1}}, \dots, \omega_N^{i_N}) \mu_0(\{\omega_1, \dots, \omega_k, \omega_{k+1}^{i_{n+1}}, \dots, \omega_N^{i_N}\})}{\sum\limits_{i_{k+1}=1, \dots, i_N=1}^{2} \mu_0(\{\omega_1, \dots, \omega_k, \omega_{k+1}^{i_{k+1}}, \dots, \omega_N^{i_N}\})}.$$

Proof. For the discount evolution (232), the representation

$$S_n((\omega_1,\ldots,\omega_n) = S_0\prod_{i=1}^{n}(1+\eta_i(\omega_i)), \quad n=\overline{1,N}, \tag{236}$$

is true, where $\eta_i(\omega_i) = \frac{\rho_i(\omega_i))-r}{(1+r)}$. Due to Theorems 30, 31, since $\eta_i(\omega_i^1) = \frac{a-r}{1+r} < 0$, $\eta_i(\omega_i^2) = \frac{b-r}{1+r} > 0$, then the measure μ_0 is unique. The rest statement of Corollary follows from Theorem 31.

Theorem 32. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset be given by the formula*

$$S_n^1((\omega_1,\ldots,\omega_n) = S_0\prod_{i=1}^{n}(1+\rho_i(\omega_i)), \quad n=\overline{1,N}, \tag{237}$$

where the random values $\rho_i(\omega_i)$, $i=\overline{1,N}$, are such that $\rho_i(\omega_i^1) = b_i^1$, $\rho_i(\omega_i^2) = b_i^2$, $i=\overline{1,N}$, and let the bank account evolution be given by the formula

$$B_n = B_0\prod_{i=1}^{n}(1+r_{i-1}(\omega_{i-1})), \quad B_0>0, \quad n=\overline{1,N}, \tag{238}$$

where the random values $r_i(\omega_i)$, $i=\overline{1,N-1}$, are such that $r_i(\omega_i^1) = r_i^1$, $r_i(\omega_i^2) = r_i^2$, $i=\overline{1,N-1}$, $r_0>0$. Then, for the discount evolution of risky asset

$$S_n((\omega_1,\ldots,\omega_n) = \frac{S_0\prod_{i=1}^{n}(1+\rho_i(\omega_i))}{B_0\prod_{i=1}^{n}(1+r_{i-1}(\omega_{i-1}))}, \quad n=\overline{1,N}, \tag{239}$$

the martingale measure μ_0 is unique, if $b_1^1 < r_0 < b_1^2$, $b_i^1 < r_{i-1}^1 < r_{i-1}^2 < b_i^2$, $i=\overline{2,N}$. It is determined by the formula (220) with

$$\eta_1(\omega_1) = \rho_1(\omega_1) - r_0, \quad \eta_i(\omega_i) = \rho_i(\omega_i) - r_{i-1}^2, \quad i=\overline{2,N},$$

$$f_1(\omega_1) = \frac{1}{1+r_0}, \quad f_i(\omega_1,\ldots,\omega_i) =$$

$$\frac{\rho_i(\omega_i) - r_{i-1}(\omega_{i-1})}{(\rho_i(\omega_i) - r_{i-1}^2)(1+r_{i-1}(\omega_{i-1}))}, \quad i=\overline{2,N}. \tag{240}$$

The fair price φ_0 of the contingent liability $\varphi_N(\omega_1,\ldots,\omega_N)$ is given by the formula

$$\varphi_0 = \int\limits_{\Omega_N}\varphi_N(\omega_1,\ldots,\omega_N)d\mu_0 =$$

$$\sum_{i_1=1}^{2}\cdots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\ldots,\omega_n^{i_n})\varphi_N(\omega_1^{i_1},\ldots,\omega_N^{i_N}), \tag{241}$$

where the number of shares is determined by the formula (242) and the number of bonds is determined by the formula (243)

$$\gamma_k(\omega_1, \ldots, \omega_{k-1}) =$$

$$\frac{m_k(\omega_1, \ldots, \omega_{k-1}, \omega_k^1) - m_k(\omega_1, \ldots, \omega_{k-1}, \omega_k^2)}{S_k(\omega_1, \ldots, \omega_{k-1}, \omega_k^1) - S_k(\omega_1, \ldots, \omega_{k-1}, \omega_k^2)}, \quad k = \overline{1, N}, \tag{242}$$

$$\beta_k(\omega_1, \ldots, \omega_{k-1}) =$$

$$m_{k-1}(\omega_1, \ldots, \omega_{k-1}) - \gamma_k(\omega_1, \ldots, \omega_{k-1}) S_{k-1}(\omega_1, \ldots, \omega_{k-1}), \tag{243}$$

$$m_k(\omega_1, \ldots, \omega_k) = E^{\mu_0}\{\varphi_N(\omega_1, \ldots, \omega_N)|\mathcal{F}_k\} =$$

$$\frac{\sum\limits_{i_{k+1}=1, \ldots, i_N=1}^{2} \varphi_N(\omega_1, \ldots, \omega_k, \omega_{k+1}^{i_{k+1}}, \ldots, \omega_N^{i_N}) \mu_0(\{\omega_1, \ldots, \omega_k, \omega_{k+1}^{i_{n+1}}, \ldots, \omega_N^{i_N}\})}{\sum\limits_{i_{k+1}=1, \ldots, i_N=1}^{2} \mu_0(\{\omega_1, \ldots, \omega_k, \omega_{k+1}^{i_{k+1}}, \ldots, \omega_N^{i_N}\})}.$$

Proof. To prove Theorem 32 it is necessary to prove the existence of unique spot measure. The discount evolution (239) can be represented in the form

$$S_n((\omega_1, \ldots, \omega_n) =$$

$$\frac{S_0}{B_0} \prod_{i=1}^{n} (1 + f_i(\omega_1, \ldots, \omega_i)\eta_i(\omega_i)), \quad n = \overline{1, N}, \tag{244}$$

where

$$\eta_1(\omega_1) = \rho_1(\omega_1) - r_0, \quad \eta_i(\omega_i) = \rho_i(\omega_i) - r_{i-1}^2, \quad i = \overline{2, N},$$

$$f_1(\omega_1) = \frac{1}{1 + r_0}, \quad f_i(\omega_1, \ldots, \omega_i) =$$

$$\frac{\rho_i(\omega_i) - r_{i-1}(\omega_{i-1})}{(\rho_i(\omega_i) - r_{i-1}^2)(1 + r_{i-1}(\omega_{i-1}))}, \quad i = \overline{2, N}. \tag{245}$$

It is evident that $\eta_i(\omega_i^1) < 0$, $\eta_i(\omega_i^2) > 0$, $f_i(\omega_1, \ldots, \omega_i) > 0$. Therefore, from the representation (244), (245) it follows that we can construct only one spot measure, which is martingale measure, being equivalent to the initial measure P_N. In accordance with Theorem 30, since $S_n(\omega_1, \ldots, \omega_n^1) \neq S_n(\omega_1, \ldots, \omega_n^2)$, $\{\omega_1, \ldots, \omega_{n-1}\} \in \Omega_{n-1}$ such a measure is unique. Theorem 32 is proved.

Theorem 33. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset be given by the formula*

$$S_n^1((\omega_1, \ldots, \omega_n) = S_0 \prod_{i=1}^{n} e^{\sigma_i(\omega_1, \ldots, \omega_{i-1})\varepsilon_i(\omega_i)}, \quad n = \overline{1, N}, \tag{246}$$

where the random values $\varepsilon_i(\omega_i)$, $i=\overline{1,N}$, *are such that* $\varepsilon_i(\omega_i^1)<0$, $\varepsilon_i(\omega_i^2)>0$, $\sigma_i(\omega_1,\ldots,\omega_{i-1})\geq\sigma_i^0>0$, $i=\overline{1,N}$, *and let the bank account evolution be given by the formula*

$$B_n=B_0\prod_{i=1}^{n}(1+r_{i-1}(\omega_{i-1})),\quad B_0>0,\quad n=\overline{1,N}, \tag{247}$$

where the random values $r_i(\omega_i)$, $i=\overline{1,N-1}$, *are such that* $r_i(\omega_i^1)=r_i^1$, $r_i(\omega_i^2)=r_i^2$, $i=\overline{1,N-1}$, $r_0>0$. *Then, for the discount evolution of risky asset*

$$S_n((\omega_1,\ldots,\omega_n)=\frac{S_0\prod\limits_{i=1}^{n}e^{\sigma_i(\omega_1,\ldots,\omega_{i-1})\varepsilon_i(\omega_i)}}{B_0\prod\limits_{i=1}^{n}(1+r_{i-1}(\omega_{i-1}))},\quad n=\overline{1,N}, \tag{248}$$

the martingale measure μ_0 *is unique, if*

$$\exp\{\sigma_1^0\varepsilon_1(\omega_1^1)\}<r_0<\exp\{\sigma_1^0\varepsilon_1(\omega_1^2)\},$$

$$\exp\{\sigma_i^0\varepsilon_i(\omega_i^1)\}<r_{i-1}^1<r_{i-1}^2<\exp\{\sigma_i^0\varepsilon_i(\omega_i^2)\},\ i=\overline{2,N}. \tag{249}$$

It is determined by the formula (220) with

$$\eta_1(\omega_1)=\exp\{\sigma_1^0\varepsilon_1(\omega_1)\}-r_0,\quad f_1(\omega_1)=\frac{1}{1+r_0},$$

$$\eta_i(\omega_i)=\exp\{\sigma_i^0\varepsilon_i(\omega_i)\}-r_{i-1}^2,\quad f_i(\omega_1,\ldots,\omega_i)=$$

$$\frac{e^{\sigma_i(\omega_1,\ldots,\omega_{i-1})\varepsilon_i(\omega_i)}-r_{i-1}(\omega_{i-1})}{(\exp\{\sigma_i^0\varepsilon_i(\omega_i)\}-r_{i-1}^2)(1+r_{i-1}(\omega_{i-1}))},\quad \{\omega_1,\ldots,\omega_i\}\in\Omega_n,\quad i=\overline{2,N}. \tag{250}$$

The fair price φ_0 *of the contingent liability* $\varphi_N(\omega_1,\ldots,\omega_N)$ *is given by the formula*

$$\varphi_0=\int\limits_{\Omega_N}\varphi_N(\omega_1,\ldots,\omega_N)d\mu_0=$$

$$\sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\ldots,\omega_n^{i_n})\varphi_N(\omega_1^{i_1},\ldots,\omega_N^{i_N}), \tag{251}$$

where the number of shares is determined by the formula (252) and the number of bonds is determined by the formula (253)

$$\gamma_k(\omega_1,\ldots,\omega_{k-1})=$$

$$\frac{m_k(\omega_1,\ldots,\omega_{k-1},\omega_k^1)-m_k(\omega_1,\ldots,\omega_{k-1},\omega_k^2)}{S_k(\omega_1,\ldots,\omega_{k-1},\omega_k^1)-S_k(\omega_1,\ldots,\omega_{k-1},\omega_k^2)},\quad k=\overline{1,N}, \tag{252}$$

$$\beta_k(\omega_1, \dots, \omega_{k-1}) =$$

$$m_{k-1}(\omega_1, \dots, \omega_{k-1}) - \gamma_k(\omega_1, \dots, \omega_{k-1}) S_{k-1}(\omega_1, \dots, \omega_{k-1}), \tag{253}$$

$$m_k(\omega_1, \dots, \omega_k) = E^{\mu_0}\{\varphi_N(\omega_1, \dots, \omega_N)|\mathcal{F}_k\} =$$

$$\frac{\sum\limits_{i_{k+1}=1,\dots,i_N=1}^{2} \varphi_N(\omega_1, \dots, \omega_k, \omega_{k+1}^{i_{k+1}}, \dots, \omega_N^{i_N}) \mu_0(\{\omega_1, \dots, \omega_k, \omega_{k+1}^{i_{n+1}}, \dots, \omega_N^{i_N}\})}{\sum\limits_{i_{k+1}=1,\dots,i_N=1}^{2} \mu_0(\{\omega_1, \dots, \omega_k, \omega_{k+1}^{i_{k+1}}, \dots, \omega_N^{i_N}\})}.$$

Proof. For the discount evolution (248), the following representation

$$S_n((\omega_1, \dots, \omega_n) =$$

$$\frac{S_0}{B_0} \prod_{i=1}^{n} (1 + f_i(\omega_1, \dots, \omega_i)\eta_i(\omega_i)), \quad n = \overline{1, N}, \tag{254}$$

is true, where

$$\eta_1(\omega_1) = \exp\{\sigma_1^0 \varepsilon_1(\omega_1)\} - r_0, \quad f_1(\omega_1) = \frac{1}{1 + r_0},$$

$$\eta_i(\omega_i) = \exp\{\sigma_i^0 \varepsilon_i(\omega_i)\} - r_{i-1}^2, \quad f_i(\omega_1, \dots, \omega_i) =$$

$$\frac{e^{\sigma_i(\omega_1, \dots, \omega_{i-1})\varepsilon_i(\omega_i)} - r_{i-1}(\omega_{i-1})}{(\exp\{\sigma_i^0 \varepsilon_i(\omega_i)\} - r_{i-1}^2)(1 + r_{i-1}(\omega_{i-1}))}, \quad \{\omega_1, \dots, \omega_i\} \in \Omega_n, \quad i = \overline{2, N}. \tag{255}$$

It is evident that $\eta_i(\omega_i^1) < 0$, $\eta_i(\omega_i^2) > 0$, $f_i(\omega_1, \dots, \omega_i) > 0$. From this, we obtain that the spot measure exists and it is unique. Theorem 33 is proved.

On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, suppose that the market consists of d assets the evolution of which is given by the law

$$S_n((\omega_1, \dots, \omega_n) = \{S_n^1((\omega_1, \dots, \omega_n), \dots, S_n^d((\omega_1, \dots, \omega_n)\}, \quad n = \overline{1, N}, \tag{256}$$

where

$$S_n^k((\omega_1, \dots, \omega_n) = S_0^k \prod_{i=1}^{n} (1 + a_i^k f_i(\omega_1, \dots, \omega_i)\eta_i(\omega_i)), \quad k = \overline{1, d}, \tag{257}$$

and the random values $\eta_i(\omega_i)$, $f_i(\omega_1, \dots, \omega_i)$, $i = \overline{1, N}$, and constants a_i^k satisfy the inequalities

$$\eta_i(\omega_i^1) < 0, \quad \eta_i(\omega_i^2) > 0, \quad f_i(\omega_1, \dots, \omega_i) > 0,$$

$$0 < a_i^k < \frac{1}{\max\limits_{\{\omega_1, \dots, \omega_{i-1}\} \in \Omega_{i-1}} f_i(\omega_1, \dots, \omega_i^1)\eta_i^-(\omega_i^1)}, \quad i = \overline{1, N}, \quad k = \overline{1, d}. \tag{258}$$

Proposition 3. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky assets be given by the formulas (256), (257), where constants a_i^k $i=\overline{1,N}$, $k=\overline{1,d}$, satisfy the inequalities (258). For such an evolution of risky asset the martingale measure μ_0 does not depend on a_i^k and is unique. It is determined by the formula (220). For the contingent claims $\varphi_N^i(\omega_1,\ldots,\omega_N)$, $i=\overline{1,d}$, the fair prices φ_0^i are given by the formulas*

$$\varphi_0^i=\sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\ldots,\omega_n^{i_n})\varphi_N^i(\omega_1^{i_1},\ldots,\omega_N^{i_N}),\quad i=\overline{1,d}, \tag{259}$$

where the number of i-th shares is determined by the formula (260) and the number of i-th bonds is determined by the formula (261)

$$\gamma_k^i(\omega_1,\ldots,\omega_{k-1})=$$

$$\frac{m_k^i(\omega_1,\ldots,\omega_{k-1},\omega_k^1)-m_k^i(\omega_1,\ldots,\omega_{k-1},\omega_k^2)}{S_k^i(\omega_1,\ldots,\omega_{k-1},\omega_k^1)-S_k^i(\omega_1,\ldots,\omega_{k-1},\omega_k^2)},\quad k=\overline{1,N}, \tag{260}$$

$$\beta_k^i(\omega_1,\ldots,\omega_{k-1})=$$

$$m_{k-1}^i(\omega_1,\ldots,\omega_{k-1})-\gamma_k^i(\omega_1,\ldots,\omega_{k-1})S_{k-1}^i(\omega_1,\ldots,\omega_{k-1}),\quad k=\overline{1,N}, \tag{261}$$

$$m_k^i(\omega_1,\ldots,\omega_k)=E^{\mu_0}\{\varphi_i(\omega_1,\ldots,\omega_N)|\mathcal{F}_k\}=$$

$$\frac{\sum\limits_{i_{k+1}=1,\ldots,i_N=1}^{2}\varphi_i(\omega_1,\ldots,\omega_k,\omega_{k+1}^{i_{k+1}},\ldots,\omega_N^{i_N})\mu_0(\{\omega_1,\ldots,\omega_k,\omega_{k+1}^{i_{n+1}},\ldots,\omega_N^{i_N}\})}{\sum\limits_{i_{k+1}=1,\ldots,i_N=1}^{2}\mu_0(\{\omega_1,\ldots,\omega_k,\omega_{k+1}^{i_{k+1}},\ldots,\omega_N^{i_N}\})}.$$

If $f_i(\omega_1,\ldots,\omega_i)=1$, $i=\overline{1,N}$, the unique martingale measure is a direct product of measures $\mu_0^i(A)$, $A\in\mathcal{F}_i^0$, given on the measurable space $\{\Omega_i^0,\mathcal{F}_i^0\}$, $i=\overline{1,N}$, where

$$\mu_0^i(\omega_i^1)=\frac{\eta_i^+(\omega_i^2)}{(\eta_i^-(\omega_i^1)+\eta_i^+(\omega_i^2))},\quad \mu_0^i(\omega_i^2)=\frac{\eta_i^-(\omega_i^1)}{(\eta_i^-(\omega_i^1)+\eta_i^+(\omega_i^2))}. \tag{262}$$

The fair prices φ_0^i, $i=\overline{1,N}$, of the contingent liability $\varphi_N^i(\omega_1,\ldots,\omega_N)$, $i=\overline{1,N}$, are given by the formula

$$\varphi_0^i=\int\limits_{\Omega_N}\varphi_N^i(\omega_1,\ldots,\omega_N)d\mu_0=$$

$$\sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\varphi_N^i(\omega_1^{i_1},\ldots,\omega_N^{i_N})\prod_{k=1}^{N}\mu_0^k(\omega_k^{i_k}), \tag{263}$$

where the number of i-th shares is determined by the formula (264) and the number of i-th bonds is determined by the formula (265)

$$\gamma_k^i(\omega_1,\ldots,\omega_{k-1})=$$

$$\frac{m_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^1)-m_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^2)}{S_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^1)-S_k^i(\omega_1,\dots,\omega_{k-1},\omega_k^2)}, \quad k=\overline{1,N}, \tag{264}$$

$$\beta_k^i(\omega_1,\dots,\omega_{k-1})=$$

$$m_{k-1}^i(\omega_1,\dots,\omega_{k-1})-\gamma_k^i(\omega_1,\dots,\omega_{k-1})S_{k-1}^i(\omega_1,\dots,\omega_{k-1}), \quad k=\overline{1,N}, \tag{265}$$

$$m_k^i(\omega_1,\dots,\omega_k)=E^{\mu_0}\{\varphi_i(\omega_1,\dots,\omega_N)|\mathcal{F}_k\}=$$

$$\frac{\sum\limits_{i_{k+1}=1,\dots,i_N=1}^{2}\varphi_i(\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{k+1}},\dots,\omega_N^{i_N})\mu_0(\{\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{n+1}},\dots,\omega_N^{i_N}\})}{\sum\limits_{i_{k+1}=1,\dots,i_N=1}^{2}\mu_0(\{\omega_1,\dots,\omega_k,\omega_{k+1}^{i_{k+1}},\dots,\omega_N^{i_N}\})}.$$

If $S_0^i,S_1^i,\dots,S_N^i$, $i=\overline{1,d}$, are the samples of the processes (256), (257) let us denote the order statistics $S_{(0)}^i,S_{(1)}^i,\dots,S_{(N)}^i$, $i=\overline{1,d}$, of this samples.

Proposition 4. *Suppose that $S_0^i,S_1^i,\dots,S_N^i$ is a sample of the random processes (256), (257). Then, for the parameters $a_1^i,\dots,a_N^i$ the estimation*

$$a_k^i=\frac{\left[1-\frac{S_{(N-k)}^i}{S_{(N-k+1)}^i}\right]}{f_k\eta_k^-(\omega_k^1)}, \quad k=\overline{1,N}, \quad i=\overline{1,d}, \tag{266}$$

is valid.

In the formulas (266) we put that $f_k=\max\limits_{\{\omega_1,\dots,\omega_{k-1}\}\in\Omega_{k-1}}f_k(\omega_1,\dots,\omega_{k-1},\omega_k^1), \quad k=\overline{1,N}$.

Proof. The proof of Proposition 4 is the same as the proof of Theorem 15.

IX. MARTINGALE MEASURES ON DISCRETE PROBABILITY SPACE

This section presents all the necessary results for constructing a non-arbitrage incomplete market on a discrete probability space. The conditions under which the entire family of martingale measures is described for the considered class of evolution of risky assets are minimal. In particular, conditions are presented under which the family of martingale measures considered is equivalent to the original measure. They are minimal. The entire set of equivalent martingale measures is a convex combination of a finite number of spot martingale measures. On this basis, new formulas were found for the fair price of the super hedge.

In this section, we put that $\Omega_i^0=\{\omega_i^1,\dots,\omega_i^M\}$, $i=\overline{1,N}$, and we assume that $2<M<\infty$, the σ-algebra $\mathcal{F}_i^0$ consists from all subsets of Ω_i^0. We suppose that $P_i^0(\omega_i^k)>0, \omega_i^k\in\Omega_i^0, k=\overline{1,M}$, $i=\overline{1,N}$. As before, the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$ is a direct product of probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}, i=\overline{1,N}$. Sometimes, any elementary event $\omega_i^k\in\Omega_i^0$ it is convenient to denote by ω_i not indicating the index k. Further, we use the both denotations. As in section 2, we introduce filtration $\mathcal{F}_n$ on the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$. As before, it is convenient to introduce the family of probability spaces $\{\Omega_n, \mathcal{F}_n, P_n\}, n = \overline{1, N}$, being a direct product of probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}, i = \overline{1, n}$.

The evolution of risky asset is given by the formula (1) with the assumptions given in the section 2. In this case

$$\Omega_n^- = \Omega_{n-1} \times \Omega_n^{0-}, \ \Omega_n^+ = \Omega_{n-1} \times \Omega_n^{0+}. \tag{267}$$

Further, we also use the measurable space with measure

$$\left\{\prod_{i=1}^N [\Omega_i^{0-} \times \Omega_i^{0+}], \prod_{i=1}^N [\mathcal{F}_i^{0-} \times \mathcal{F}_i^{0+}], \prod_{i=1}^N [P_i^{0-} \times P_i^{0+}]\right\}. \tag{268}$$

The measure P_n^{0-} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0-} = \Omega_n^{0-} \cap \mathcal{F}_n^0$, P_n^{0+} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0+} = \Omega_n^{0+} \cap \mathcal{F}_n^0$. Additionally, we assume

$$P_n^0(\{\omega_n \in \Omega_n^0, |\eta_n(\omega_n)| < \infty\}) = 1. \tag{269}$$

Let us consider the random values

$$\psi_n(\omega_1, \dots, \omega_n) = \chi_{\Omega_n^{0-}}(\omega_n)\, \psi_n^1(\omega_1, \dots, \omega_n) +$$

$$\chi_{\Omega_n^{0+}}(\omega_n)\, \psi_n^2(\omega_1, \dots, \omega_n), \quad n = \overline{1, N}, \tag{270}$$

where

$$\psi_n^1(\omega_1, \dots, \omega_{n-1}, \omega_n) =$$

$$\chi_{\Omega_n^{0+}}(\omega_n^2) \frac{\Delta S_n^+(\omega_1, \dots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2)}, \quad (\omega_1, \dots, \omega_{n-1}) \in \Omega_{n-1}, \quad \omega_n^2 \in \Omega_n^{0+}, \tag{271}$$

$$\psi_n^2(\omega_1, \dots, \omega_{n-1}, \omega_n) =$$

$$\chi_{\Omega_n^{0-}}(\omega_n^1) \frac{\Delta S_n^-(\omega_1, \dots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2)}, \quad (\omega_1, \dots, \omega_{n-1}) \in \Omega_{n-1}, \quad \omega_n^1 \in \Omega_n^{0-}. \tag{272}$$

Definition 1. *Let the evolution of risky asset be given by the formula (1). On the measurable space $\{\Omega_N, \mathcal{F}_N\}$, being the direct product of the measurable spaces $\{\Omega_i^0, \mathcal{F}_i^0\}$, for every point $\{\{\omega_1^1, \omega_1^2\}, \dots, \{\omega_N^1, \omega_N^2\}\} \in \prod_{i=1}^N [\Omega_i^{0-} \times \Omega_i^{0+}]$ let us introduce the spot measure*

$$\mu_{\{\omega_1^1, \omega_1^2\}, \dots, \{\omega_N^1, \omega_N^2\}}(A) =$$

$$\sum_{i_1=1}^2 \dots \sum_{i_N=1}^2 \prod_{n=1}^N \psi_n(\omega_1^{i_1}, \dots, \omega_n^{i_n}) \chi_A(\omega_1^{i_1}, \dots, \omega_N^{i_N}), \quad A \in \mathcal{F}_N, \tag{273}$$

where $\psi_n(\omega_1, \dots, \omega_n)$ is determined by the formulas (270) - (272).

Lemma 4. *The spot measure* $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$*, given by the formula (273), is a martingale measure for the evolution of risky asset, given by the formula (1), for every point* $\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\} \in \prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}]$*. If the point* $\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}$ *is such that* $\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1) < 0,\ \Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2) > 0,\ \{\omega_1,\ldots,\omega_{n-1}\} \in \Omega_{n-1},\ n = \overline{1,N}$*, then the spot measure* $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ *is a martingale measure, being equivalent to the measure* P_N*.*

Proof. Let us prove that $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ is a probability measure. Let us calculate

$$\sum_{i_j=1}^{2} \psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j}) = \psi_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1) + \psi_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2) =$$

$$\chi_{\Omega_j^{0-}}(\omega_j^1)\,\psi_j^1(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^1)+$$

$$\chi_{\Omega_j^{0+}}(\omega_j^1)\psi_j^2(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^1)+$$

$$\chi_{\Omega_j^{0-}}(\omega_j^2)\psi_j^1(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^2)+$$

$$\chi_{\Omega_j^{0+}}(\omega_j^2)\,\psi_j^2(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^2) =$$

$$\chi_{\Omega_j^{0-}}(\omega_j^1)\chi_{\Omega_j^{0+}}(\omega_j^2)\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^{0+}}(\omega_j^1)\chi_{\Omega_j^{0-}}(\omega_j^1)\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)}+$$

$$\chi_{\Omega_j^{0-}}(\omega_j^2)\chi_{\Omega_j^{0+}}(\omega_j^2)\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^{0+}}(\omega_j^2)\chi_{\Omega_j^{0-}}(\omega_j^1)\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)} =$$

$$\chi_{\Omega_j^{0-}}(\omega_j^1)\chi_{\Omega_j^{0+}}(\omega_j^2)\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^{0+}}(\omega_j^2)\chi_{\Omega_j^{0-}}(\omega_j^1)\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)} = \chi_{\Omega_j^{0-}}(\omega_j^1)\chi_{\Omega_j^{0+}}(\omega_j^2) = 1.$$

The last equalities proves that $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(\Omega_N) = 1$ for every point $\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\} \in \prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}]$. Further,

$$\sum_{i_j=1}^{2} \psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\Delta S_j(\omega_1^{i_1},\ldots,\omega_j^{i_j}) =$$

$$\psi_j(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\Delta S_j(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^1)+$$
$$\psi_j(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\Delta S_j(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^2)=$$
$$\chi_{\Omega_j^{0-}}(\omega_j^1)\chi_{\Omega_j^{0+}}(\omega_j^2)\times$$
$$\left[-\frac{\Delta S_j^+(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}\Delta S_j^-(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^1)+\right.$$
$$\left.\frac{\Delta S_j^-(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)}\Delta S_j^+(\omega_1^{i_1},\dots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\right]=0,\quad j=\overline{1,N}. \tag{274}$$

Let us prove that the set of measures $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ is a set of martingale measures. Really, for A, belonging to the σ-algebra $\mathcal{F}_{n-1}$ of the filtration we have $A=B\times\prod_{i=n}^{N}\Omega_i^0$, where B belongs to σ-algebra $\mathcal{F}_{n-1}$ of the measurable space $\{\Omega_{n-1},\mathcal{F}_{n-1}\}$. Then,

$$\int\limits_A \Delta S_n(\omega_1,\dots,\omega_n)d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}=$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\chi_B(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}})\Delta S_n(\omega_1^{i_1},\dots,\omega_n^{i_n})=$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_n=1}^{2}\prod_{j=1}^{n}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\chi_B(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}})\Delta S_n(\omega_1^{i_1},\dots,\omega_n^{i_n})=$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_{n-1}=1}^{2}\prod_{j=1}^{n-1}\psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\chi_B(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}})\times$$

$$\sum_{i_n=1}^{2}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\Delta S_n(\omega_1^{i_1},\dots,\omega_n^{i_n})=0,\quad A\in\mathcal{F}_{n-1}. \tag{275}$$

To prove the last statement it needs to prove that $\psi_n(\omega_1,\dots,\omega_n)>0,\ n=\overline{1,N}$. But,

$$\psi_n(\omega_1,\dots,\omega_n)=\chi_{\Omega_n^{0-}}(\omega_n)\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}+$$

$$\chi_{\Omega_n^{0+}}(\omega_n)\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}>0,\quad n=\overline{1,N}. \tag{276}$$

The last means the needed statement.

Suppose that the random values $a_i(\omega_1,\dots,\omega_i)$, $\eta_i(\omega_i)$ satisfy the inequalities

$$a_i(\omega_1,\dots,\omega_i) > 0, \qquad \sup_{\{\omega_1,\dots,\omega_i\}\in\Omega_i} a_i(\omega_1,\dots,\omega_i) < \frac{1}{\sup\limits_{\omega_i\in\Omega_i^0,\eta_i(\omega_i)<0} \eta_i^-(\omega_i)},$$

$$P_i^0(\eta_i(\omega_i) < 0) > 0, \quad P_i^0(\eta_i(\omega_i) > 0) > 0, \quad i=\overline{1,N}, \tag{277}$$

the evolution $S_n(\omega_1,\dots,\omega_n)$ is given by the formula

$$S_n(\omega_1,\dots,\omega_n) = S_0\prod_{i=1}^{n}(1+a_i(\omega_1,\dots,\omega_i)\eta_i(\omega_i)), \quad n=\overline{1,N}, \quad S_0>0. \tag{278}$$

Below, we describe the convex set of equivalent martingale measures.
We use for $\alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\})$ the denotation $\alpha_N(\{\omega\}_N^1;\{\omega\}_N^2)$.

Theorem 34. *Let the evolution of risky asset be given by the formula (278). On the measurable space with measure* $\{\prod_{i=1}^N[\Omega_i^{0-}\times\Omega_i^{0+}], \prod_{i=1}^N[\mathcal{F}_i^{0-}\times\mathcal{F}_i^{0+}], \prod_{i=1}^N[P_i^{0-}\times P_i^{0+}]\}$, suppose that the random value $\alpha_N(\{\omega\}_N^1;\{\omega\}_N^2)$ satisfies the conditions

$$\alpha_N(\{\omega\}_N^1;\{\omega\}_N^2) > 0, \quad \{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}\in\prod_{i=1}^N[\Omega_i^{0-}\times\Omega_i^{0+}], \tag{279}$$

$$\int\limits_{\prod_{i=1}^N[\Omega_i^{0-}\times\Omega_i^{0+}]} \alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\})\prod_{i=1}^N dP_i^0(\omega_i^1)dP_i^0(\omega_i^2) = 1. \tag{280}$$

The measure $\mu_0(A)$, given by the formula

$$\mu_0(A) =$$

$$\int\limits_{\prod_{i=1}^N[\Omega_i^{0-}\times\Omega_i^{0+}]} \alpha_N(\{\omega\}_N^1;\{\omega\}_N^2)\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)d\prod_{i=1}^N[P_i^{0-}\times P_i^{0+}], \tag{281}$$

is a martingale measure, being equivalent to the measure P_N.

Proof. Introduce the denotations

$$\alpha_n(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\}) =$$

$$\frac{\int\limits_{\prod_{i=n+1}^N[\Omega_i^{0-}\times\Omega_i^{0+}]} \alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\})\prod\limits_{i=n+1}^N dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)}{\int\limits_{\prod_{i=n}^N[\Omega_i^{0-}\times\Omega_i^{0+}]} \alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\})\prod\limits_{i=n}^N dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)}, \quad n=\overline{1,N-1},$$

$$\alpha_N(\{\omega_1^1,\dots,\omega_{N-1}^1,\omega_N^1\};\{\omega_1^2,\dots,\omega_{N-1}^2,\omega_N^2\}) =$$

$$\frac{\alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\})}{\int\limits_{\Omega_N^{0-}\times\Omega_N^{0+}}\alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\})dP_N^0(\omega_N^1)dP_N^0(\omega_N^2)}. \quad (282)$$

It is not difficult to note that

$$\prod_{n=1}^N \alpha_n(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\}) = \alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\}).$$

Since the random values $\alpha_n(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})$ are finite valued, then

$$\int\limits_{\Omega_n^{0-}\times\Omega_n^{0+}}\alpha_n(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1)dP_n^0(\omega_n^2) < \infty,$$

$$(\omega_1,\dots,\omega_{n-1}) \in \Omega_{n-1}. \quad (283)$$

It is evident that the set of strictly positive finite valued random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2), n=\overline{1,N}$, given by the formula (282), satisfy the conditions

$$E^{\mu_0}|\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)| =$$

$$\int\limits_{\Omega_N}\prod_{i=1}^N\psi_i(\omega_1,\dots,\omega_i)|\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)|\prod_{i=1}^N dP_i^0(\omega_i) < \infty, \quad n=\overline{1,N}. \quad (284)$$

Moreover, for the measure (281) the representation (32) is true, meaning that it is equivalent to the measure P_N. The last proves Theorem 34.

Let us define the integral for the random value $f_N(\omega_1,\dots,\omega_{N-1},\omega_N)$ relative to the measure $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ by the formula

$$\int\limits_{\Omega_N} f_N(\omega_1,\dots,\omega_{N-1},\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}} =$$

$$\sum_{i_1=1}^2\dots\sum_{i_N=1}^2\prod_{n=1}^N\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})f_N(\omega_1^{i_1},\dots,\omega_N^{i_N}). \quad (285)$$

Theorem 35. *Let the evolution of risky asset be given by the formula (278). If the conditions of Theorem 34 are true, then the fair price of super-hedge f_0 for the nonnegative payoff function $f(x)$ is given by the formula*

$$f_0 = \sup_{P\in M} E^P f(S_N) = \max_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}}\int\limits_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}. \quad (286)$$

Moreover,

$$\inf_{P\in M} E^P f(S_N) = \min_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+}, i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}. \tag{287}$$

Proof. Let us prove the formula (286). Denote M the set of all martingale measure, being equivalent to P_N. If an equivalent martingale measure $P_0 \in M$, then $\alpha P_0 + (1-\alpha)\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \in M$ for arbitrary $0 < \alpha \leq 1$. We have the inequality

$$\alpha E^{P_0} f(S_N) + (1-\alpha)\int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \leq \sup_{P\in M} E^P f(S_N).$$

Since $\alpha > 0$ is arbitrary, we obtain the inequality

$$\int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \leq \sup_{P\in M} E^P f(S_N).$$

From here, we obtain the inequality

$$\max_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+}, i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \leq \sup_{P\in M} E^P f(S_N).$$

The inverse inequality follows from the representation (281) for any martingale measure, being equivalent to the measure P_N. Really,

$$E^P f_N = \int\limits_{\prod_{i=1}^N [\Omega_i^{0-}\times\Omega_i^{0+}]} \alpha_N(\{\omega\}_N^1; \{\omega\}_N^2)\times$$

$$\int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_{N-1},\omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} d\prod_{i=1}^N [P_i^{0-}\times P_i^{0+}]. \tag{288}$$

From the formula (288) it follows the inequality

$$E^P f_N \leq \max_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+}, i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}.$$

Or,

$$\sup_{P\in M} E^P f_N \leq \max_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+}, i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}.$$

The proof of (287) is analogous. We have the inequality

$$\alpha E^{P_0} f(S_N) + (1-\alpha)\int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \geq \inf_{P\in M} E^P f(S_N).$$

Tending α to zero and taking the minimum all over the spot measures we obtain

$$\min_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}}\int\limits_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \geq \inf_{P\in M} E^P f(S_N).$$

Using the representation (288) we have

$$E^P f_N \geq \min_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}}\int\limits_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}.$$

Taking the infimum all over the martingale measures we obtain

$$\inf_{P\in M} E^P f_N \geq \min_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}}\int\limits_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}.$$

Theorem 35 is proved.

X. MODELS OF NON-ARBITRAGE INCOMPLETE FINANCIAL MARKETS

Using the construction of the family of spot measures introduced in the previous section, this section presents the conditions under which the considered family of spot measures is invariant with respect to a certain class of evolutions of risky assets. For a certain class of contingent liabilities including a standard call option, the fair price of the super hedge is shown to be less than the spot price of the underlying asset. Specific applications of the results obtained for the previously known evolutions of risky assets are considered. New formulas is found for the non-arbitrage price range. A model of a non-arbitrage incomplete market is proposed and estimates are obtained in the case of a multi-parameter model of a non-arbitrage market.

On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let us assume that the random values $b_i(\omega_1,\ldots,\omega_{i-1})$, $f_i(\omega_1,\ldots,\omega_i)$, $\eta_i(\omega_i)$, $i=\overline{1,N}$, satisfy the inequalities

$$b_i(\omega_1,\ldots,\omega_{i-1})>0, \quad f_i(\omega_1,\ldots,\omega_i)>0,$$

$$\max_{\{\omega_1,\ldots,\omega_{i-1}\}\in\Omega_{i-1}} b_i(\omega_1,\ldots,\omega_{i-1}) <$$

$$\frac{1}{\max\limits_{\{\omega_1,\ldots,\omega_{i-1}\}\in\Omega_{i-1}}\ \max\limits_{\{\omega_i,\eta_i(\omega_i)<0\}} f_i(\omega_1,\ldots,\omega_i)\eta_i^-(\omega_i)},$$

$$P_i^0(\eta_i(\omega_i)<0)>0, \quad P_i^0(\eta_i(\omega_i)>0)>0, \quad i=\overline{1,N}. \tag{289}$$

As before, we put $\Omega_i^{0-}=\{\omega_i\in\Omega_i^0,\ \eta_i(\omega_i)\leq 0\}$, $\Omega_i^{0+}=\{\omega_i\in\Omega_i^0,\ \eta_i(\omega_i)>0\}$. We assume that the evolution $S_n(\omega_1,\ldots,\omega_n)$ of risky asset is given by the formula

$$S_n(\omega_1,\ldots,\omega_n)=S_0\prod_{i=1}^{n}(1+b_i(\omega_1,\ldots,\omega_{i-1})f_i(\omega_1,\ldots,\omega_i)\eta_i(\omega_i)), \quad n=\overline{1,N}. \tag{290}$$

With every point $v = \{(\omega_1^1, \omega_1^2), \dots, (\omega_N^1, \omega_N^2)\} \in \mathcal{V}$, where $\mathcal{V} = \prod_{i=1}^{N} [\Omega_i^{0-} \times \Omega_i^{0+}]$, we connect the spot measure

$$\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A) =$$

$$\sum_{i_1=1}^{2} \dots \sum_{i_N=1}^{2} \prod_{n=1}^{N} \psi_n(\omega_1^{i_1}, \dots, \omega_n^{i_n}) \chi_A(\omega_1^{i_1}, \dots, \omega_N^{i_N}), \quad A \in \mathcal{F}_N. \tag{291}$$

Let us denote $\nu_v(A) = \prod_{i=1}^{N} \nu_{\omega_i^1,\omega_i^2}(A_i)$, $A = \prod_{i=1}^{N} A_i, \in \mathcal{F}_N$, the direct product of the measures $\nu_{\omega_i^1,\omega_i^2}(A_i)$, $A_i \in \mathcal{F}_i^0$, $i = \overline{1,N}$, where $v = \{(\omega_1^1, \omega_1^2), \dots, (\omega_N^1, \omega_N^2)\} \in \mathcal{V}$, $\mathcal{V} = \prod_{i=1}^{N} [\Omega_i^{0-} \times \Omega_i^{0+}]$, and

$$\nu_{\omega_i^1,\omega_i^2}(A_i) = \chi_{A_i}(\omega_i^1) \frac{\eta_i^+(\omega_i^2)}{\eta_i^-(\omega_i^1) + \eta_i^+(\omega_i^2)} + \chi_{A_i}(\omega_i^2) \frac{\eta_i^-(\omega_i^1)}{\eta_i^-(\omega_i^1) + \eta_i^+(\omega_i^2)}, \tag{292}$$

for $\omega_i^1 \in \Omega_i^{0-}$, $\omega_i^2 \in \Omega_i^{0+}$, $A_i \in \mathcal{F}_i^0$. Then, there exists a countable additive function $\nu_v(A), A \in \mathcal{F}_N$, on the σ-algebra $\mathcal{F}_N$ for every $v \in \mathcal{V}$.

Theorem 36. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset be given by the formula (290). For every point $v = \{(\omega_1^1, \omega_1^2), \dots, (\omega_N^1, \omega_N^2)\} \in \mathcal{V}$, the spot measure $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ given by the formula (291) does not depend on the random values $b_i(\omega_1, \dots, \omega_{i-1})$, $i = \overline{1,N}$. In the case as $f_i(\omega_1, \dots, \omega_i) = 1$, $i = \overline{1,N}$, the formula*

$$\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A) = \nu_v(A) \tag{293}$$

is true. For the evolution of risky asset (290), the set of martingale measures being equivalent to the measure P_N does not depend on the random values $b_i(\omega_1, \dots, \omega_{i-1})$, $i = \overline{1,N}$.

Proof. The proof of Theorem 36 is the same as proof of the Theorem 8.

Theorem 37. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset be given by the formula (290). Suppose that the nonnegative convex down payoff function $f(x)$ on the set $0 \leq x < \infty$ satisfies the inequality $0 \leq f(x) < x$. Then, the inequalities*

$$f(S_0) \leq \sup_{P \in M} E^P f(S_N) =$$

$$\max_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}} < S_0 \tag{294}$$

are true.

Proof. Since the set of points $v = \{(\omega_1^1, \omega_1^2), \ldots, (\omega_N^1, \omega_N^2)\}$ in the set $\mathcal{V}$ is finite then the minimum in the formula

$$\min_{\omega_1,\ldots,\omega_N} [S_N(\omega_1, \ldots, \omega_N) - f(S_N(\omega_1, \ldots, \omega_N))] = d > 0 \tag{295}$$

is reached at a certain point $v_0 = \{(\omega_1^{1,0}, \omega_1^{2,0}), \ldots, (\omega_N^{1,0}, \omega_N^{2,0})\}$. Therefore, the inequality

$$S_N(\omega_1, \ldots, \omega_N) - f(S_N(\omega_1, \ldots, \omega_N)) \geq d, \quad \{\omega_1, \ldots, \omega_N\} \in \Omega_N, \tag{296}$$

is true

Integrating left and right parts of inequality over the measure $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$, we have

$$\int\limits_{\Omega_N} S_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} -$$

$$\int\limits_{\Omega_N} d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} f(S_N(\omega_1, \ldots, \omega_N)) \geq d. \tag{297}$$

Since

$$\int\limits_{\Omega_N} S_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} = S_0 \tag{298}$$

we obtain the needed. It is evident that from the convexity down of payoff function $f(x)$ and Jensen inequality we obtain the inequality

$$\int\limits_{\Omega_N} f(S_N(\omega_1, \ldots, \omega_N)) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \geq f(S_0). \tag{299}$$

Theorem 37 is proved.

Let us note that the interval of non arbitrage prices for a certain processes was found in the papers [26], [27].

Corollary 4. *For the standard call option of European type with payoff function* $f(x) = (x - K)^+, K > 0$, *the conditions of Theorem 37 are true. Therefore, the inequalities (294) are valid.*

Theorem 38. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *being the direct product of the probability spaces* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, *let the evolution of risky asset be given by the formula (290). Suppose that the nonnegative convex down payoff function* $f(x)$ *on the set* $0 \leq x < \infty$ *satisfies the inequality* $0 \leq f(x) \leq K, \ K > 0$. *Then, the inequalities*

$$f(S_0) \leq \sup_{P \in M} E^P f(S_N) = \max_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \leq K \tag{300}$$

are true.

Proof. The proof is evident.

Corollary 5. *For the standard put option of European type with payoff function* $f(x) = (K - x)^+, K > 0$*, the conditions of Theorem 38 are true. Therefore, the inequalities (300) are valid.*

Corollary 6 For the standard call option of European type with payoff function $f(x) = (x - K)^+, K > 0$, *the interval of non arbitrage prices coincide with the interval*

$$\left(\min_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}, \right.$$

$$\left. \max_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} \right). \tag{301}$$

Corollary 7. *For the standard put option of European type with payoff function* $f(x) = (K - x)^+, K > 0$*, the interval of non arbitrage prices coincide with the interval*

$$\left(\min_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}, \right.$$

$$\left. \max_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} \right). \tag{302}$$

Corollary 8. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$*, being the direct product of the probability spaces* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$*, let the evolution of risky asset is given by the formula*

$$S_n^1((\omega_1, \ldots, \omega_n) = S_0 \prod_{i=1}^{n} (1 + \rho_i(\omega_i)), \quad n = \overline{1, N}, \quad S_0 > 0, \tag{303}$$

where the random value $\rho_i(\omega_i)$ *is given on the probability space* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}, i = \overline{1, N}$*, and let the bank account evolution be given by the formula*

$$B_n = B_0(1 + r)^n, \quad r > 0, \quad B_0 > 0, \quad n = \overline{1, N}. \tag{304}$$

Then, for the discount evolution of risky asset

$$S_n((\omega_1, \ldots, \omega_n) = \frac{S_0 \prod_{i=1}^{n} (1 + \rho_i(\omega_i))}{B_0(1 + r)^n}, \quad n = \overline{1, N}, \tag{305}$$

the set of martingale measure is nonempty one if the following conditions are true

$$P_i^0(\rho_i(\omega_i) - r < 0) > 0, \; P_i^0(\rho_i(\omega_i) - r > 0) > 0,$$

$$P_i^0(\rho_i(\omega_i) - r < 0) + P_i^0(\rho_i(\omega_i) - r > 0) = 1, \quad i = \overline{1,N}.$$

For every point $v = \{(\omega_1^1, \omega_1^2), \ldots, (\omega_N^1, \omega_N^2)\}$ *in the set* $\mathcal{V}$ *the spot measure* $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ *is a direct product of measures* $\mu_0^i(A_i)$, $A_i \in \mathcal{F}_i^0$, $i = \overline{1,N}$, *given on the measurable space* $\{\Omega_i^0, \mathcal{F}_i^0\}$, *where* $\mu_0^i(A_i) = \nu_{\omega_i^1,\omega_i^2}(A_i)$, *and* $\nu_{\omega_i^1,\omega_i^2}(A_i)$ *is given by the formula (292) with* $\eta_i(\omega_i) = \frac{\rho_i(\omega_i)-r}{1+r}$, $i = \overline{1,N}$. *The fair price* φ_0 *of super-hedge of the nonnegative contingent liability* $\varphi_N(\omega_1, \ldots, \omega_N)$ *is given by the formula*

$$\varphi_0 = \max_{v \in \mathcal{V}} \int\limits_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\nu_v.$$

The interval of non-arbitrage prices is written in the form

$$\left(\min_{v \in \mathcal{V}} \int\limits_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\nu_v, \ \max_{v \in \mathcal{V}} \int\limits_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\nu_v \right).$$

Theorem 39. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *being the direct product of the probability spaces* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, *let the evolution of risky asset be given by the formula*

$$S_n^1((\omega_1, \ldots, \omega_n) = S_0 \prod_{i=1}^{n} (1 + \rho_i(\omega_i)), \quad n = \overline{1,N}, \tag{306}$$

where the random value $\rho_i(\omega_i)$, *is given on the probability space* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, $P_i^0(\{\rho_i(\omega_i) < 0\}) > 0$, $P_i^0(\{\rho_i(\omega_i) > 0\}) > 0$, $i = \overline{1,N}$, *and let the bank account evolution be given by the formula*

$$B_n = B_0 \prod_{i=1}^{n} (1 + r_{i-1}(\omega_{i-1})), \quad B_0 > 0, \quad n = \overline{1,N}, \tag{307}$$

where the strictly positive random values $r_i(\omega_i)$ *are given on the probability* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}, i = \overline{1,N}$. *Then, for the discount evolution of risky asset*

$$S_n((\omega_1, \ldots, \omega_n) = \frac{S_0 \prod\limits_{i=1}^{n} (1 + \rho_i(\omega_i))}{B_0 \prod\limits_{i=1}^{n} (1 + r_{i-1}(\omega_{i-1}))}, \quad n = \overline{1,N}, \tag{308}$$

the set of martingale measure is nonempty one if the following conditions are true

$$\max_{\omega_{i-1} \in \Omega_{i-1}} r_{i-1}(\omega_{i-1}) < \min_{\omega_i \in \Omega_i, \rho_i(\omega_i) > 0} \rho_i(\omega_i),$$

$$\min_{\omega_{i-1} \in \Omega_{i-1}} r_{i-1}(\omega_{i-1}) > 0, \ i = \overline{2,N}$$

$$0 < r_0 < \min_{\omega_1 \in \Omega_1, \rho_1(\omega_1) > 0} \rho_1(\omega_1). \tag{309}$$

The fair price φ_0 of super-hedge of the nonnegative contingent liability $\varphi_N(\omega_1, \ldots, \omega_N)$ is given by the formula

$$\varphi_0 = \max_{\omega_i^1 \in \Omega_i^{0-},\ \omega_i^2 \in \Omega_i^{0+},\ i=\overline{1,N}} \int_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}.$$

The interval of non-arbitrage prices is written in the form

$$\left(\min_{\omega_i^1 \in \Omega_i^{0-},\ \omega_i^2 \in \Omega_i^{0+},\ i=\overline{1,N}} \int_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}, \right.$$

$$\left. \max_{\omega_i^1 \in \Omega_i^{0-},\ \omega_i^2 \in \Omega_i^{0+},\ i=\overline{1,N}} \int_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} \right).$$

Proof. The discount evolution (308) can be represented in the form

$$S_n(\omega_1, \ldots, \omega_n) =$$

$$\frac{S_0}{B_0} \left(1 + \frac{(\rho_1(\omega_1) - r_0)}{1 + r_0} \right) \prod_{i=2}^{n} \left(1 + \frac{\rho_i(\omega_i) - r_{i-1}(\omega_{i-1})}{1 + r_{i-1}(\omega_{i-1})} \right) =$$

$$\frac{S_0}{B_0} \prod_{i=1}^{n} (1 + f_i(\omega_1, \ldots, \omega_i) \eta_i(\omega_i)), \tag{310}$$

where

$$f_1(\omega_1) = \frac{1}{1 + r_0}, \quad \eta_1(\omega_1) = \rho_1(\omega_1) - r_0, \tag{311}$$

$$f_i(\omega_1, \ldots, \omega_i) = \frac{\rho_i(\omega_i) - r_{i-1}(\omega_{i-1})}{(\rho_i(\omega_i) - \min_{\omega_{i-1} \in \Omega_{i-1}} r_{i-1}(\omega_{i-1})(1 + r_{i-1}(\omega_{i-1}))},$$

$$\eta_i(\omega_i) = \rho_i(\omega_i) - \min_{\omega_{i-1} \in \Omega_{i-1}} r_{i-1}(\omega_{i-1}) \quad i = \overline{2, N}. \tag{312}$$

Since

$$f_i(\omega_1, \ldots, \omega_i) > 0, \quad i = \overline{1, N}, \tag{313}$$

$$P_i^0(\eta_i(\omega_i) < 0) > 0, \quad P_i^0(\eta_i(\omega_i) > 0) > 0, \quad i = \overline{1, N}, \tag{314}$$

then it means that the set of martingale measures, being equivalent to R_N, is a nonempty set. Theorem 39 is proved.

Theorem 40. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset be given by the formula*

$$S_n^1((\omega_1, \ldots, \omega_n) = S_0 \prod_{i=1}^{n} e^{\sigma_i(\omega_1,\ldots,\omega_{i-1})\varepsilon_i(\omega_i)}, \quad n = \overline{1, N}, \tag{315}$$

where the random values $\varepsilon_i(\omega_i)$, $i = \overline{1, N}$, are such that

$$P_i^0(\varepsilon_i(\omega_i) < 0) > 0, \quad P_i^0(\varepsilon_i(\omega_i) > 0) > 0,$$

$$P_i^0(\varepsilon_i(\omega_i) < 0) + P_i^0(\varepsilon_i(\omega_i) > 0) = 1,$$

$$\sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0, \quad i = \overline{1, N},$$

and let the bank account evolution be given by the formula

$$B_n = B_0 \prod_{i=1}^{n}(1 + r_{i-1}(\omega_{i-1})), \quad B_0 > 0, \quad n = \overline{1, N}, \tag{316}$$

where the random values $r_i(\omega_i)$, $i = \overline{1, N-1}$, are strictly positive ones, $r_0 > 0$. Then, for the discount evolution of risky asset

$$S_n((\omega_1, \ldots, \omega_n) = \frac{S_0 \prod_{i=1}^{n} e^{\sigma_i(\omega_1,\ldots,\omega_{i-1})\varepsilon_i(\omega_i)}}{B_0 \prod_{i=1}^{n}(1 + r_{i-1}(\omega_{i-1}))}, \quad n = \overline{1, N}, \tag{317}$$

the set of martingale measure is nonempty one, if

$$\exp\{\sigma_1^0 \max_{\{\omega_1, \varepsilon_1(\omega_1)<0\}} \varepsilon_1(\omega_1)\} < r_0 < \exp\{\sigma_1^0 \min_{\{\omega_1, \varepsilon_1(\omega_1)>0\}} \varepsilon_1(\omega_1)\},$$

$$\exp\{\sigma_i^0 \max_{\{\omega_i, \varepsilon_i(\omega_i)<0\}} \varepsilon_i(\omega_i)\} < \min_{\{\omega_{i-1} \in \Omega_{i-1}\}} r_{i-1}(\omega_{i-1}) <$$

$$\max_{\{\omega_{i-1} \in \Omega_{i-1}\}} r_{i-1}(\omega_{i-1}) < \exp\{\sigma_i^0 \min_{\{\omega_i, \varepsilon_i(\omega_i)>0\}} \varepsilon_i(\omega_i)\}, \quad i = \overline{2, N}. \tag{318}$$

Then, the fair price of super-hedge φ_0 of the nonnegative contingent liability $\varphi_N(\omega_1, \ldots, \omega_N)$ is given by the formula

$$\varphi_0 = \max_{v \in \mathcal{V}} \int_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} =$$

$$\max_{v \in \mathcal{V}} \sum_{i_1=1}^{2} \ldots \sum_{i_N=1}^{2} \prod_{n=1}^{N} \psi_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) \varphi_N(\omega_1^{i_1}, \ldots, \omega_N^{i_N}). \tag{319}$$

Proof. For the discount evolution (317), the following representation

$$S_n((\omega_1,\dots,\omega_n) =$$

$$\frac{S_0}{B_0}\prod_{i=1}^{n}(1+f_i(\omega_1,\dots,\omega_i)\eta_i(\omega_i)), \quad n=\overline{1,N}, \tag{320}$$

is true, where

$$\eta_1(\omega_1)=\exp\{\sigma_1^0\varepsilon_1(\omega_1)\}-r_0, \quad f_1(\omega_1)=\frac{1}{1+r_0},$$

$$\eta_i(\omega_i)=\exp\{\sigma_i^0\varepsilon_i(\omega_i)\}-\max_{\{\omega_{i-1}\in\Omega_{i-1}\}} r_{i-1}(\omega_{i-1}),$$

$$f_i(\omega_1,\dots,\omega_i)=$$

$$\frac{e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)}-r_{i-1}(\omega_{i-1})}{(\exp\{\sigma_i^0\varepsilon_i(\omega_i)\}-\max_{\{\omega_{i-1}\in\Omega_{i-1}\}} r_{i-1}(\omega_{i-1}))(1+r_{i-1}(\omega_{i-1}))}>0,$$

$$\{\omega_1,\dots,\omega_i\}\in\Omega_i, \quad i=\overline{2,N}. \tag{321}$$

In this case, the spot measures

$$\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)=$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\chi_A(\omega_1^{i_1},\dots,\omega_N^{i_N}), \quad A\in\mathcal{F}_N, \tag{322}$$

figuring in the formula (319), are determined by the formulas

$$\psi_n(\omega_1,\dots,\omega_n)=\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n)\,\psi_n^1(\omega_1,\dots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n)\,\psi_n^2(\omega_1,\dots,\omega_n), \tag{323}$$

$$\psi_n^1(\omega_1,\dots,\omega_{n-1},\omega_n)=$$

$$\chi_{\Omega_n^{0+}}(\omega_n^2)\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\dots,\omega_{n-1})\in\Omega_{n-1}, \quad \omega_n^2\in\Omega_n^{0+}, \tag{324}$$

$$\psi_n^2(\omega_1,\dots,\omega_{n-1},\omega_n)=$$

$$\chi_{\Omega_n^{0-}}(\omega_n^1)\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\dots,\omega_{n-1})\in\Omega_{n-1}, \quad \omega_n^1\in\Omega_n^{0-}, \tag{325}$$

where

$$\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}=$$

$$\frac{f_n(\omega_1,\dots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)}{f_n(\omega_1,\dots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)+f_n(\omega_1,\dots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}, \tag{326}$$

$$\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}=$$

$$\frac{f_n(\omega_1,\dots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}{f_n(\omega_1,\dots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)+f_n(\omega_1,\dots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}, \tag{327}$$

$$(\omega_1,\dots,\omega_{n-1})\in\Omega_{n-1}.$$

and the random values $\eta_i(\omega_i)$, $f_i(\omega_1,\dots,\omega_i)$, $i=\overline{1,N}$, are given by the formulas (321). The obtained representation (320) proves Theorem 40.

Suppose that the random values $\eta_k(\omega_k)$, $f_k(\omega_1,\dots,\omega_k)$, $k=\overline{1,N}$, and constants a_k^i satisfy the inequalities

$$0<a_k^i<\frac{1}{\max\limits_{\{\omega_1,\dots,\omega_{k-1}\}\in\Omega_{k-1}}\ \max\limits_{\{\omega_k,\eta_k(\omega_k)<0\}} f_k(\omega_1,\dots,\omega_k)\eta_k^-(\omega_k)},\quad k=\overline{1,N},\quad i=\overline{1,d},$$

$$f_i(\omega_1,\dots,\omega_i)>0,\quad P_i^0(\eta_i(\omega_i)<0)>0,\quad P_i^0(\eta_i(\omega_i)>0)>0,\quad i=\overline{1,N}. \tag{328}$$

We assume that the evolutions of d risky assets $S_n(\omega_1,\dots,\omega_n)$ is given by the formula

$$S_n(\omega_1,\dots,\omega_n)=\{S_n^i(\omega_1,\dots,\omega_n)\}_{i=1}^d, \tag{329}$$

where

$$S_n^i(\omega_1,\dots,\omega_n)=S_0^i\prod_{k=1}^n(1+a_k^if_k(\omega_1,\dots,\omega_k)\eta_k(\omega_k)),\quad n=\overline{1,N},\quad i=\overline{1,d}. \tag{330}$$

Proposition 5. *On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, let the evolution of risky assets be given by the formulas (329), (330), where the random values $\eta_k(\omega_k)$, $f_k(\omega_1,\dots,\omega_k)$ and constants a_k^i, $k=\overline{1,N}$, $i=\overline{1,d}$ satisfy the inequalities (328). For such an evolution of risky assets the set of martingale measures μ_0 does not depend on a_k^i. The spot measures $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ are determined by the formulas (322) - (327). The fair price φ_0^i of super-hedge of the nonnegative contingent liability $\varphi_N^i(\omega_1,\dots,\omega_N)$ is given by the formula*

$$\varphi_0^i=\max_{v\in\mathcal{V}}\int\limits_{\Omega_N}\varphi_N^i(\omega_1,\dots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}},\quad i=\overline{1,d}.$$

The interval of non-arbitrage prices is written in the form

$$\left(\min_{v \in \mathcal{V}} \int\limits_{\Omega_N} \varphi_N^i(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} , \right.$$

$$\left. \max_{v \in \mathcal{V}} \int\limits_{\Omega_N} \varphi_N^i(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} \right), \quad i = \overline{1, d}.$$

In the case $f_k(\omega_1, \ldots, \omega_k) = 1, k = \overline{1, N}$, for every point $v = \{(\omega_1^1, \omega_1^2), \ldots, (\omega_N^1, \omega_N^2)\}$ in the set $\mathcal{V}$ the spot measure $\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}(A)$ is a direct product of measures $\mu_0^i(A_i)$, $A_i \in \mathcal{F}_i^0$, $i = \overline{1, N}$, given on the measurable space $\{\Omega_i^0, \mathcal{F}_i^0\}$, where $\mu_0^i(A_i) = \nu_{\omega_i^1, \omega_i^2}(A_i)$, and $\nu_{\omega_i^1, \omega_i^2}(A_i)$ is given by the formula (292).

If $S_0^i, S_1^i, \ldots, S_N^i$, $i = \overline{1, d}$, are the samples of the processes (329), (330), let us denote the order statistics $S_{(0)}^i, S_{(1)}^i, \ldots, S_{(N)}^i$, $i = \overline{1, d}$, of this samples. Introduce the denotations

$$f_k^1 = \max_{\{\omega_1, \ldots, \omega_{k-1}\} \in \Omega_{k-1}, \ \omega_k^1 \in \Omega_1^{0-}} f_k(\omega_1, \ldots, \omega_{k-1}, \omega_k^1), \ k = \overline{1, N}.$$

Proposition 6. *Suppose that $S_0^i, S_1^i, \ldots, S_N^i$ is a sample of the random processes (329), (330). Then, for the parameters $a_1^i, \ldots, a_N^i$ the estimation*

$$a_k^i = \frac{\left[1 - \frac{S_{(N-k)}^i}{S_{(N-k+1)}^i}\right]}{f_k^1 \max_{\omega_k^1 \in \Omega_k^{0-}} \eta_k^-(\omega_k^1)}, \quad k = \overline{1, N}, \quad i = \overline{1, d}, \tag{331}$$

is valid.

XI. CONCLUSIONS

Section 1 provides an overview of the achievements and formulates the main problem that has been solved. Section 2 contains the formulation of conditions which must satisfy the evolution of risky asset. In Section 3, the conditions for the evolution of risky asset and random variables are formulated, on the basis of which a recursive method of constructing a family of martingale measures equivalent to the original measure is proposed. Lemma 1 gives a simple proof of the non-emptiness of the set of random variables satisfying conditions (20) - (22), in contrast to similar results in [2]. In Lemma 2, an integral representation is obtained for the measure constructed by the recursive method (28) - (30), from which it follows that it is equivalent to the original measure. In Theorem 1, the conditions under which the recursively constructed measure is martingale one and equivalent to the original measure are formulated.

The Section 4 introduces a family of spot measures and a measure built on the basis of these spot measures and a family of random variables. In Theorem 2, an integral representation is found for the introduced family of measures, which means that this family of measures is absolutely continuous to the original measure. Theorem 3 guarantees the conditions under which the constructed family of measures are martingale and equivalent to the original measure. Theorem 4 gives a complete description of martingale measures equivalent to the original measure. Theorems

5 and 6 are auxiliary. Theorem 7 guarantees conditions when the infimum and supremum of the average value of the payment function over the set of martingale measures coincide with the infimum and supremum of the average value of the payment function over the set of all spot measures. Theorem 8 proves that the family of martingale measures is invariant with respect to a certain class of transformations.

In Section 5, based on Theorem 8, a parametric family of evolutions of risky assets based on some evolution of risky asset is introduced. The proposed parametric model based on the canonical model of the evolution of risky assets, which takes into account both memory and price clustering, takes into account the fact that the price of a risky asset cannot fall to zero.

For a wide class of payment functions, in Theorem 9, an estimate is obtained both from above and from below for the supremum of the average value of the payment functions over the set of all martingale measures. A similar result as in Theorem 9 is obtained in Theorem 10 only for another class of payment functions. For the considered parametric evolution, in Theorem 11, a fair superhedge price is found for the payment function of a standard European-type call option. The same Theorem 11 specifies the interval of non-arbitrage prices. In Theorem 12 , for the considered parametric evolution of the risky asset, a fair superhedge price is found for the payment function of a standard European-type put option. In Theorems 13 and 14, similar results are obtained as in Theorems 11, 12 only for the payment functions of Asian call and put options. On the basis of the sample, in Theorem 15, the estimates of the parameters of the introduced parametric model of the evolution of risky assets are obtained.

In Theorems 16, 17 the fair price of the superhedge for the payment functions of the standard call and put options are given in terms of the obtained parameter estimates. Analogous results are given in Theorems 18 and 19 for fair superhedge prices for Asian-type call and put option payment functions.

Another parametric model of the evolution of risky assets is considered in Section 6. It differs from the previous one in that it considers the discounted evolution of risky asset. Theorems 20 - 21 are proved, in which estimates are obtained both from above and from below and established. Theorems 22 - 23 derive formulas for the fair price of a superhedge for the payment functions of call and put options, respectively. A similar result is obtained in Theorems 24 - 25 for the payment functions of Asian-type put and call options. In Theorems 26 - 29, based on the sample for the evolution of the risky asset, the formulas for the fair price of the superhedge through parameter estimation are presented.

Section 7 contains Theorems 30 and 31, which give the necessary and sufficient conditions for the evolution of risky assets for which the martingale measure is unique. Formulas for the fair price of option contracts and investor hedging strategies are found. A clear construction of such martingale measures and hedging strategies of the investor is given.

In section 8, Proposition 1 establishes the invariance of a single martingale measure with respect to a certain class of evolutions of risky asset. On this basis, proposition 2 builds a parametric model of the financial market and finds formulas for the fair price of an option contract and the investor's hedging strategies. In Corollary 3 and Theorems 32, 33 examples of various evolutions of risky asset are given and the conditions for the existence of a single martingale measure are established. An explicit construction of a single martingale measure is given and formulas for the fair price of an option contract and investor's hedging strategies are constructed.

Proposition 3 constructs a parametric securities market model with a single martingale measure and provides formulas for the fair prices of options contracts and investor hedging strategies. Proposition 4 provides an estimate of the parameters of the introduced parametric models through realizations of risky assets.

Section 9 contains models of incomplete financial markets in discrete probability space. Theorem 34 gives a complete description of all martingale measures equivalent to the original one. Theorem 35 establishes formulas for both the lower and upper limits of the interval of non-arbitrage prices for the evolution of risky assets through the minimum and maximum of the average value of the payment functions over a finite set of spot measures.

Section 10 considers models of the evolution of risky assets that are invariant with respect to a certain class of evolutions of risky assets. Theorem 37 establishes that for a certain class of payment functions and for a wide class of evolutions of risky assets, the fair price of the superhedge is strictly less than the price of the underlying asset. Among such payment functions is the payment function of the standard call option of the European type. Theorems 39, 40 give various examples of discounted evolutions of risky assets that satisfy the conditions of the proved theorems 35 - 37, and find the conditions under which the family of martingale measures is nonempty. Formulas for a fair superhedge price have been found. Proposition 5 contains the construction of a parametric model of an incomplete financial market, a family of martingale measures of which does not depend on the considered parameters. Proposition 6 provides an estimates of the parameters of the constructed models of incomplete markets through realizations of the considered evolutions of risky assets.

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