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G. A. Grigorian

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Keywords: systems of equations with piecewise constant coecients, the Riccati equation, normal and extremal solutions, main, nonprincipal and ordinary solutions of the system, the theorem of Wazevski.

I. INTRODUCTION

Let $a_{jk}(t)\ \ (j,k=1,2)$ be complex-valued continuous functions on the interval $[t_0,+\infty)$. Consider the system

$$\begin{cases} \phi'(t) = a_{11}(t)\phi(t) + a_{12}(t)\psi(t); \\ \psi'(t) = a_{21}(t)\phi(t) + a_{22}(t)\psi(t), \end{cases} \tag{1.1}$$

$t \geq t_0$. Study of the question of stability of solutions of differential equations and systems of differential equations, in particular of system (1.1), is an important problem of qualitative theory of differential equations and many works are devoted to it (see [1], [2] and cited works therein [3 - 10]). The fundamental theorem of R. Bellman (see [2], pp. 168, 169) reduces the study of boundedness of solutions of wide class of nonlinear systems to the study of stability of linear systems of differential equations. Many problems of mechanics, physics and other natural sciences are connected with the study of stability of the linear systems of differential equations (in particular of the linear differential equations) too (see for example [6,7]). One of ways to study the mentioned above question is the use of different methods of estimations of solutions of systems of being studied equations (see [4]).

In this paper some estimates of solutions of the system (1.1) in terms of its coefficients are obtained. To obtain them it was used the method of approximation of solutions of (1.1) by solutions of system with piecewise constant coefficients (the discrete ordinates method).

Denote: $P(t) \equiv a_{12}(t)\exp\Big\{\int\limits_{t_0}^{t}\big[a_{22}(\tau) - a_{11}(\tau)\big]d\tau\Big\}$, $Q(t) \equiv a_{21}(t)\exp\Big\{\int\limits_{t_0}^{t}\big[a_{11}(\tau) - a_{22}(\tau)\big]d\tau\Big\}$. In this article we will study the following two principal cases:

$A)$ $P(t) > 0, \quad Q(t) < 0, \quad t \geq t_0;$

$B)$ $P(t) > 0, \quad Q(t) > 0, \quad t \geq t_0$

(the case $P(t) < 0,\ Q(t) > 0,\ t \geq t_0$, is similar to the case A), and the case $P(t) < 0$, $Q(t) < 0,\ t \geq t_0$, is reducible to the case B) by the simple substitution $\phi(t) \to -\phi(t)$). The case A) can be geometrically interpreted as a case, when the origin of coordinates of phase plane of variables u, v is a "center" or a "focus" and the case B) as a a "saddle" with respect to the curves $\{(u(t), v(t))\},\ t \geq t_0$, where $\{(u(t), v(t))\}$ are the solutions of the system

$$\begin{cases} u'(t) = \qquad P(t)v(t); \\ v'(t) = Q(t)u(t), \end{cases} \tag{1.2}$$

$t \geq t_0$. On examples the obtained results are compared with the results obtained by methods of Liapunov, Yu. S. Bogdanov, T. Wazevski, estimates of solutions by logarithmic norm of S. M. Lozinski and of freezing.

II. AUXILIARY PROPOSITIONS

Lemma 2.1. *For each solution $(\phi(t), \psi(t))$ of the system (1.1) and for each $\varepsilon > 0$ and $t_1 > t_0$ there exists piecewise constant functions $\widetilde{a}_{jk}$, $t \geq t_0$, $j, k = 1, 2$, such, that the solutions $(\widetilde{\phi}(t), \widetilde{\ }(t))$ of the system*

$$\begin{cases} \phi'(t) = \widetilde{a}_{11}(t)\phi(t) + \widetilde{a}_{12}(t)\boldsymbol{\psi}(t); \\ \boldsymbol{\psi}'(t) = \widetilde{a}_{21}(t)\phi(t) + \widetilde{a}_{22}(t)\boldsymbol{\psi}(t), \end{cases} \tag{2.0}$$

$t \geq t_0$. with $\widetilde{\phi}(t_0) = \phi(t_0)$, $\widetilde{\psi}(t_0) = \psi(t_0)$ satisfy the inequalities: $|\widetilde{\phi}(t_1) - \phi(t_1)| \leq \varepsilon$, $|\widetilde{\psi}(t_1) - \psi(t_1)| \leq \varepsilon$.

The proof of this lemma is not difficult, and we omit it.

Remark 2.1. *By a solution of the system (2.0) we will mean a pair of absolutely continuous functions $\phi(t)$ and $\psi(t)$, satisfying (2.0) almost everywhere on $[t_0, +\infty)$.*

Let $t_0 < t_1 < ... < t_n < ...$ be a finite or infinite sequens, and let $\widetilde{p}(t) = p_j > 0$, $\widetilde{q}(t) = = q_j > 0$, $t \in [t_j; t_{j+1})$, $j = 0, 1, 2, ...$ Consider the Cauchy problem

$$\begin{cases} u'(t) = \widetilde{p}(t)v(t); \\ v'(t) = -\widetilde{q}(t)u(t); \\ u(t_0) = u_{(0)}, \ \ v(t_0) = v_{(0)}, \end{cases} \tag{2.1}$$

$t \geq t_0$. Any solution of this system we will seek in the form

$$u(t) = A_j \sin\left(\sqrt{l_j}t + \omega_j\right), \quad v(t) = A_j\sqrt{h_j}\cos\left(\sqrt{l_j}t + \omega_j\right), \quad t \in [t_j, t_{j+1}), \tag{2.2}$$

where $l_j \equiv p_j q_j$, $h_j \equiv \frac{q_j}{p_j}$ and A_j, ω_j are the sought constants, $j = 0, 1, 2, ...$ By virtue of initial conditions of problem (2.1) the unknowns A_0 and ω_0 we determine by solving the system

$$\begin{cases} A_0 \sin\left(\sqrt{l_0}t_0 + \omega_0\right) = u_{(0)}; \\ A_0\sqrt{h_0}\cos\left(\sqrt{l_0}t_0 + \omega_0\right) = v_{(0)}, \end{cases} \tag{2.3}$$

and the remaining unknowns by successive solving of the systems

$$\begin{cases} A_{j+1}\sin\left(\sqrt{l_{j+1}}t_{j+1} + \omega_{j+1}\right) = A_j \sin\left(\sqrt{l_j}t_{j+1} + \omega_j\right); \\ A_{j+1}\sqrt{h_{j+1}}\cos\left(\sqrt{l_{j+1}}t_{j+1} + \omega_{j+1}\right) = A_j\sqrt{h_j}\cos\left(\sqrt{l_j}t_{j+1} + \omega_j\right), \end{cases} \tag{2.4}$$

$j = 1, 2, ...$ From (2.3) it follows

$$A_0 = u_{(0)}^2 + \frac{v_{(0)}^2}{h_0}. \tag{2.5}$$

Denote: $\alpha_j \equiv \sqrt{l_j}t_{j+1} + \omega_j$, $\beta_j \equiv \sqrt{l_{j+1}}t_{j+1} + \omega_{j+1}$, $j = 0, 1, 2, ...$ From (2.4) it is easy to derive the equalities:

$$A_{j+1}^2 = A_j^2\left[\frac{h_j + h_{j+1}}{2h_{j+1}} + \frac{h_j - h_{j+1}}{2h_{j+1}}\cos 2\alpha_j\right], \quad A_j^2 = A_{j+1}^2\left[\frac{h_j + h_{j+1}}{2h_j} + \frac{h_{j+1} - h_j}{2h_j}\cos 2\beta_j\right],$$

$j = 0, 1, 2, ...$ From here it follows

$$|A_{j+1}| \leq |A_j| \leq \sqrt{\frac{h_{j+1}}{h_j}}|A_{j+1}| \quad for \quad h_j \leq h_{j+1}; \tag{2.6}$$

$$|A_j| \leq |A_{j+1}| \leq \sqrt{\frac{h_j}{h_{j+1}}}|A_j| \quad for \quad h_j \geq h_{j+1}; \tag{2.7}$$

$j = 0, 1, 2, ...$ From (2.2) it follows:

$$u^2(t)\frac{1}{h_j} + v^2(t) = A_j^2, \ \ t \in [t_j, t_{j+1}), \ \ j = 0, 1, 2, ... \tag{2.8}$$

Let the initial values $u_{(0)}$ and $v_{(0)}$ be real. Then $u(t)$ and $v(t)$ will be real valued. Therefore, from (2.8) we will have:

$$\min\{1, h_j^1\}A_j^2 \leq u^2(t) + v^2(t) \leq \max\{1, h_j^1\}A_j^2, \ \ t \in [t_j, t_{j+1}), \tag{2.9}$$

where $h_j^1 \equiv \frac{p_j}{q_j}, \ \ j = 0, 1, 2, ...$

Definition 2.1. We shall say, that a continuous on the interval $[t_0; +\infty)$ function $f(t)$ belongs to the class $C_\sim = C_\sim[t_0, +\infty)$, if there exists an infinitely large sequence $\xi_0 = t_0 \leq \xi_1 \leq ... \leq \xi_n \leq ...$ such, that $f(t)$ is a nondecreasing on the interval $[\xi_{2n}, \xi_{2n+1}]$ and nonincreasing on the interval $[\xi_{2n+1}, \xi_{2n+2}]$ function, $n = 0, 1, 2, ...$ The numbers $\xi_n, \ \ n = 0, 1, 2, ...$ we shall call points of possible extremums of the function $f(t)$.

Let $S(t) \in C_\sim$, and let $\xi_n, \ \ n = 0, 1, 2, ...$ be the points of possible extremums of $S(t)$. Note, that if $\xi_{2n+1} = \xi_{2n+2}, \ \ n = 0, 1, 2, ...,$ then $S(t)$ is a nondecreasing function and if $\xi_{2n} = \xi_{2n+1}, \ \ n = 0, 1, 2, ...,$ then $S(t)$ is a nonincreasing function. Let $S(t) > 0, \ \ t \geq t_0$. Consider the functions

$$r_S^-(t) \equiv \begin{cases} 1, & t \in [\xi_0; \xi_1]; \\ \sqrt{\frac{S(\xi_1)}{S(t)}}, & t \in [\xi_1; \xi_2]; \\ \prod\limits_{k=1}^{n} \sqrt{\frac{S(\xi_{2k-1})}{S(\xi_{2k})}}, & t \in [\xi_{2n}; \xi_{2n+1}], \ \ n = 1, 2, ...; \\ \left[\prod\limits_{k=1}^{n-1} \sqrt{\frac{S(\xi_{2k-1})}{S(\xi_{2k})}}\right] \sqrt{\frac{S(\xi_{2n-1})}{S(t)}}, & t \in [\xi_{2n-1}; \xi_{2n}], \ \ n = 2, 3, ..., \end{cases}$$

$$r_S^+(t) \equiv \begin{cases} \sqrt{\frac{S(t)}{S(\xi_0)}}, & t \in [\xi_0; \xi_1]; \\ \prod\limits_{k=0}^{n} \sqrt{\frac{S(\xi_{2k+1})}{S(\xi_{2k})}}, & t \in [\xi_{2n+1}; \xi_{2n+2}], \quad n = 0, 1, ...; \\ \left[\prod\limits_{k=9}^{n-1} \sqrt{\frac{S(\xi_{2k-1})}{S(\xi_{2k})}}\right] \sqrt{\frac{S(t)}{S(\xi_{2n})}}, & t \in [\xi_{2n}; \xi_{2n+1}], \quad n = 1, 2, ..., \end{cases}$$

Let $S(t)$ be absolutely continuous. Then $\sqrt{\frac{S(\xi_{2n+1})}{S(t)}} = \exp\Big\{-\frac{1}{2} \int\limits_{\xi_{2n+1}}^{t} \frac{S'(\tau)}{S(\tau)} d\tau\Big\}, \quad n = 0, 1, 2, ...,$

$\sqrt{\frac{S(\xi_{2k-1})}{S(\xi_{2k})}} = \exp\Big\{-\frac{1}{2} \int\limits_{\xi_{2k-1}}^{\xi_{2k}} \frac{S'(\tau)}{S(\tau)} d\tau\Big\}, \quad k = 1, 2, ...$ Therefore

$$r_S^-(t) = \exp\bigg\{\frac{1}{2} \int\limits_{t_0}^{t} \frac{S'_{(-)}(\tau)}{S(\tau)} d\tau\bigg\}, \qquad t \geq t_0, \tag{2.10}$$

where $S_{(-)}(t) \equiv \begin{cases} -S(t), & \text{if exists } \ S'(t) > 0; \\ 0, & \text{otherwise,} \end{cases}$ $t \geq t_0$. By analogy it shows, that

$$r_S^+(t) = \exp\bigg\{\frac{1}{2} \int\limits_{t_0}^{t} \frac{S'_{(+)}(\tau)}{S(\tau)} d\tau\bigg\}, \qquad t \geq t_0, \tag{2.11}$$

where $S_{(+)}(t) \equiv \begin{cases} S(t), & \text{if exists } \ S'(t) > 0; \\ 0, & \text{otherwise,} \end{cases}$ $t \geq t_0$. It is clear that $S'(t) = S'_{(+)}(t) -$ $- S'_{(-)}(t), \ |S'(t)| = S'_{(+)}(t) + S'_{(-)}(t)$ in all the points of existence of $S'(t)$. From here, from (2.10) and (2.11) it follows:

$$r_S^-(t) = \sqrt[4]{\frac{S(t_0)}{S(t)}} \exp\bigg\{\frac{1}{4} \int\limits_{t_0}^{t} \frac{|S'(\tau)|}{S(\tau)} d\tau\bigg\}, \qquad t \geq t_0, \tag{2.12}$$

$$r_S^+(t) = \sqrt[4]{\frac{S(t)}{S(t_0)}} \exp\bigg\{\frac{1}{4} \int\limits_{t_0}^{t} \frac{|S'(\tau)|}{S(\tau)} d\tau\bigg\}, \qquad t \geq t_0, \tag{2.13}$$

Let $p(t)$ and $q(t)$ be positive and continuous functions on the interval $[t_0; +\infty)$. Consider the system of equations

$$\begin{cases} u'(t) = & p(t)v(t); \\ v'(t) = -q(t)u(t), \end{cases} \tag{2.14}$$

$t \geq t_0$. Let us introduce some notations, necessary in the sequel: $h(t) \equiv \frac{q(t)}{p(t)}$, $h_1(t) \equiv \frac{p(t)}{q(t)}$, $g(t) \equiv \min\{\sqrt[4]{h(t)}, \sqrt[4]{h_1(t)}\}$, $G(t) \equiv \max\{\sqrt[4]{h(t)}, \sqrt[4]{h_1(t)}\}$, $||(x(t), y(t))|| \equiv \equiv \sqrt{|x(t)|^2 + |y(t)|^2}$, $r_z(t) \equiv \exp\left\{\frac{1}{4}\int\limits_{t_0}^{t} \frac{|z'(\tau)|}{z(\tau)} d\tau\right\}$, $t \geq t_0$, where $x(t)$ and $y(t)$ are continuous functions on the interval $[t_0; +\infty)$, $z(t)$ is a absolutely continuous and positive function on the interval $[t_0; +\infty)$ with locally finite variation.

Lemma 2.2. *Let $h(t)$ be an absolutely continuous function with locally finite variation. Then for every solution $(u(t), v(t))$ of the system (2.14) the following inequalities hold:*

$$\frac{g(t_0)g(t)}{r_h(t)}||(u(t_0), v(t_0)|| \leq ||(u(t), v(t))|| \leq G(t_0)G(t)||(u(t_0), v(t_0))||r_h(t), \tag{2.15}$$

$t \geq t_0$.

Proof. Let us consider first the case, when $h(t)$ has the additional property: $h(t) \in C_\sim$. Let $(u_0(t), v_0(t))$ be a nontrivial real valued solution of the system (2.14), and let $\xi_0 = t_0 \leq \xi_1 \leq ... \leq \xi_n \leq ...$ be the possible extremums of the function $h(t)$. Let $t_1(> t_0)$ and $\varepsilon(> 0)$ be fixed. By virtue of Lemma 2.1 there exist piecewise constant functions $\widetilde{\widetilde{p}}(t)$ and $\widetilde{\widetilde{q}}(t)$ such, that the solution $(\widetilde{u}(t), \widetilde{v}(t))$ of the system

$$\begin{cases} u'(t) = & \widetilde{\widetilde{p}}(t)v(t); \\ v'(t) = -\widetilde{\widetilde{q}}(t)u(t), \end{cases}$$

$t \geq t_0$, with $\widetilde{u}(t_0) = u_0(t_0)$, $\widetilde{v}(t_0) = v_0(t_0)$ satisfies the inequalities:

$$|\widetilde{u}(t_1) - u_0(t_1)| \leq \varepsilon, \qquad |\widetilde{v}(t_1) - v_0(t_1)| \leq \varepsilon. \tag{2.16}$$

Without loss of generality we will assume that $\widetilde{\widetilde{p}}(\xi_k) = p(\xi_k)$, $\widetilde{\widetilde{q}}(\xi_k) = q(\xi_k)$, $k = 0, 1, ...$, $\widetilde{\widetilde{p}}(t_1) = p(t_1)$, $\widetilde{\widetilde{q}}(t_1) = q(t_1)$; $\widetilde{\widetilde{p}}(t) > 0$, $\widetilde{\widetilde{q}}(t) > 0$, $t \geq t_0$; the function $\frac{\widetilde{\widetilde{q}}(t)}{\widetilde{\widetilde{p}}(t)}$ is nondecreasing on the intervals $[\xi_{2k}, \xi_{2k+1}]$ and nonincreasing on the intervals $[\xi_{2k+1}, \xi_{2k+2}]$, $k = 0, 1, 2, ...$ Then by (2.5) - (2.7), (2.9) the following inequalities hold

$$g_1(t_0)g_1(t_1)||(u_0(t_0), v_0(t_0))||\sqrt{\frac{h(\xi_0)}{h(t_1)}} \leq ||(\widetilde{u}(t_1), \widetilde{v}(t_1)|| \leq G_1(t_0)G_1(t_1)||(u_0(t_0), v_0(t_0))||,$$

$$\text{if } t_1 \in [\xi_0, \xi_1];$$

$$g_1(t_0)g_1(t_1)||(u_0(t_0), v_0(t_0))||\sqrt{\frac{h(\xi_0)}{h(t_1)}} \le ||(\widetilde{u}(t_1), \widetilde{v}(t_1))|| \le$$

$$\le G_1(t_0)G_1(t_1)||(u_0(t_0), v_0(t_0))||\sqrt{\frac{h(\xi_1)}{h(t_1)}}, \ \ if \ \ t_1 \in [\xi_1, \xi_2];$$

$$g_1(t_0)g_1(t_1)||(u_0(t_0), v_0(t_0))|| \left[\prod_{k=0}^{n}\sqrt{\frac{h(\xi_{2k})}{h(\xi_{2k+1})}}\right]\sqrt{\frac{h(\xi_{2n})}{h(t_1)}} \le ||(\widetilde{u}(t_1), \widetilde{v}(t_1))|| \le$$

$$\le G_1(t_0)G_1(t_1)||(u_0(t_0), v_0(t_0))||\prod_{k=1}^{n}\sqrt{\frac{h(\xi_{2k-1})}{h(\xi_{2k})}}, \ \ if \ \ t_1 \in [\xi_{2n}, \xi_{2n+1}], \ \ n = 1, 2, ...;$$

$$g_1(t_0)g_1(t_1)||(u_0(t_0), v_0(t_0))||\prod_{k=0}^{n}\sqrt{\frac{h(\xi_{2k})}{h(\xi_{2k+1})}} \le ||(\widetilde{u}(t_1), \widetilde{v}(t_1))|| \le$$

$$\le G_1(t_0)G_1(t_1)||(u_0(t_0), v_0(t_0))|| \left[\prod_{k=0}^{n-1}, \sqrt{\frac{h(\xi_{2k+1})}{h(\xi_{2k+2})}}\right]\sqrt{\frac{h(\xi_{2n+1})}{h(t_1)}}, \ \ if \ \ t_1 \in [\xi_{2n+1}, \xi_{2n+2}],$$

$n = 1, 2, ...,$ where $g_1(t) \equiv \min\{1, \sqrt{h_1(t)}\}$, $G_1(t) \equiv \max\{1, \sqrt{h_1(t)}\}$, $t \ge t_0$. It follows from here, that

$$\frac{g_1(t_0)g_1(t_1)||(u_0(t_0), v_0(t_0))||}{r_h^+(t_1)} \le ||(\widetilde{u}(t_1), \widetilde{v}(t_1)|| \le$$

$$\le G_1(t_0)G_1(t_1)||(u_0(t_0), v_0(t_0))||r_h^-(t_1). \qquad (2.17)$$

By analogy (making the substitution $u(t) \to -u(t)$, interchanging $p(t)$ and $q(t)$, as well as interchanging $u(t)$ and $v(t)$) we come to the inequalities

$$\frac{g_2(t_0)g_2(t_1)||(u_0(t_0), v_0(t_0))||}{r_{h_1}^+(t_1)} \le ||(\widetilde{u}(t_1), \widetilde{v}(t_1))|| \le$$

$$\le G_2(t_0)G_2(t_1)||(u_0(t_0), v_0(t_0))||r_{h_1}^-(t_1),$$

where $g_2(t) \equiv \min\{1, \sqrt{h(t)}\}$, $G_2(t) \equiv \max\{1, \sqrt{h(t)}\}$, $t \ge t_0$. From here and from (2.17) we obtain:

$$\sqrt{\frac{g_1(t_0)g_1(t_1)g_2(t_0)g_2(t_1)}{r_h^+(t_1)r_{h_1}^+(t_1)}}||(u_0(t_0), v_0(t_0))|| \le ||(\widetilde{u}(t_1), \widetilde{v}(t_1)|| \le$$

$$\leq \sqrt{G_1(t_0)G_1(t_1)G_2(t_0)G_2(t_1)}||(u_0(t_0), v_0(t_0))||\sqrt{r_h^-(t_1)r_{h_1}^-(t_1)}. \qquad (2.18)$$

Obviously,

$$g_1(t)g_2(t) = g^2(t), \;\; G_1(t)G_2(t) = G^2(t), \qquad t \geq t_0. \qquad (2.19)$$

Note (due to (2.12) and (2.13)), that $r_{h_1}^{\pm}(t) = r_h^{\mp}(t), \;\; t \geq t_0$. Then $r_{h_1}^{\pm}(t_1)r_h^{\mp}(t_1) = r_h^2(t_1)$. From here, from (2.18) and (2.19) we obtain:

$$\frac{g(t_0)g(t_1)||(u_0(t_0), v_0(t_0))||}{r_h(t_1)} \leq ||(\widetilde{u}(t_1), \widetilde{v}(t_1)|| \leq$$

$$\leq G(t_0)G(t_1)||(u_0(t_0), v_0(t_0))||r_h(t_1).$$

From here and from (2.16) it follows:

$$\frac{g(t_0)g(t_1)||(u_0(t_0), v_0(t_0))||}{r_h(t_1)} - \sqrt{2}\varepsilon \leq ||(u_0(t_1), v_0(t_1)|| \leq$$

$$\leq G(t_0)G(t_1)||(u_0(t_0), v_0(t_0))||r_h(t_1) + \sqrt{2}\varepsilon.$$

By the arbitrariness of $\varepsilon(> 0)$ from here we will have:

$$\frac{g(t_0)g(t_1)||(u_0(t_0), v_0(t_0))||}{r_h(t_1)} \leq ||(u_0(t_1), v_0(t_1)|| \leq$$

$$\leq G(t_0)G(t_1)||(u_0(t_0), v_0(t_0))||r_h(t_1). \qquad (2.20)$$

Let $(u(t), v(t))$ be an arbitrary (complex) solution of the system (2.14). Since $(u(t), v(t)) =$

$= (u_1(t), v_1(t)) + i(u_2(t), v_2(t))$, where $(u_j(t), v_j(t)), \;\; (j = 1, 2)$ are some real solutions of the system (2.14), by (2.20) we will get:

$$\left[\frac{g(t_0)g(t_1)}{r_h(t_1)}\right]^2 \sum_{j=1}^{2} ||(u_j(t_0), v_j(t_0))||^2 \leq \sum_{j=1}^{2} ||(u_j(t_1), v_j(t_1))||^2 \leq$$

$$\leq [G(t_0)G(t_1)r_h(t_1)]^2 \sum_{j=1}^{2} ||(u_j(t_0), v_j(t_0))||^2.$$

Taking into account the equality $||(u(t), v(t))||^2 = \sum\limits_{j=1}^{2} ||(u_j(t), v_j(t))||^2, \;\; t \geq t_0$, from here we will have:

$$\frac{g(t_0)g(t_1)||(u(t_0), v(t_0))||}{r_h(t_1)} \leq ||(u(t_1), v(t_1)|| \leq G(t_0)G(t_1)||(u(t_0), v(t_0))||r_h(t_1).$$

By virtue of arbitrariness of $t_1(> t_0)$ from here it follows (2.15). Thus, we have proved (2.15) under the additional assumption $h(t) \in C_\sim$. Let us prove it in the general case. Let $A_C[t_0;T]$ be the space of absolutely continuous functions $f(t)$ of finite variation on the interval $[t_0;T]$ with the norm $||f(t)|| \equiv \max\limits_{t\in[t_0;T]} |f(t)| + \int\limits_{t_0}^{T} |f'(\tau)|d\tau$. Obviously the set of rational functions is everywhere dense in $A_C[t_0,T]$. In view of this we choose polynomials $p_1(t)$ and $q_1(t)$ such, that $p_1(t) > 0$, $q_1(t) > 0$, $t \ge t_0$, and such, that for each fixed $t_1 \in (t_0,T]$ and $\varepsilon(> 0)$ the following inequalities hold

$$\left|\frac{g(t_0)g(t_1)}{r_h(t_1)} - \frac{\widetilde{g}(t_0)\widetilde{g}(t_1)}{r_{\widetilde{h}}(t_1)}\right| \le \varepsilon, \quad \left|G(t_0)G(t_1)r_h(t_1) - \widetilde{G}(t_0)\widetilde{G}(t_1)r_{\widetilde{h}}(t_1)\right| \le \varepsilon, \tag{2.21}$$

where $\widetilde{g}(t) \equiv \min\left\{\sqrt[4]{\widetilde{h}(t)}, \sqrt[4]{\widetilde{h}_1(t)}\right\}$, $\widetilde{G}(t) \equiv \max\left\{\sqrt[4]{\widetilde{h}(t)}, \sqrt[4]{\widetilde{h}_1(t)}\right\}$, $\widetilde{h}(t) \equiv \frac{q_1(t)}{p_1(t)}$, $\widetilde{h}_1(t) \equiv \frac{p_1(t)}{q_1(t)}$, $t \ge t_0$, as well as (by Lemma 2.1) the following inequalities hold

$$|u(t_1) - \widetilde{u}(t_1)| \le \varepsilon, \qquad |v(t_1) - \widetilde{v}(t_1)| \le \varepsilon, \tag{2.22}$$

where $(\widetilde{u}(t), \widetilde{v}(t))$ is the solution of the system

$$\begin{cases} u'(t) = \qquad\qquad p_1(t)v(t); \\ v'(t) = q_1(t)u(t), \end{cases}$$

$t \ge t_0$, with $\widetilde{u}(t_0) = u(t_0)$, $\widetilde{u}(t_0) = u(t_0)$. Since obviously $\widetilde{h}(t)|_{[t_0;+\infty)} \in C_\sim$, by already proven

$$\frac{\widetilde{g}(t_0)\widetilde{g}(t_1)||(u(t_0),v(t_0))||}{r_{\widetilde{h}}(t_1)} \le ||(\widetilde{u}(t_1),\widetilde{v}(t_1)|| \le \widetilde{G}(t_0)\widetilde{G}(t_1)||(u(t_0),v(t_0))||r_{\widetilde{h}}(t_1).$$

From here, from (2.21) and (2.22) it follows

$$\frac{g(t_0)g(t_1)||(u(t_0),v(t_0))||}{r_h(t_1)} - [\sqrt{2} + ||(u(t_0),v(t_0))||]\varepsilon \le ||(u(t_1),v(t_1)|| \le$$

$$\le G(t_0)G(t_1)||(u(t_0),v(t_0))||r_h(t_1) + [\sqrt{2} + ||(u(t_0),v(t_0))||]\varepsilon.$$

By virtue of arbitrariness of $t_1 \in (t_0;T]$, $T(> t_0)$ and $\varepsilon(> 0)$ from here it follows (2.15). The lemma is proved.

Let us consider the Cauchy problem

$$\begin{cases} u'(t) = \qquad \widetilde{p}(t)v(t); \\ v'(t) = \widetilde{q}(t)u(t); \\ u(t_0) = u_{(0)}, \;\; v(t_0) = v_{(0)}, \end{cases} \tag{2.23}$$

$t \geq t_0$. Its solution we will seek in the form

$$\begin{cases} u(t) = A_j\sqrt{p_j}\exp\{\sqrt{l_j}t\} + B_j\sqrt{p_j}\exp\{-\sqrt{l_j}t\}; \\ v(t) = A_j\sqrt{q_j}\exp\{\sqrt{l_j}t\} - B_j\sqrt{q_j}\exp\{-\sqrt{l_j}t\}, \end{cases} \tag{2.24}$$

$t \in [t_j; t_{j+1}), \;\; j = 0, 1, 2, ...$ The unknowns A_0 and B_0 we can find by solving the system

$$\begin{cases} A_0\sqrt{p_0}\exp\{\sqrt{l_0}t)_0\} + B_0\sqrt{p_0}\exp\{-\sqrt{l_0}t_0\} = u_{(0)}; \\ A_0\sqrt{q_0}\exp\{\sqrt{l_0}t_0\} - B_0\sqrt{q_0}\exp\{-\sqrt{l_0}t_0\} = v_{(0)}, \end{cases} \tag{2.25}$$

and the remaining unknowns $A_j, B_j (j = 1, 2, ...)$ we can find by successive solving the systems

$$\begin{cases} A_j\sqrt{p_j}\exp\{\sqrt{l_j}t_j\} + B_j\sqrt{p_j}\exp\{-\sqrt{l_j}t_j\} = u(t_j); \\ A_j\sqrt{q_j}\exp\{\sqrt{l_j}t_j\} - B_j\sqrt{q_j}\exp\{-\sqrt{l_j}t_j\} = v(t_j), \end{cases}$$

$j = 1, 2, ...$ We have:

$$A_j = \frac{u(t_j)\sqrt{q_j} + v(t_j)\sqrt{p_j}}{2\sqrt{l_j}}\exp\{-\sqrt{l_j}t_j\}, \qquad B_j = \frac{u(t_j)\sqrt{q_j} - v(t_j)\sqrt{p_j}}{2\sqrt{l_j}}\exp\{-\sqrt{l_j}t_j\}.$$

From here, from (2.24) and (2.25) it follows

$$\begin{cases} u(t) = u(t_j)\,\mathrm{ch}\{\sqrt{l_j}(t - t_j)\} + v(t_j)\sqrt{\frac{p_j}{q_j}}\,\mathrm{sh}\{\sqrt{l_j}(t - t_j)\}; \\ v(t) = v(t_j)\,\mathrm{ch}\{\sqrt{l_j}(t - t_j)\} + u(t_j)\sqrt{\frac{q_j}{p_j}}\,\mathrm{sh}\{\sqrt{l_j}(t - t_j)\}, \end{cases} \tag{2.26}$$

$t \in [t_j; t_{j+1}), \;\; j = 0, 1, 2, ...$ From here it is easy to derive the equalities

$$\sqrt{q_j}u(t) + \sqrt{p_j}v(t) = [\sqrt{q_j}u(t_j) + \sqrt{p_j}v(t_j)]\exp\{\sqrt{l_j}(t - t_j)\}, \qquad t \in [t_j; t_{j+1}), \tag{2.27}$$

$j = 0, 1, ...$ From here it follows

$$\sqrt{q_{j+1}}u(t_{j+1})+\sqrt{p_{j+1}}v(t_{j+1}) = \sqrt{\frac{q_{j+1}}{q_j}}\left[\left(\sqrt{q_j}u(t_j) + \sqrt{\frac{h_j}{h_{j+1}}}\sqrt{p_j}v(t_j)\right) \times\right.$$

$$\left.\times \operatorname{ch}\{\sqrt{l_j}(t_{j+1} - t_j)\} + \left(\sqrt{\frac{h_j}{h_{j+1}}}\sqrt{q_j}u(t_j) + \sqrt{p_j}v(t_j)\right)\operatorname{sh}\{\sqrt{l_j}(t_{j+1} - t_j)\}\right], \quad (2.28)$$

$j = o, 1, 2, ...$ Let $u(t_0) \geq 0, \;\; v(t_0) \geq 0, \;\; u(t_0)^2 + v(t_0)^2 \neq 0$. Then from (2.26) it follows that

$$u(t) > 0, \qquad v(t) > 0, \quad t > t_0. \quad (2.29)$$

From here, from (2.27) and (2.28) we get

$$\sqrt{\frac{q_{j+1}}{q_j}}\left[\sqrt{q_j}u(t_j) + \sqrt{p_j}v(t_j)\right]\exp\{\sqrt{l_j}(t_{j+1}-t_j)\} \leq \sqrt{q_{j+1}}u(t_{j+1})+\sqrt{p_{j+1}}v(t_{j+1}) \leq$$

$$\leq \sqrt{\frac{p_{j+1}}{p_j}}\left[\sqrt{q_j}u(t_j) + \sqrt{p_j}v(t_j)\right]\exp\{\sqrt{l_j}(t_{j+1} - t_j)\}, \quad (2.30)$$

for $h_j \geq h_{j+1}, \;\; j = 0, 1, ...$ and

$$\sqrt{\frac{p_{j+1}}{p_j}}\left[\sqrt{q_j}u(t_j) + \sqrt{p_j}v(t_j)\right]\exp\{\sqrt{l_j}(t_{j+1}-t_j)\} \leq \sqrt{q_{j+1}}u(t_{j+1})+\sqrt{p_{j+1}}v(t_{j+1}) \leq$$

$$\leq \sqrt{\frac{q_{j+1}}{q_j}}\left[\sqrt{q_j}u(t_j) + \sqrt{q_j}v(t_j)\right]\exp\{\sqrt{l_j}(t_{j+1} - t_j)\}, \quad (2.31)$$

for $h_j \leq h_{j+1}, \;\; j = 0, 1, ...$ Let us consider the system

$$\begin{cases} u'(t) = \qquad\qquad p(t)v(t); \\ v'(t) = q(t)u(t), \end{cases} \quad (2.32)$$

$t \geq t_0$. Denote: $e(t) \equiv \exp\left\{\int\limits_{t_0}^{t}\sqrt{p(\tau)q(\tau)}d\tau\right\}, \;\; t \geq t_0.$

Lemma 2.3. *Let $h(t)$ be absolutely continuous and has a local finite variation. Then for each solution $(u(t), v(t))$ of the system (2.32) with $u(t_0) \geq 0, \;\; v(t_0) \geq 0$ the following inequalities hold*

$$\frac{d(u,v)e(t)\sqrt{p(t)}}{\sqrt{p(t_0)}r_h^-(t)} \leq \sqrt{q(t)}u(t) + \sqrt{p(t)}v(t) \leq \frac{d(u,v)e(t)\sqrt{q(t)}}{\sqrt{q(t_0)}}r_h^-(t), \qquad t \geq t_0, \quad (2.33)$$

where $d(u,v) \equiv \sqrt{q(t_0)}u(t_0) + \sqrt{p(t_0)}v(t_0)$.

Proof. We prove the lemma only in the particular case when $h(t) \in C_\sim$. The proof in the general case by analogy of the last part of the proof of Lemma 2.2. Let $\varepsilon(>0)$ and $t_1(>t_0)$ be fixed, and let $\xi_0 = t_0 \le \xi_1 \le ... \le \xi_n \le ...$ be the possible extremums of the function $h(t)$. Let $(u(t), v(t))$ be a solution of the system (2.32) with $u(t_0) \ge 0, \;\; v(t_0) \ge 0$. By virtue of Lemma 2.1 there exist piecewise constant functions $\widetilde{\widetilde{p}}(t)$ and $\widetilde{\widetilde{q}}(t)$ such, that the solution $(\widetilde{u}(t), \widetilde{v}(t))$ of the system

$$\begin{cases} u'(t) = \qquad \widetilde{\widetilde{p}}(t)v(t); \\ v'(t) = \widetilde{\widetilde{q}}(t)u(t), \end{cases}$$

$t \ge t_0$, with $\widetilde{u}(t_0) = u_0(t_0), \;\; \widetilde{v}(t_0) = v_0(t_0)$ satisfies the inequalities

$$|u(t_1) - \widetilde{u}(t_1)| \le \frac{\varepsilon}{\sqrt{p(t_1)} + \sqrt{q(t_1)}}, \;\; |v(t_1) - \widetilde{v}(t_1)| \le \frac{\varepsilon}{\sqrt{p(t_1)} + \sqrt{q(t_1)}}. \tag{2.34}$$

Without loss of generality we will assume that $\widetilde{\widetilde{p}}(\xi_k) = p(\xi_k), \;\; \widetilde{\widetilde{q}}(\xi_k) = q(\xi_k), \;\; k = 0, 1, ..., \;\; \widetilde{\widetilde{p}}(t_1) = p(t_1), \;\; \widetilde{\widetilde{q}}(t_1) = q(t_1)$; the function $\frac{\widetilde{\widetilde{q}}(t)}{\widetilde{\widetilde{q}}(t)}$ is nondecreasing on the intervals $[\xi_{2k}; \xi_{2k+1}]$ and nonincreasing on the intervals $[\xi_{2k+1}; \xi_{2k+2}], \;\; k = 0, 1, ...$;

$$\left| \frac{d(u,v)\widetilde{e}(t_1)\sqrt{p(t_1)}}{\sqrt{p(t_0)}r_h^-(t_1)} - \frac{d(u,v)e(t_1)\sqrt{p(t_1)}}{\sqrt{p(t_0)}r_h^-(t_1)} \right| \le \varepsilon, \tag{2.35}$$

$$\left| \frac{d(u,v)\widetilde{e}(t_1)\sqrt{q(t_1)}r_h^-(t_1)}{\sqrt{q(t_0)}} - \frac{d(u,v)e(t_1)\sqrt{q(t_1)}r_h^-(t_1)}{\sqrt{q(t_0)}} \right| \le \varepsilon, \tag{2.36}$$

where $\widetilde{e}(t) \equiv \exp\{\int\limits_{t_0}^{t} \sqrt{\widetilde{\widetilde{p}}(\tau)\widetilde{\widetilde{q}}(\tau)}d\tau\}, \;\; t \ge t_0$. Then by (2.30) and (2.31) we will have:

$$\frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{p(t_0)}}\sqrt{p(t_1)} \le \sqrt{q_1(t)}\widetilde{u}(t_1) + \sqrt{p_1(t)}\widetilde{v}(t_1) \le \frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{q(t_0)}}\sqrt{q(t_1)}, \;\; if \; t_1 \in [\xi_0 l \xi_1];$$

$$\frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{p(t_0)}}\sqrt{\frac{h(t_1)}{h(\xi_1)}}\sqrt{p(t_1)} \le \sqrt{q_1(t)}\widetilde{u}(t_1) + \sqrt{p_1(t)}\widetilde{v}(t_1) \le \frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{q(t_0)}}\sqrt{\frac{h(\xi_1)}{h(t_1)}}\sqrt{q(t_1)},$$

if $t_1 \in [\xi_0; \xi_1]$;

$$\frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{p(t_0)}}\left(\prod_{k=0}^{n-1}\sqrt{\frac{h(\xi_{2k+1})}{h(\xi_{2k+2})}}\right)\sqrt{p(t_1)} \le \sqrt{q_1(t)}\widetilde{u}(t_1) + \sqrt{p_1(t)}\widetilde{v}(t_1) \le$$

$$\leq \frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{q(t_0)}}\left(\prod_{k=0}^{n-1}\sqrt{\frac{h(\xi_{2k+2})}{h(\xi_{2k+1})}}\right)\sqrt{q(t_1)}, \quad if \quad t_1 \in [\xi_{2n};\xi_{2n+1}], \quad n=1,2,....;$$

$$\frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{p(t_0)}}\left(\prod_{k=0}^{n-1}\sqrt{\frac{h(\xi_{2k+1})}{h(\xi_{2k+2})}}\right)\sqrt{\frac{h(t_1)}{h(\xi_{2n+1})}}\sqrt{p(t_1)} \leq \sqrt{q_1(t)}\widetilde{u}(t_1)+\sqrt{p_1(t)}\widetilde{v}(t_1) \leq$$

$$\leq \frac{d(u,v)\widetilde{e}(t_1)}{\sqrt{q(t_0)}}\left(\prod_{k=0}^{n-1}\sqrt{\frac{h(\xi_{2k+2})}{h(\xi_{2k+1})}}\right)\sqrt{\frac{h(\xi_{2n+1})}{h(t_1)}}\sqrt{q(t_1)}, \quad if \quad t_1 \in [\xi_{2n+1};\xi_{2n+2}], \quad n=1,2,....;$$

Therefore

$$\frac{d(u,v)\widetilde{e}(t_1)\sqrt{p(t)}}{\sqrt{p(t_0)}r_h^-(t_1)} \leq \sqrt{q(t_1)}\widetilde{u}(t_1)+\sqrt{p(t_1)}\widetilde{v}(t_1) \leq \frac{d(u,v)\widetilde{e}(t_1)\sqrt{q(t_1)}}{q(t_0)}r_h^-(t_1).$$

From here and from (2.34) - (2.36) it follows that

$$\frac{d(u,v)e(t_1)\sqrt{p(t_1)}}{\sqrt{p(t_0)}r_h^-(t_1)} - 2\varepsilon \leq \sqrt{q(t_1)}u(t_1)+\sqrt{p(t_1)}v(t_1) \leq \frac{d(u,v)e(t_1)\sqrt{q(t_1)}}{q(t_0)}r_h^-(t_1)+2\varepsilon.$$

By virtue of arbitrariness of $\varepsilon(>0)$ and $t_1(>t_0)$ from here it follows (2.33). The lemma is proved.

Consider the Riccati equation

$$y'(t)+p(t)y^2(t)-q(t)=0, \qquad t \geq t_0. \tag{2.37}$$

The solutions $y(t)$ of this equation, existing on the interval $[t_1,t_2)(t_0 \leq t_1 < t_2 \leq +\infty)$ are connected with the solutions $(u(t),v(t))$ of the system (2.32) by equalities (see [11], pp. 153, 154):

$$u(t)=u(t_1)\exp\left\{\int_{t_2}^{t}p(\tau)y(\tau)d\tau\right\}, \qquad v(t)=y(t)u(t), \qquad t\in[t_1,t_2). \tag{2.38}$$

Let $y_0(t)$ be a solution of Eq. (2.37) with $y_0(t_0)>0$. It follows from Theorem 4.1 of work [12] (see [12],p. 26) that $y_0(t)$ exists on the interval $[t_0,+\infty)$ and

$$y_0(t)>0, \qquad t\geq t_0. \tag{2.39}$$

Since $p(t)>0, \ q(t)>0, \ t\geq t_0$, then from Theorem 3.1 of work [13] (see [13], p. 4) it follows that

$$y_0(s) > \frac{y_0(t)}{1+y_0(t)\int_t^s p(\zeta)d\zeta}, \qquad s\geq t\geq t_0. \tag{2.40}$$

Consider the integral

$$\nu_{y_0}(t) \equiv \int_t^{+\infty} p(\tau)\exp\Big\{-2\int_t^{\tau} p(\xi)y_0(\xi)d\xi\Big\}d\tau, \qquad t \ge t_0.$$

From (2.39) and (2.40) it follows that

$$\nu_{y_0}(t) \le \int_t^{+\infty} p(\tau)\exp\Big\{-2\int_t^{\tau} \frac{p(s)y_0(t)}{1 + y_0(t)\int_t^s p(\zeta)d\zeta}ds\Big\}d\tau = \frac{1}{y_0(t)}, \qquad t \ge t_0. \tag{2.41}$$

The function $y_*(t) \equiv y_0(t) - \frac{1}{\nu_{y_0}(t)}$, $t \ge t_0$, is an extremal solution of Eq. (2.37) (see [14], p. 194, Theorem 4). From (2.41) it follows that

$$y_*(t) \le 0, \qquad t \ge t_0. \tag{2.42}$$

Let us show that

$$y_*(t) < 0, \qquad t \ge t_0. \tag{2.43}$$

Suppose that $y_*(t_1) = 0$ for some $t_1 \ge t_0$. Then by virtue of Theorem 4.1 of work [13] the following inequality holds $y_*(t) \ge 0$, $t \ge t_1$. From here and from (2.42) it follows that $y_*(t) \equiv 0$ on the interval $[t_1, +\infty)$, which is impossible. The obtained contradiction proves (2.43). Since $y_0(t_0) > 0$, then from (2.42) ((2.43)) and from Theorem 4 of work [14] it follows that $y_0(t)$ is a normal solution (a solution $y(t)$ of Eq. (2.37) is said to be normal if there exists a neighborhood of the point $y(t_0)$ such that every solution of Eq. (2.37) with initial value from this neighborhood exists on the interval $[t_0, +\infty)$). Then (see [14], p. 195)

$$\int_{t_0}^{+\infty} p(\tau)[y_0(\tau) - y_*(\tau)]d\tau = +\infty. \tag{2.44}$$

Definition 2.1. *The solution $(u_*(t), v_*(t))$ of the system (2.32), satisfying the initial conditions $u_*(t_0) = 1$, $v_*(t_0) = y_*(t_0)$, will be called the canonical main solution of Eq. (2.32). The (a) solution $(u_0(t), v_0(t))$ of the system (2.32), satisfying the initial conditions $u_0(t_0) = v_0(t_0) = 1$ ($u_0(t_0) \ge 0$, $v_0(t_0) \ge 0$, $u_0^2(t_0) + v_0^2(t_0) \ne 0$), will be called the canonical nonprincipal (a real nonprincipal) solution of the system (2.32). The solutions $\lambda(u_*(t), v_*(t))$ and $\lambda(u_0(t), v_0(t))$, where λ is an arbitrary constant and $(u_0(t), v_0(t))$ is an real nonprincipal solution of the system (2.32), will be called a main and a principal* *solutions of the system (2.32) respectively. A solution of the system (2.32), which is not main solution will be called an ordinary solution of the system (2.32).*

From (2.43) it follows that the canonical main and nonprincipal solutions of the system (2.32) are linearly independent. Therefore for general solution $(u(t), v(t))$ of the system (2.32) the following representation holds

$$(u(t), v(t)) = \lambda_0(u_0(t), v_0(t)) + \lambda_*(u_*(t), v_*(t)), \quad \lambda_0 = const, \quad \lambda_* = const, \quad t \geq t_0. \tag{2.45}$$

On the strength of (2.38) we have

$$u_*(t) = \exp\Big\{\int\limits_{t_0}^{t} p(\tau)y_*(\tau)d\tau\Big\}, \qquad v_*(t) = y_*(t)u_*(t), \qquad t \geq t_0; \tag{2.46}$$

$$u_0(t) = \exp\Big\{\int\limits_{t_0}^{t} p(\tau)y_0(\tau)d\tau\Big\}, \qquad t \geq t_0, \tag{2.47}$$

where $y_0(t)$ is the solution of eq. (2.37) with $y_0(t_0) = 1$. From here and from (2.44) it follows that

$$\frac{u_*(t)}{u_0(t)} \to 0 \;\; for \;\; t \to +\infty. \tag{2.48}$$

Let us show that

$$\frac{v_*(t)}{v_0(t)} \to 0 \;\; for \;\; t \to +\infty. \tag{2.49}$$

In Eq. (2.37) we make the change $y(t) = \frac{1}{z(t)}$. We will come to the equation

$$z'(t) + q(t)z^2(t) - p(t) = 0, \qquad t \geq t_0. \tag{2.50}$$

To prove (2.49) we need to the following

Lemma 2.4. *Suppose* $\int\limits_{t_0}^{+\infty} p(t)dt = +\infty$ *or* $\int\limits_{t_0}^{+\infty} q(t)dt = +\infty$. *Then* $z_*(t) \equiv \frac{1}{y_*(t)}, \;\; t \geq t_0$, *is the extremal solution of Eq. (2.50), where* $y_*(t)$ *is the extremal solution of Eq. (2.37).*

Proof. Above it was shown that $y_*(t) < 0, \;\; t \geq t_0$. Therefore $z_*(t)$ is defined correct. Obviously $z_*(t)$ is a solution to Eq. (2.50). Suppose $\int\limits_{t_0}^{+\infty} p(t)dt = +\infty$. Let $\widetilde{z}_*(t)$ be the extremal solution of Eq. (2.50). Suppose $z_*(t) \neq \widetilde{z}_*(t)$. Then the solution $z_N(t)$ of Eq. (2.50) with $z_N(t_0) = \frac{z_*(t_0)+\widetilde{z}_*(t_0)}{2}$ is a normal solution to Eq. (2.50). and $z_N(t) < 0, \;\; t \geq t_0$ (the graph of $z_N(t)$ is between the graphs of $z_*(t)$ and $\widetilde{z}_*(t)$). Therefore $y_N(t) \equiv \frac{1}{z_N(t)}, \;\; t \geq$

t_0 is a normal solution of Eq. (2.37). Hence $\nu_{y_N}(t_0) < +\infty$. On the other hand since $y_N(t) < 0,\ t \ge t_0$ we have $\nu_{y_N}(t_0) \ge \int\limits_{t_0}^{+\infty} p(t)dt = +\infty$. The obtained contradiction shows that $z_*(t)$ is extremal. Suppose now $\int\limits_{t_0}^{+\infty} q(t)dt = +\infty$. Let $z_*(t)$ not be extremal. the the integral $\int\limits_{t_0}^{+\infty} q(t)\exp\Big\{-\int\limits_{t_0}^{t} q(s)z_*(s)ds\Big\}dt$ is convergent. On the other hand since $z_*(t) < 0,\ t \ge t_0$, we have $\int\limits_{t_0}^{+\infty} q(t)\exp\Big\{-\int\limits_{t_0}^{t} q(s)z_*(s)ds\Big\}dt \ge \int\limits_{t_0}^{+\infty} q(t)dt = +\infty$. The obtained contradiction completes the proof of the lemma.

Obviously $z_0(t) \equiv \frac{1}{y_0(t)}$ is a normal solutions of Eq. (2.50). Then since $z_*(t)$ is extremal we have

$$\int\limits_{t_0}^{+\infty} q(\tau)[z_0(\tau) - z_*(\tau)]d\tau = +\infty. \tag{2.51}$$

Let $\widetilde{v}_0(t) \equiv \exp\Big\{\int\limits_{t_0}^{t} q(\tau)z_0(\tau)d\tau\Big\},\ \ \widetilde{u}_0(t) = z_0(t)\widetilde{v}_0(t),\ \ \widetilde{v}_*(t) \equiv \exp\Big\{\int\limits_{t_0}^{t} q(\tau)z_*(\tau)d\tau\Big\},$ $\widetilde{u}_*(t) = z_*(t)\widetilde{v}_*(t),\ \ t \ge t_0$. By virtue of (2.38) $(\widetilde{u}_0(t), \widetilde{v}_0(t))$ and $(\widetilde{u}_*(t), \widetilde{v}_*(t))$ are solutions of the system (2.32). From (2.51) it follows that

$$\frac{\widetilde{v}_*(t)}{\widetilde{v}_0(t)} \to 0 \ \ for \ \ t \to +\infty. \tag{2.52}$$

Since $\widetilde{u}_0(t_0) = \widetilde{v}_0(t_0) = u_0(t_0) = v_0(t_0) = 1,\ \widetilde{u}_*(t_0) = \frac{1}{y_*(t_0)},\ \widetilde{v}_*(t_0) = 1$, we have $(\widetilde{u}_0(t), \widetilde{v}_0(t)) = (u_0(t), v_0(t)),\ \ (\widetilde{u}_*(t), \widetilde{v}_*(t)) = \frac{1}{y_*(t_0)}(u_*(t), v_*(t)),\ \ t \ge t_0$. From here and from (2.52) it follows (2.49). From (2.44), (2.48) and (2.49) it follows

$$(u(t), v(t)) = \lambda_0(u_0(t), v_0(t))[1 + o(1)], \qquad t \to +\infty. \tag{2.53}$$

By (2.32) from (2.43) and (2.46) we will have

$$0 < u_*(t) \le u_*(t_0), \qquad v_*(t_0) \le v_*(t) < 0, \qquad t \ge t_0. \tag{2.54}$$

III. ESTIMATES OF THE SOLUTIONS OF THE SYSTEM (1.1)

In the system (1.1) we make the substitutions

$$\phi(t) = \exp\Big\{\int\limits_{t_0}^{t} a_{11}(\tau)d\tau\Big\}u(t), \qquad \psi(t) = \exp\Big\{\int\limits_{t_0}^{t} a_{22}(\tau)d\tau\Big\}v(t), \tag{3.1}$$

We will come to the system (1.2). In the sequel we will assume that the function $\frac{a_{12}(t)}{a_{21}(t)}$ is absolutely continuous and has a locally finite variation. Denote:

$$m(t) \equiv \min\Big\{\sqrt[4]{\Big|\frac{a_{12}(t)}{a_{21}(t)}\Big|}, \sqrt[4]{\Big|\frac{a_{21}(t)}{a_{12}(t)}\Big|}\Big\}, \qquad M(t) \equiv \max\Big\{\sqrt[4]{\Big|\frac{a_{12}(t)}{a_{21}(t)}\Big|}, \sqrt[4]{\Big|\frac{a_{21}(t)}{a_{12}(t)}\Big|}\Big\},$$

$$\mathcal{F}(t) \equiv \Big|\int\limits_{t_0}^{t}\big[Re\, a_{11}(\tau) - Re\, a_{22}(\tau)\big]d\tau\Big| + \int\limits_{t_0}^{t}\Big|\frac{1}{4}\Big(\frac{a_{12}(\tau)}{a_{21}(\tau)}\Big)'\frac{a_{21}(\tau)}{a_{12}(\tau)} + \frac{a_{22}(\tau) - a_{11}(\tau)}{2}\Big|d\tau, \ \ t \ge t_0.$$

Theorem 3.1. *Let the condition A) be satisfied. Then for each solution $(\phi(t), \psi(t))$ of the system (1.1) the following inequalities hold*

$$m(t_0)||(\phi(t_0), \psi(t_0))||m(t)\exp\Big\{\int\limits_{t_0}^{t}\Big[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)\Big]d\tau - \mathcal{F}(t)\Big\} \le ||(\phi(t), \psi(t))|| \le$$

$$\le M(t_0)||(\phi(t_0), \psi(t_0))||M(t)\exp\Big\{\int\limits_{t_0}^{t}\Big[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)\Big]d\tau + \mathcal{F}(t)\Big\}, \qquad t \ge t_0. \tag{3.2}$$

Proof. Let $(\phi(t), \psi(t))$ be a solution of the system (1.1), and let $(u(t), v(t))$ be the solution of the system (1.2), satisfying the initial conditions $u(t_0) = \phi(t_0), \ \ v(t_0) = \Psi(t_0)$. Then by virtue of (3.1) we have

$$|\phi(t)| = \exp\Big\{\int\limits_{t_0}^{t} Re\, a_{11}(\tau)d\tau\Big\}|u(t)|, \qquad |\Psi(t)| = \exp\Big\{\int\limits_{t_0}^{t} Re\, a_{22}(\tau)d\tau\Big\}|v(t)|, \qquad t \ge t_0.$$

From here it follows

$$||(\phi(t), \psi(t))|| = \sqrt{\exp\Big\{2\int\limits_{t_0}^{t} Re\, a_{11}(\tau)d\tau\Big\}|u(t)|^2 + \exp\Big\{2\int\limits_{t_0}^{t} Re\, a_{22}(\tau)d\tau\Big\}|v(t)|^2}, \qquad t \ge t_0.$$

Therefore

$$\exp\Big\{\min\Big\{\int_{t_0}^{t} Re\, a_{11}(\tau)d\tau, \int_{t_0}^{t} Re\, a_{22}(\tau)d\tau\Big\}\Big\}||(u(t),v(t))|| \le ||(\phi(t),\psi(t))|| \le, \qquad t \ge t_0.$$

$$\le \exp\Big\{\max\Big\{\int_{t_0}^{t} Re\, a_{11}(\tau)d\tau, \int_{t_0}^{t} Re\, a_{22}(\tau)d\tau\Big\}\Big\}||(u(t),v(t))||, \qquad t \ge t_0. \qquad (3.3)$$

Denote $H(t) \equiv \Big|\frac{Q(t)}{P(t)}\Big|$, $H_1(t) \equiv \Big|\frac{P(t)}{Q(t)}\Big|$, $w(t) \equiv \min\{\sqrt[4]{H(t)}, \sqrt[4]{H_1(t)}\}$, $W(t) \equiv$ $\equiv \max\{\sqrt[4]{H(t)}, \sqrt[4]{H_1(t)}\}$, $t \ge t_0$. By virtue of Lemma 2.2 from the condition of the theorem it follows

$$w(t_0)||(\phi(t_0),\psi(t_0))||\frac{w(t)}{r_H(t)} \le ||(u(t),v(t))|| \le W(t_0)W(t)||(\phi(t_0),\psi(t_0))||r_H(t), \qquad t \ge t_0.$$

From here and from (3.3) we will get

$$w(t_0)||(\phi(t_0),\psi(t_0))||\frac{w(t)}{r_H(t)}\exp\Big\{\min\Big\{\int_{t_0}^{t} Re\, a_{11}(\tau)d\tau, \int_{t_0}^{t} Re\, a_{22}(\tau)d\tau\Big\}\Big\} \le ||(\phi(t),\psi(t))|| \le$$

$$\le W(t_0)||(\phi(t_0),\psi(t_0))||W(t)\exp\Big\{\max\Big\{\int_{t_0}^{t} Re\, a_{11}(\tau)d\tau, \int_{t_0}^{t} Re\, a_{22}(\tau)d\tau\Big\}\Big\}r_H(t), \qquad t \ge t_0.$$

Since $w(t_0) = m(t_0)$, $W(t_0) = M(t_0)$,

$$m(t)\exp\Big\{\min\Big\{\int_{t_0}^{t} \frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau, \int_{t_0}^{t} \frac{Re\, a_{22}(\tau) - Re\, a_{11}(\tau)}{2}d\tau\Big\}\Big\} \le w(t),$$

$$W(t) \le M(t)\exp\Big\{\max\Big\{\int_{t_0}^{t} \frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau, \int_{t_0}^{t} \frac{Re\, a_{22}(\tau) - Re\, a_{11}(\tau)}{2}d\tau\Big\}\Big\}, \qquad t \ge t_0,$$

taking into account the equalities

$$\min\Big\{\int_{t_0}^{t} Re\, a_{11}(\tau)d\tau, \int_{t_0}^{t} Re\, a_{22}(\tau)d\tau\Big\} =$$

$$= \int_{t_0}^{t}\Big[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)\Big]d\tau - \Big|\int_{t_0}^{t} \frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau\Big|,$$

$$\max\left\{\int_{t_0}^{t} Re\, a_{11}(\tau)d\tau, \int_{t_0}^{t} Re\, a_{22}(\tau)d\tau\right\} =$$

$$= \int_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)\right]d\tau + \left|\int_{t_0}^{t} \frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau\right|,$$

$$\min\left\{\int_{t_0}^{t} \frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau, \int_{t_0}^{t} \frac{Re\, a_{22}(\tau) - Re\, a_{11}(\tau)}{2}d\tau\right\} =$$

$$= -\left|\int_{t_0}^{t} \frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau\right|,$$

$$\max\left\{\int_{t_0}^{t} \frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau, \int_{t_0}^{t} \frac{Re\, a_{22}(\tau) - Re\, a_{11}(\tau)}{2}d\tau\right\} =$$

$$= \left|\int_{t_0}^{t} \frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau\right|,$$

$t \geq t_0$. from (3.4) we will get (3.2). The theorem is proved.

Remark 3.1. *Let $a(t)$ and $b(t)$ be some continuous functions on $[t_0; +\infty)$ and let $b(t) > 0,\ \ t \geq t_0$. Consider the system*

$$\begin{cases} \phi'(t) = a(t)\phi(t) + b(t)\psi(t); \\ \psi'(t) = -b(t)\phi(t) + a(t)\psi(t), \ \ t \geq t_0. \end{cases}$$

For this system we have $\mathcal{F}(t) \equiv 0,\ \ m(t) = M(t) \equiv 1$. Therefore by Theorem 3.1 for its general solution $(\phi(t), \psi(t))$ the inequalities

$$||(\phi(t_0), \psi(t_0))|| \exp\left\{\int_{t_0}^{t} Re\, a(\tau)d\tau\right\} \leq ||(\phi(t), \psi(t))|| \leq ||(\phi(t_0), \psi(t_0))|| \exp\left\{\int_{t_0}^{t} Re\, a(\tau)d\tau\right\},$$

$t \geq t_0$, are fulfilled. Hence

$$||(\phi(t), \psi(t))|| = ||(\phi(t_0), \psi(t_0))|| \exp\left\{\int_{t_0}^{t} Re\, a(\tau)d\tau\right\}, \qquad t \geq t_0,$$

and in this sense the estimates (3.2) are sharp.

Example 3.1. Consider the system

$$\begin{cases} \phi'(t) = (-\lambda + \sin t)\phi(t) + t^{\alpha}\psi(t); \\ \psi'(t) = -t^{\beta}\phi(t) + (-\mu + \cos t)\psi(t), \end{cases} \tag{3.5}$$

$t \geq \frac{\pi}{4}$, where $\lambda,\ \mu,\ \alpha$ and β are some real numbers. For this system $m(t) = t^{-\frac{|\alpha-\beta|}{2}}$, $M(t) = t^{\frac{|\alpha-\beta|}{2}}$, $\mathcal{F}(t) = |\sqrt{2} + \lambda - \mu + (\lambda - \mu)t| + \int\limits_{\pi/4}^{t} \left| \frac{\alpha-\beta}{4\tau} + \frac{\lambda-\mu}{2} + \frac{\sqrt{2}}{2}\cos(\tau + \frac{\pi}{4})\right| d\tau$, $t \geq \frac{\pi}{4}$. Using Theorem 3.1 it is easy to find the following regions of values of the parameters $\lambda,\ \mu,\ \alpha,\ \beta$ for which the system (3.5) is asymptotically stable:

$O_1 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda + \mu > 3|\lambda - \mu| + \sqrt{2},\ \ \lambda > 0,\ \ \mu > 0\}$;

$O_2 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda = \mu > \frac{\sqrt{2}}{2\pi}\}$;

$O_3 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda + \mu > 3|\lambda - \mu| > 3\sqrt{2},\ \ \lambda > 0,\ \ \mu > 0\}$;

$O_3 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda + \mu > 3|\lambda - \mu| = 3\sqrt{2},\ \ \lambda > 0,\ \ \mu > 0,\ \ \alpha = \beta\}$; and the following regions of values of parameters $\lambda,\ \mu,\ \alpha,\ \beta$ for which system (3.5) is unstable:

$O_5 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda + \mu + 3|\lambda - \mu| + \sqrt{2} < 0,\ \ \lambda < 0,\ \ \mu < 0\}$;

$O_6 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda = \mu < -\frac{\sqrt{2}}{2\pi}\}$;

$O_7 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda + \mu + 3|\lambda - \mu| < 0,\ \ |\lambda - \mu| > \sqrt{2},\ \ \lambda < 0,\ \ \mu < 0\}$;

$O_7 \equiv \{(\lambda,\mu,\alpha,\beta) : \lambda + \mu + 3|\lambda - \mu| < 0,\ \ |\lambda - \mu| = \sqrt{2},\ \ \lambda < 0,\ \ \mu < 0,\ \ \alpha = \beta\}$;

It is not difficult to verify that the application of the estimates of Liapunov (see [4], p. 432), Yu. S. Bogdanov (see [4], p. 433) and estimate by freezing method (see [4], p 441) to the system (3.5) give no result. The estimates by logarithmic norms γ_I and γ_{II} of S. M. Lozinski (see [4], pp. 435, 436) give result only for $\lambda > 0,\ \ \mu > 0,\ \ \alpha < -1,\ \ \beta < -1$. For comparison now we use the theorem of Wazevski to the system (3.5) (see [4], p. 434). By virtue of this theorem for each solution $(\phi(t), \psi(t))$ of the system (3.5) the following inequalities hold

$$||(\phi(t),\psi(t))|| \exp\Big\{\int\limits_{\pi/4}^{t} \omega_-(\tau)d\tau\Big\} \leq ||(\phi(t),\psi(t))|| \leq$$

$$\leq ||(\phi(t),\psi(t))|| \exp\Big\{\int\limits_{\pi/4}^{t} \omega_+(\tau)d\tau\Big\}, \qquad t \geq \frac{\pi}{4}, \tag{3.6}$$

where $\omega_\pm \equiv \frac{-\lambda-\mu+\sin t+\cos t\pm\sqrt{(\lambda-\mu+cost-\sin t)^2+(t^\alpha-t^\beta)^2}}{2}$. If $\alpha \neq \beta > 0$ or $\beta \neq \alpha > 0$, then from (3.6) does not follow neither asymptotic stability nor instability of the system (3.5) for every values of λ and μ.

Definition 3.1. *A solution $(\phi(t), \psi(t))$ of the system (1.1), satisfying the condition B), is said to be a main (a nonprincipal, an ordinary) solution of the system (1.1), if $\phi(t_0) = u(t_0)$, $\psi(t_0) = v(t_0)$, where $(u(t), v(t))$ is a main (a nonprincipal, an ordinary) solution of the system (1.2).*

Theorem 3.2. *Let the condition B) be satisfied and let*

C) $\int\limits_{t_0}^{+\infty} a_{12}(t)\exp\left\{\int\limits_{t_0}^{t}\Big[a_{22}(s)-a_{11}(s)\Big]ds\right\}dt = +\infty$ *or*

$$\int\limits_{t_0}^{+\infty} a_{21}(t)\exp\left\{\int\limits_{t_0}^{t}\Big[a_{11}(s)-a_{22}(s)\Big]ds\right\}dt = +\infty.$$

Then if:
i) $(\phi(t), \psi(t))$ is a nonprincipal solution of the system (1.1), then

$$D(\phi,\psi)m(t)\exp\left\{\int\limits_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2}Re\,a_{jj}(\tau)+\sqrt{a_{12}(\tau)a_{21}(\tau)}\right]d\tau-\mathcal{F}(t)\right\} \leq |\phi(t)|+|\psi(t)| \leq$$

$$\leq D(\phi,\psi)M(t)\exp\left\{\int\limits_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2}Re\,a_{jj}(\tau)+\sqrt{a_{12}(\tau)a_{21}(\tau)}\right]d\tau+\mathcal{F}(t)\right\}, \quad \forall t \geq t_0, \quad (3.7)$$

where $D(\phi,\psi) \equiv \sqrt[4]{\left|\frac{a_{21}(t_0)}{a_{12}(t_0)}\right|}|\phi(t_0)| + \sqrt[4]{\left|\frac{a_{12}(t_0)}{a_{21}(t_0)}\right|}|\psi(t_0)|;$

ii) $(\phi(t), \psi(t))$ is a main solution of the system (1.1), then

$|\phi(t)|+|\quad(t)| \leq$

$$\leq (|\phi(t_0)|+|\psi(t_0)|)\exp\left\{\int\limits_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2}Re\,a_{jj}(\tau)\right]d\tau+\left|\int\limits_{t_0}^{t}\frac{Re\,a_{11}(\tau)-Re\,a_{22}(\tau)}{2}d\tau\right|\right\}, \quad (3.8)$$

$t \geq t_0;$
iii) $(\phi(t), \psi(t))$ is an ordinary solution of the system (1.1), then

$$c_1 m(t)\exp\left\{\int\limits_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2}Re\,a_{jj}(\tau)+\sqrt{a_{12}(\tau)a_{21}(\tau)}\right]d\tau-\mathcal{F}(t)\right\} \leq |\phi(t)|+|\psi(t)| \leq$$

$$\leq c_2 M(t)\exp\left\{\int\limits_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)+\sqrt{a_{12}(\tau)a_{21}(\tau)}\right]d\tau+\mathcal{F}(t)\right\},\qquad t\geq t_0, \qquad (3.9)$$

where $c_j = const > 0, \;\; j = 1, 2.$

Proof. Let $(\phi(t), \psi(t))$ be a solution of the system (1.1), and $(u(t), v(t))$ be the solution of the system (1.2) with $u(t_0) = \phi(t_0), \;\; v(t_0) = \psi(t_0)$. Then by (3.1) we have

$$|\phi(t)|+|\psi(t)|=\exp\left\{\int\limits_{t_0}^{t} Re\, a_{11}(\tau)d\tau\right\}|u(t)|+\exp\left\{\int\limits_{t_0}^{t} Re\, a_{22}(\tau)d\tau\right\}|v(t)|,\qquad t\geq t_0.$$

From here it follows

$$\exp\left\{\min\left\{\int\limits_{t_0}^{t} Re\, a_{11}(\tau)d\tau,\int\limits_{t_0}^{t} Re\, a_{22}(\tau)d\tau\right\}\right\}(|u(t)|+|v(t)|)\leq|\phi(t)|+|\psi(t)|\leq$$

$$\leq\exp\left\{\max\left\{\int\limits_{t_0}^{t} Re\, a_{11}(\tau)d\tau,\int\limits_{t_0}^{t} Re\, a_{22}(\tau)d\tau\right\}\right\}(|u(t)|+|v(t)|),\qquad t\geq t_0,$$

or, which is the same,

$$\exp\left\{\int\limits_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)\right]d\tau-\left|\int\limits_{t_0}^{t}\frac{Re\, a_{11}(\tau)-Re\, a_{22}(\tau)}{2}d\tau\right|\right\}(|u(t)|+|v(t)|)\leq$$

$$\leq|\phi(t)|+|\psi(t)|\leq$$

$$\leq\exp\left\{\int\limits_{t_0}^{t}\left[\frac{1}{2}\sum_{j=1}^{2} Re\, a_{jj}(\tau)\right]d\tau+\left|\int\limits_{t_0}^{t}\frac{Re\, a_{11}(\tau)-Re\, a_{22}(\tau)}{2}d\tau\right|\right\}(|u(t)|+|v(t)|), \quad (3.10)$$

$t \geq t_0$. Let $(u_0(t), v_0(t))$ be a real nonprincipal solution of the system (1.2). Then by virtue of Lemma 2.3 and (2.29) we have

$$\frac{\widetilde{D}E(t)\sqrt{P(t)}}{\sqrt{P(t_0)}r_H^-(t)}\leq\sqrt{Q(t)}u_0(t)+\sqrt{P(t)}v_0(t)\leq\frac{\widetilde{D}E(t)\sqrt{Q(t)}}{\sqrt{Q(t_0)}}r_H^-(t),\qquad t\geq t_0,\qquad (3.11)$$

where $\widetilde{D} \equiv \sqrt{Q(t_0)}u_0(t) + \sqrt{P(t_0)}v_0(t) = \sqrt{a_{21}(t_0)}\phi(t_0) + \sqrt{a_{12}(t_0)}\psi(t_0), \quad E(t) \equiv$

$\equiv \exp\left\{\int\limits_{t_0}^{t} \sqrt{P(\tau)Q(\tau)}d\tau\right\} = \exp\left\{\int\limits_{t_0}^{t} \sqrt{a_{12}(\tau)a_{21}(\tau)}d\tau\right\}, \qquad t \geq t_0$. Obviously

$$\min\{\sqrt{P(t)}, \sqrt{Q(t)}\}[u_0(t)+v_0(t)] \leq \sqrt{Q(t)}u_0(t)+\sqrt{P(t)}v_0(t) \leq$$
$$\leq \max\{\sqrt{P(t)}, \sqrt{Q(t)}\}[u_0(t) + v_0(t)], \qquad t \geq t_0.$$

Therefore,

$$\min\left\{\frac{1}{\sqrt{P(t)}}, \frac{1}{\sqrt{Q(t)}}\right\}\left[\sqrt{Q(t)}u_0(t)+\sqrt{P(t)}v_0(t)\right] \leq u_0(t)+v_0(t) \leq$$
$$\leq \max\left\{\frac{1}{\sqrt{P(t)}}, \frac{1}{\sqrt{Q(t)}}\right\}\left[\sqrt{Q(t)}u_0(t) + \sqrt{P(t)}v_0(t)\right], \qquad t \geq t_0.$$

From here and from (3.11) we will get

$$\widetilde{D}\min\left\{\sqrt[4]{\frac{P(t)}{Q(t)}}, \sqrt[4]{\frac{Q(t)}{P(t)}}\right\}\frac{E(t)}{\sqrt[4]{P(t_0)Q(t_0)}r_H(t)} \leq u_0(t)+v_0(t) \leq$$

$$\leq \widetilde{D}\max\left\{\sqrt[4]{\frac{P(t)}{Q(t)}}, \sqrt[4]{\frac{Q(t)}{P(t)}}\right\}\frac{E(t)}{\sqrt[4]{P(t_0)Q(t_0)}}r_H(t), \qquad t \geq t_0.$$

Therefore,

$$\frac{\widetilde{D}m(t)}{\sqrt[4]{a_{12}(t_0)a_{21}(t_0)}}\exp\left\{-\left|\int\limits_{t_0}^{t}\frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau\right|\right\}\frac{E(t)}{r_H(t)} \leq u_0(t)+v_0(t) \leq$$

$$\leq \frac{\widetilde{D}M(t)}{\sqrt[4]{a_{12}(t_0)a_{21}(t_0)}}\exp\left\{\left|\int\limits_{t_0}^{t}\frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau\right|\right\}E(t)r_H(t), \qquad t \geq t_0.$$

Taking into account the equalities $(u(t), v(t)) = \lambda_0(u_0(t), v_0(t)), \quad \lambda_0 = const \neq 0, \quad |\phi(t_0)| =$

$= |\lambda_0|u_0(t_0), \quad |\psi(t_0)| = |\lambda_0|v_0(t_0)$ from here we will get

$$D(\phi, \psi)\exp\left\{-\left|\int\limits_{t_0}^{t}\frac{Re\, a_{11}(\tau) - Re\, a_{22}(\tau)}{2}d\tau\right|\right\}\frac{m(t)E(t)}{r_H(t)} \leq |u(t)| + |v(t)$$

$$\leq D(\phi,\psi)\exp\Big\{\Big|\int_{t_0}^{t}\frac{Re\,a_{11}(\tau)-Re\,a_{22}(\tau)}{2}d\tau\Big|\Big\}M(t)E(t)r_H(t),\qquad t\geq t_0. \quad (3.12)$$

From here and from (3.10) it follows (3.7). The assertion i) is proved. Let us prove ii). Let $(\phi(t),\psi(t))$ be a main solution of the system (1.1). Then by (3.1) we have

$$\phi(t)=\phi(t_0)\exp\Big\{\int_{t_0}^{t}a_{11}(\tau)d\tau\Big\}u_0(t),\qquad \psi(t)=\boldsymbol{\psi}(t_0)\exp\Big\{\int_{t_0}^{t}a_{22}(\tau)d\tau\Big\}v_0(t), \quad (3.13)$$

$t\geq t_0$, where $(u_0(t),v_0(t))$ is the canonical main solution of the system (1.2). By (2.54) from C) it follows $|\phi_0(t_0)|u_*(t)+|\phi_0(t_0)||v_*(t)|\leq|\phi_0(t_0)|u_*(t_0)+|\phi_0(t_0)||v_*(t_0)|,\ \ t\geq t_0$. By (3.10) from here and from (3.13) it follows (3.8). The assertion ii) is proved. Let us prove iii). Let $(\phi(t),\psi(t))$ be an ordinary solution of the system (1.1). By (3.1) we have

$$\phi(t)=\exp\Big\{\int_{t_0}^{t}a_{11}(\tau)d\tau\Big\}\Big[\lambda_0u_0(t)+\lambda_*u_*(t)\Big],\qquad t\geq t_0, \quad (3.14)$$

$$(t)=\exp\Big\{\int_{t_0}^{t}a_{11}(\tau)d\tau\Big\}\Big[\lambda_0v_0(t)+\lambda_*v_*(t)\Big],\qquad t\geq t_0, \quad (3.15)$$

where $(u_*(t),v_*(t))$ and $(u_0(t),v_0(t))$ are the canonical main and canonical nonprincipal solutions of the system (1.2) respectively, and $\lambda_0\neq 0$. Then by (2.29) and (2.52) we can deduce from C) that $\widetilde{c}_1[u_0(t)+v_0(t)]\leq|\lambda_0u_0(t)+\lambda_*u_0(t)|+|\lambda_0v_0(t)+\lambda_*u_*(t)|\leq$ $\leq\widetilde{c}_2[u_0(t)+v_0(t)],\ t\geq t_0,\ \widetilde{c}_j=const,\ j=1,2$. By virtue of (3.10) from here, from (3.14) and (3.15) we obtain

$$\widetilde{c}_1\exp\Big\{\int_{t_0}^{t}\Big[\frac{1}{2}\sum_{j=1}^{2}Re\,a_{jj}(\tau)\Big]d\tau-\Big|\int_{t_0}^{t}\frac{Re\,a_{11}(\tau)-Re\,a_{22}(\tau)}{2}d\tau\Big|\Big\}[u_0(t)+v_0(t)]\leq$$

$$\leq|\phi(t)|+|\boldsymbol{\psi}(t)|\leq$$

$$\leq\widetilde{c}_2\exp\Big\{\int_{t_0}^{t}\Big[\frac{1}{2}\sum_{j=1}^{2}Re\,a_{jj}(\tau)\Big]d\tau+\Big|\int_{t_0}^{t}\frac{Re\,a_{11}(\tau)-Re\,a_{22}(\tau)}{2}d\tau\Big|\Big\}[u_0(t)+v_0(t)],$$

$\boldsymbol{\psi} t\geq t_0$. By (3.12) from here it follows (3.9). The assertion iii), and therefore, the theorem are proved.

Remark 3.2. *Let $a(t)$ and $b(t)$ be the same as in Remark 3.1. Consider the system*

$$\begin{cases} \phi'(t) = a(t)\phi(t) + b(t)\psi(t); \\ \psi'(t) = b(t)\phi(t) + a(t)\psi(t), \quad t \geq t_0. \end{cases}$$

For this system we have $\mathcal{F}(t) \equiv 0$, $m(t) = M(t) \equiv 1$. Therefore by Theorem 3.2 for its each nonprincipal solution $(\phi(t), \psi(t))$ the inequalities

$$|\phi(t)| + |\psi(t)| = (|\phi(t_0)| + |\psi(t_0)|) \exp\left\{\int_{t_0}^{t} \left[Re\, a(\tau) + b(\tau)\right] d\tau\right\} \leq |\phi(t)| + |\psi(t)| \leq$$

$$\leq (|\phi(t_0)| + |\ (t_0)|) \exp\left\{\int_{t_0}^{t} \left[Re\, a(\tau) + b(\tau)\right] d\tau\right\}, \qquad t \geq t_0,$$

are fulfilled. Hence

$$|\phi(t)| + |\psi(t)| = (|\phi(t_0)| + |\psi(t_0)|) \exp\left\{\int_{t_0}^{t} \left[Re\, a(\tau) + b(\tau)\right] d\tau\right\}, \qquad t \geq t_0,$$

and in this sense the estimates (3.7) are sharp.

Example 3.2. Let us consider the system

$$\begin{cases} \phi'(t) = (-\lambda + \sin t)\phi(t) + t^{\alpha}\psi(t); \\ \psi'(t) = t^{\beta}\phi(t) + (-\mu + \cos t)\psi(t), \end{cases} \tag{3.16}$$

$t \geq \frac{\pi}{4}$, where λ, μ, α and β are some real constants. For this system the functions $m(t)$, $M(t)$ and $\mathcal{F}(t)$ are the same, which are in the example 3.1. Applying Theorem 3.2 to (3.16) it is easy to find the following regions of parameters λ, μ, α, β for which Eq. (3.16) is asymptotically stable:

$O_1^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| + \sqrt{2},\ \lambda > 0,\ \mu > 0,\ \lambda \neq \mu,\ \alpha + \beta < 0\}$;
$O_2^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| + 2 + \sqrt{2},\ \lambda > 0,\ \mu > 0,\ \lambda \neq \mu,\ \alpha + \beta = 0\}$;
$O_3^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| > 3\sqrt{2},\ \lambda > 0,\ \mu > 0,\ \lambda \neq \mu,\ \alpha + \beta < 0\}$;
$O_4^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| + 2 > 3\sqrt{2} + 2,\ \lambda > 0,\ \mu > 0,\ \lambda \neq \mu,\ \alpha + \beta = 0\}$;
$O_5^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| = 3\sqrt{2},\ \lambda > 0,\ \mu > 0,\ \lambda \neq \mu,\ \alpha = \beta < 0\}$;
$O_6^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| + 1 = 3\sqrt{2} + 2,\ \lambda > 0,\ \mu > 0,\ \lambda \neq \mu,\ \alpha = \beta = 0\}$;

and the following regions of parameters λ, μ, α, β for which eq. (3.16) is instable:

$O_7^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda \neq \mu, \ \ \alpha + \beta > 0\};$
$O_8^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu + 3|\lambda - \mu| + \sqrt{2} < 2, \ \ \lambda < 0, \ \ \mu < 0, \ \ \alpha + \beta = 0\};$
$O_9^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda = \mu < -\frac{\sqrt{2}}{2\pi}\};$
$O_{10}^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu + 3|\lambda - \mu| < 2, \ \ |\lambda - \mu| \geq \sqrt{2}, \ \ \alpha + \beta = 0\}.$

As in the case of the system (3.5) the application of the estimates of Liapunov, Yu. S. Bogdanov and estimate by freezing method to the system (3.16) give no result and the estimates by logarithmic norms γ_I and γ_{II} of S. M. Lozinski give result only for $\lambda > 0, \ \ \mu > 0, \ \ \alpha < -1, \ \ \beta < -1$. For the case $\alpha > 0$ or $\beta > 0$ it is impossible by use of the theorem of Wazevski to verify neither asymptotic stability nor instability of system (3,16).

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