Scan to know paper details and author's profile

On the Well-Posedness for the 3-D Micropolar Fluid System in Critical Fourier-Besov-Morrey Spaces

Fatima Ouidirne, Mohamed Oukessou & Jalila El ghordaf

Sultan Moulay Slimane University

ABSTRACT

In the present paper, we study the Cauchy problem of the incompressible micropolar fluid system in $\mathbb{R}^3$. We show that this problem is locally well-posed in Fourier-Besov-Morrey spaces $\mathcal{F}\mathcal{N}_{1,\bar{\lambda},q}$ for $1 \leq q \leq \infty$, and is globally well-posed in these spaces with small initial data.

Keywords: 3-D micropolar fluid system, Fourier-Besov-Morrey spaces, well-posedness.

Classification: MSC Code: 35Q35

Language: English

Great Britain Journals Press

LJP Copyright ID: 925612
Print ISSN: 2631-8490
Online ISSN: 2631-8504

London Journal of Research in Science: Natural & Formal

Volume 24 | Issue 10 | Compilation 1.0

On the Well-Posedness for the 3-D Micropolar Fluid System in Critical Fourier-Besov-Morrey Spaces

Fatima Ouidirne[α], Mohamed Oukessou[σ] & Jalila El ghordaf [ρ]

ABSTRACT

In the present paper, we study the Cauchy problem of the incompressible micropolar fluid system in $\mathbb{R}^3$. ***We show that this problem is locally well-posed in Fourier-Besov-Morrey spaces*** $\mathcal{FN}^{-}_{1,\lambda,q}$ for $1 \leq q \leq \infty$, ***and is globally well-posed in these spaces with small initial data.***

Keywords: 3-D micropolar fluid system, Fourier-Besov-Morrey spaces, well-posedness.

Author α σ ρ: Laboratory LMACS, FST of Beni-Mellal, Sultan Moulay Slimane University, Morocco.

I. INTRODUCTION

In this paper, we are intersted the following initial value problem for the system of partial differential equations describing the motion of incompressible micropolar fluid:

$$\begin{cases} \partial_t u - (\chi+\nu)\Delta u + u\cdot\nabla u + \nabla\pi - 2\chi\nabla\times\omega = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ \partial_t\omega - \mu\Delta\omega + u\cdot\nabla\omega + 4\chi\omega - \kappa\nabla\operatorname{div}\omega - 2\chi\nabla\times u = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}+, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}+, \\ (u,\omega)|_{t=0} = (u_0,\omega_0) & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $u = u(x,t), \pi = \pi(x,t)$ and $\omega = \omega(x,t)$ are unknown functions representing the linear velocity field, the pressure field of the fluid and the micro-rotation velocity field, respectively. κ,μ,ν and χ are positive constants reflecting various viscosity of the fluid. Throughout this paper we only consider the situation with $\kappa = \mu = 1$ and $\chi = \nu = 1/2$.

Theory of micropolar fluid was proposed by Eringen [8] in 1996, his idea allows us to consider several physical phenomena which cannot be treated by the classical Navier-Stokes system for the viscous incompressible fluid, then the problem (1) was presented as a necessary modification to the traditional Navier-Stokes equations in order to better characterize the motion of real-world fluids consisting of rigid but randomly oriented particles (such as blood) by examining the influence of micro-rotation of the particles suspended in the fluid.

There are several results on the weak and strong solvency of the micropolar fluid system and some related topics. The weak solution of (1) was firstly considered by Galdi and Rionero [11]. The existence theorem of the micropolar fuid system with sufficiently regular initial data has been showed by Lukaszewicz [15]. Inoue et al. [12] proved similar result for the magneto-micropolar fluid system. Many authors obtained the well-posedness of the problem (1) in various function spaces.

London Journal of Research in Science: Natural & Formal

For instance in the Besov spaces $\dot{B}_{p,q}^{-1+\frac{3}{p}}$ for $p \in [1,6)$ and $q = \infty$, Chen and Miao [5] obtained global well-posedness of the problem (1) for small initial data. Zhu and Zhao [21] proved that the Cauchy problem (1) is locally well-posed in the Fourier-Besov spaces $\mathcal{F}\dot{B}_{p,q}^{-1+\frac{3}{p}}$ for $1 < p \leq \infty$ and $1 \leq q < \infty$ and globally well posed in these spaces with small initial data. Recently, Weipeng Zhu [22] considered a critical case $p = 1$ and showed that this problem is locally well-posed in $\mathcal{F}\dot{B}_{1,q}^{-1}$ for $1 \leq q \leq 2$, and is globally well-posed in these spaces with small initial data. Also, Zhu proved the ill-posedness of (1) in $\mathcal{F}\dot{B}_{1,q}^{-1}$ for $2 < q \leq \infty$. In addition, by using a similar argument Zhu established the ill-posedness of (1) in Besov spaces $\dot{B}_{\infty,q}^{-1}$ with $2 < q \leq \infty$. The well-posedness of a more general model than (1) is established by Ferreira and Villamizar-Roa [9] in pseudo-measure spaces. For the other studies of the problem (1), we refer to the monographs [6, 16, 17, 20] .

We remark that if $\chi = 0$ and $\omega = 0$, then we have the classical Navier-Stokes equations:

$$\begin{cases} u_t - \mu\Delta u + (u.\nabla)u + \nabla p = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\ \nabla \cdot u = 0, & \\ u(0,x) = u_0(x) & x \in \mathbb{R}^3. \end{cases}$$

The local and global well-posedness of the classical Navier-Stokes equations have been established by a lot of researches in various function spaces, we refer to [13, 14] and references cited therein.

In the present paper, we show that the problem (1) is locally well-posed in Fourier-Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-1}$ for $1 \leq q \leq \infty$, and globally well-posed in these spaces with small initial data. Before stating the main result of this paper, we first recall the definitions of Morrey spaces, Besov-Morrey spaces and Fourier-Besov-Morrey spaces and present some properties about these spaces. Our results on well-posedness of solutions are stated in Section 3. In Section 4, we obtain the needed linear and nonlinear estimates and we prove the well-posedness result.

II. GENERAL NOTATION

Before stating our main result, we shall introduce the notations used throughout this paper.

We denote by C a positive constant such that whose value may change with each appearance, $x \lesssim y$ means that there exists a positive constant such that $x \leq Cy$, we write $(a,b) \in X$ for $a \in X$ and $b \in X$ and $\|\cdot\|_{E\cap F} = \|\cdot\|_E + \|\cdot\|_F$. The symbol $\mathcal{S}\left(\mathbb{R}^3\right)$ is the usual Schwartz space of infinitely differentiable rapidly decreasing complex-valued functions on $\mathbb{R}^3$.

By $\hat{\varphi}$ we denote the Fourier transform of $\varphi \in \mathcal{S}\left(\mathbb{R}^3\right)$ in the version

$$\hat{\varphi}(x) := \mathcal{F}\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix\xi}\varphi(\xi)d\xi, \quad x \in \mathbb{R}^3.$$

and we define its inverse Fourier transform by

$$\check{\varphi}(\xi) = \mathcal{F}^{-1}\varphi(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{ix\cdot\xi}\varphi(x)dx.$$

For two complex or extended real-valued measurable functions f, g on $\mathbb{R}^3$, the convolution $f * g$ is given by

$$(f*g)(x) := \int_{\mathbb{R}^3} f(x-y)g(y)dy, \quad for\ x \in \mathbb{R}^3.$$

III. PRELIMINARIES AND MAIN RESULTS

Let us introduce some basic knowledge on the Littlewood-Paley theory and Fourier-Besov-Morrey spaces.

Let φ, ψ be two radial positive functions such that $\mathrm{supp}(\varphi) \subset \left\{\xi \in \mathbb{R}^3 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$ and $\mathrm{supp}(\psi) \subset \left\{\xi \in \mathbb{R}^3 : |\xi| \leq \frac{4}{3}\right\}$
and

$$\sum_{j\in\mathbb{Z}} \varphi\left(2^{-j}\xi\right) = 1, \qquad \text{for all } \xi \neq 0$$

and

$$\psi(\xi) + \sum_{j\geq 0} \varphi(2^{-j\xi}) = 1 \text{ for all } \xi \in \mathbb{R}^3.$$

We denote

$$\varphi_j(\xi) = \varphi\left(2^{-j}\xi\right), \quad \psi_j(\xi) = \sum_{k\leq j-1} \varphi_k(\xi)$$

and

$$h(x) = \mathcal{F}^{-1}\varphi(x), \quad g(x) = \mathcal{F}^{-1}\psi(x).$$

We define the homogeneous dyadic blocks $\dot{\Delta}_j$ and $\dot{S}_j$ for all $u \in S'(\mathbb{R}^3)$ as follows:

$$\dot{\Delta}_j u := \mathcal{F}^{-1}\left(\varphi(2^{-j}\xi)\mathcal{F}(u)\right) = 2^{3j}\int_{\mathbb{R}^3} h\left(2^j y\right) u(x-y)dy,$$
$$\dot{S}_j u := \sum_{k\leq j-1} \Delta_k f = \mathcal{F}^{-1}\left(\psi(2^{-j}\xi)\mathcal{F}(u)\right) = 2^{3j}\int_{\mathbb{R}^3} g\left(2^j y\right) u(x-y)dy,$$

where $\dot{\Delta}_j = \dot{S}_j - \dot{S}_{j-1}$ is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$ and S_j is a frequency to the ball $\{|\xi| \lesssim 2^j\}$.

Then for any $u \in S'(\mathbb{R}^3)/\mathcal{P}(\mathbb{R}^3)$ wehre $\mathcal{P}(\mathbb{R}^3)$ is the set of polynomials (See. [19]) we have the Littlewood-Paley decomposition:

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \;\text{ and }\; \dot{S}_j u = \sum_{k\leq j-1} \dot{\Delta}_k u.$$

By using the definition of $\dot{\Delta}_j$ and $\dot{S}_j$, one easily obtains that

$$\dot{\Delta}_j\dot{\Delta}_k u = 0, \quad \text{if } |j-k| \geq 2$$
$$\dot{\Delta}_j\left(\dot{S}_{k-1} f \dot{\Delta}_k u\right) = 0, \quad \text{if } |j-k| \geq 5.$$

Now, we define the Morrey spaces $\mathrm{M}_p^\lambda\left(\mathbb{R}^3\right)$.

Definition 3.1. ([22]) For $1 \leq p < \infty, 0 \leq \lambda < 3$, the Morrey space $\mathrm{M}_p^\lambda = \mathrm{M}_p^\lambda\left(\mathbb{R}^3\right)$ is defined by $\mathrm{M}_p^\lambda\left(\mathbb{R}^3\right) = \left\{f \in L^p_{loc}\left(\mathbb{R}^3\right); \|f\|_{\mathrm{M}_p^\lambda} < \infty\right\}$, where

$$\|f\|_{\mathrm{M}_p^\lambda} = \sup_{x_0\in\mathbb{R}^3}\sup_{r>0} r^{-\frac{\lambda}{p}}\|f\|_{L^p(B(x_0,r))}.$$

Remark 3.2. ([10])

1) The space M_p^λ equipped with the norm $\|\cdot\|_{\mathrm{M}_p^\lambda}$ is a Banach space.

2) If $1\le p_1,p_2,p_3<\infty$, $0\le\lambda_1,\lambda_2,\lambda_3<3$ with $\frac{1}{p_3}=\frac{1}{p_1}+\frac{1}{p_2}$, and $\frac{\lambda_3}{p_3}=\frac{\lambda_1}{p_1}+\frac{\lambda_2}{p_2}$, then we have the Hölder inequality

$$\|fg\|_{\mathrm{M}_{p_3}^{\lambda_3}} \le \|f\|_{\mathrm{M}_{p_1}^{\lambda_1}}\|g\|_{\mathrm{M}_{p_2}^{\lambda_2}}.$$

3) For $1\le p<\infty$ and $0\le\lambda<3$,

$$\|\varphi * g\|_{\mathrm{M}_p^\lambda} \le \|\varphi\|_{L^1}\|g\|_{\mathrm{M}_p^\lambda}, \tag{2}$$

for all $\varphi\in L^1$ and $g\in\mathrm{M}_p^\lambda$.

Lemma 3.3. *([10]) Let* $1\le p_2\le p_1<\infty, 0\le\lambda_1,\lambda_2<3$, $\frac{3-\lambda_1}{p_1}\le\frac{3-\lambda_2}{p_2}$ *and let* γ *be a multi-index. If* $\sup p(\widehat{f})\subset\left\{|\xi|\le A2^j\right\}$, *then there is a constant* $C>0$ *independent of* f *and* j *such that*

$$\left\|(i\xi)^\gamma\widehat{f}\right\|_{\mathrm{M}_{p_2}^{\lambda_2}} \le C2^{j|\gamma|+j\left(\frac{3-\lambda_2}{p_2}-\frac{3-\lambda_1}{p_1}\right)}\|\widehat{f}\|_{\mathrm{M}_{p_1}^{\lambda_1}}. \tag{3}$$

Then, we define the function spaces $\dot{\mathcal{N}}^s_{p,\lambda,q}\left(\mathbb{R}^3\right)$.

Definition 3.4. ([7]) (Homogeneous Besov-Morrey spaces) Let $s\in\mathbb{R}, 1\le p<+\infty, 1\le q\le+\infty$, and $0\le\lambda<3$. The space $\dot{\mathcal{N}}^s_{p,\lambda,q}\left(\mathbb{R}^3\right)$ is defined by $\dot{\mathcal{N}}^s_{p,\lambda,q}\left(\mathbb{R}^3\right)=\Big\{u\in\mathcal{Z}'\left(\mathbb{R}^3\right); \quad \|u\|_{\dot{\mathcal{N}}^s_{p,\lambda,q}}\left(\mathbb{R}^3\right)<\infty\Big\}$, where

$$\|u\|_{\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^3)} = \begin{cases} \left\{\sum_{j\in\mathbb{Z}} 2^{jqs}\left\|\dot{\Delta}_j u\right\|^q_{\mathrm{M}_p^\lambda}\right\}^{1/q}, & \text{for } q<\infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\left\|\dot{\Delta}_j u\right\|_{\mathrm{M}_p^\lambda}, & \text{for } q=\infty, \end{cases}$$

with appropriate modifications made when $q=\infty$. The space $\mathcal{Z}'(\mathbb{R}^3)$ is the dual space of

$$\mathcal{Z}(\mathbb{R}^3)=\left\{f\in S(\mathbb{R}^3):(\partial^\beta\hat{f})(0)=0, \text{ for every multi-index } \beta\right\}.$$

Definition 3.5. ([7]) (Homogeneous Fourier-Besov-Morrey spaces) Let $s\in\mathbb{R}, 0\le\lambda<3, 1\le p<+\infty$, and $1\le q\le+\infty$. The space $\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}\left(\mathbb{R}^3\right)$ denotes the set of all $u\in\mathcal{Z}'\left(\mathbb{R}^3\right)$ such that

$$\|u\|_{\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^3)} = \left\{\sum_{j\in\mathbb{Z}} 2^{jqs}\left\|\widehat{\dot{\Delta}_j u}\right\|^q_{M_p^\lambda}\right\}^{1/q} < +\infty, \tag{4}$$

with appropriate modifications made when $q=\infty$.

Note that the space $\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}\left(\mathbb{R}^3\right)$ equipped with the norm (4) is a Banach space. Since $\mathrm{M}_p^0=L^p$, we have $\mathcal{F}\dot{\mathcal{N}}^s_{p,0,q}=\mathcal{F}\dot{B}^s_{p,q}$, $\mathcal{F}\dot{\mathcal{N}}^s_{1,0,q}=\mathcal{F}\dot{B}^s_{1,q}=\dot{B}^s_q$ and $\mathcal{F}\dot{\mathcal{N}}^{-1}_{1,0,1}=\chi^{-1}$, where $\dot{B}^s_q$ is the Fourier-Herz space, and χ^{-1} is the Lei-Lin space.

Now, we give the definition of the mixed space-time spaces.

Definition 3.6. ([7]) Let $s \in \mathbb{R}, 1 \leq p < \infty, 1 \leq q, \rho \leq \infty, 0 \leq \lambda < 3$ and $T \in (0, \infty]$. The space-time norm is defined on $u(t,x)$ by

$$\|u(t,x)\|_{\mathcal{L}^{\rho}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}\right)} = \left\{ \sum_{j \in \mathbb{Z}} 2^{jqs} \left\| \widehat{\dot{\Delta}_j u} \right\|^{q}_{L^{\rho}\left(0,T;\mathrm{M}^{\lambda}_{p}\right)} \right\}^{1/q}.$$

We denote by $\mathcal{L}^{\rho}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}\right)$ the set of distributions in $S'\left(\mathbb{R} \times \mathbb{R}^3\right)/\mathcal{P}(\mathbb{R}^3)$ with finite $\|\cdot\|_{\mathcal{L}^{\rho}(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q})}$ norm.

We will use the next lemma to prove our main theorem.

Lemma 3.7. *([22]) Let X be a Banach space, $\mathfrak{B}$ a continuous bilinear map from $X \times X$ to X, and ε a positive real number such that*

$$\varepsilon < \frac{1}{4\|\mathfrak{B}\|} \text{ with } \|\mathfrak{B}\| := \sup_{\|u\| \cdot \|v\| \leq 1} \|\mathfrak{B}(u,v)\|.$$

For any y in the ball $B(0,\varepsilon)$ (ie., with center 0 and radius ε) in X, then there exists a unique x in $B(0,2\varepsilon)$ such that

$$x = y + \mathfrak{B}(x,x).$$

Below, we shall present our main result that establishes the local and global existence.

Theorem 3.8. *Let $q \in [1, +\infty]$, $\alpha \in (0,1)$ and $0 < \lambda < 3$.*
λ 1

(1) For any initial data $(u_0, \omega_0) \in \mathcal{F}\dot{\mathcal{N}}^{-}_{1,\lambda,q}\left(\mathbb{R}^3\right)$ satisfying div $u_0 = 0$, there exists a positive T such that the system (1) has a unique mild solution such that

$$(u,\omega) \in \mathcal{L}^{\frac{2}{1+\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}\left(\mathbb{R}^3\right)\right) \cap \mathcal{L}^{\frac{2}{1-\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}\left(\mathbb{R}^3\right)\right).$$

(2) There exists a positive constant ε such that for any initial data $(u_0, \omega_0) \in \mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}\left(\mathbb{R}^3\right)$ satisfying div $u_0 = 0$ and

$$\|(u_0, \omega_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}(\mathbb{R}^3)} < \varepsilon,$$

the system (1) has a unique global mild solution such that

$$(u,\omega) \in \mathcal{L}^{\frac{2}{1+\alpha}}\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}\left(\mathbb{R}^3\right)\right) \cap \mathcal{L}^{\frac{2}{1-\alpha}}\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}\left(\mathbb{R}^3\right)\right).$$

Before proving our main result we will present the corresponding linear system of the nonlinear system (1).

$$\begin{cases} \partial_t u - \Delta u - \nabla \times \omega = 0 \\ \partial_t \omega - \Delta \omega + 2\omega - \nabla \operatorname{div} \omega - \nabla \times u = 0, \\ \operatorname{div} u = 0, \\ (u,\omega)|_{t=0} = (u_0, \omega_0). \end{cases} \tag{5}$$

The solution operator of the above problem is denoted by the notation $G(t)$, i.e., for specified initial data (u_0, ω_0) in suitable function space, $(u,\omega)^T = G(t)\left(u_0, \omega_0\right)^T$ is the unique solution of the problem (5). The operator $G(t)$ has the following expression, as shown by a simple calculation:

$$\widehat{(G(t)f)}(\xi) = e^{-\mathcal{A}(\xi)t}\hat{f}(\xi) \quad \text{for } f(x) = (f_1(x), f_2(x))^T,$$

where

$$\mathcal{A}(\xi) = \begin{bmatrix} |\xi|^2 I & \mathcal{B}(\xi) \\ \mathcal{B}(\xi) & (|\xi|^2+2)I + \mathcal{C}(\xi) \end{bmatrix},$$

with

$$\mathcal{B}(\xi) = i\begin{bmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{bmatrix} \text{ and } \mathcal{C}(\xi) = \begin{bmatrix} {\xi_1}^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & {\xi_2}^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & {\xi_3}^2 \end{bmatrix}.$$

On the other hand, by applying the Leray projection $\mathbf{P}$ to both sides of the first equations of (1), one can eliminate the pressure π and one check

$$\begin{cases} \partial_t u - \Delta u + \mathbf{P}(u\cdot\nabla u) - \nabla\times\omega = 0 \\ \partial_t\omega - \Delta\omega + u\cdot\nabla\omega + 2\omega - \nabla\operatorname{div}\omega - \nabla\times u = 0 \\ \operatorname{div} u = 0 \\ (u,\omega)|_{t=0} = (u_0,\omega_0), \end{cases} \tag{6}$$

where $\mathbf{P} = T + \nabla(-\Delta)^{-1}$ div is the 3×3 matrix pseudo-differential operator in $\mathbb{R}^3$ with the symbol $\left(\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2}\right)_{i,j=1}^3$. We denote

$$U(x,t) = \begin{pmatrix} u(x,t) \\ \omega(x,t) \end{pmatrix}, \quad U_0 = \begin{pmatrix} u(x,0) \\ \omega(x,0) \end{pmatrix} = \begin{pmatrix} u_0 \\ \omega_0 \end{pmatrix}, \quad U_i(x,t) = \begin{pmatrix} u_i(x,t) \\ \omega_i(x,t) \end{pmatrix}, \quad i = 1,2$$

and

$$U_1\tilde{\otimes}U_2 = \begin{pmatrix} u_1\otimes u_2 \\ u_1\otimes\omega_2 \end{pmatrix}, \quad \widetilde{\mathbf{P}}\nabla\cdot(U_1\tilde{\otimes}U_2) = \begin{pmatrix} \mathbf{P}\nabla\cdot(u_1\otimes u_2) \\ \nabla\cdot(u_1\otimes\omega_2) \end{pmatrix}.$$

To solve system (6) it suffices to find the solution U of the following integral equations:

$$U(t) = G(t)U_0 - \int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U\otimes U)(\tau)d\tau. \tag{7}$$

A solution of (7) is called a mild solution of (1).

IV. PROOF OF MAIN RESULT

In this section, we will establish the local and global existence and uniqueness of solution for the problem (1). For that, we prove some estimates for the semigroup $G(\cdot)$.

4.1 Linear estimates

Firstly we give the property of semigroup $G(\cdot)$.

Lemma 4.1. *[9] For $t\geq 0$ and $|\xi|\neq 0$. We have*

$$\|e^{-t\mathcal{A}(\xi)}\| \leq e^{-|\xi|^2 t} \text{ with } \left\|e^{-t\mathcal{A}(\xi)}\right\| = \sup_{\|f\|\leq 1}\left\|e^{-t\mathcal{A}(\xi)}f\right\|. \tag{8}$$

London Journal of Research in Science: Natural & Formal

Here $\|f\| = \max_i |a_i|$ *with* $\|f\| = \sum_{i=1}^{6} a_i v_i, v_1, v_2, \ldots, v_6$ *are the eigenvectors for* $\mathcal{A}(\xi)$.

Next, we present the linear esimates for the semigroup $G(\cdot)$.

Lemma 4.2. *Let* $q \in [1, +\infty]$, $0 < \lambda < 3$. *Then there exists a positive constant* C *such that*

$$\|G(t)U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}} \leq C\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}} \tag{9}$$

for all $t \geq 0$ *and all* $U_0 \in \mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}$.

Proof. By Lemma 4.1, we have

$$\begin{aligned}
\|G(t)U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}} &= \sum_{i\in Z} 2^{(\lambda-1)jq} \left\|\mathcal{F}\left[G(t)\dot{\Delta}_j U_0\right]\right\|^q_{M^\lambda_1}\Big)^{\frac{1}{q}} \\
&= \sum_{\in Z} 2^{(\lambda-1)jq} \left\|e^{-t\mathcal{A}(\xi)}\mathcal{F}\left[\dot{\Delta}_j U_0\right]\right\|^q_{M^\lambda_1}\Big)^{\frac{1}{q}} \\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{(\lambda-1)q} e^{-|\xi|^2 qt} \left\|\mathcal{F}\left[\dot{\Delta}_j U_0\right]\right\|^q_{M^\lambda_1}\Big)^{\frac{1}{q}} \\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{(\lambda-1)q} \left\|\mathcal{F}\left[\dot{\Delta}_j U_0\right]\right\|^q_{M^\lambda_1}\Big)^{\frac{1}{q}} \\
&\lesssim \|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}}.
\end{aligned}$$

This completes the proof of Lemma 4.2 . □

Lemma 4.3. *Let* $q \in [1, +\infty]$, $0 < \lambda < 3$, $\alpha \in (0;1)$ *and* $T \in (0, \infty]$. *Then there exists a positive constant* $C = C(\alpha)$ *such that*

$$\|G(t)U_0\|_{L^{\frac{2}{1\pm\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda\pm\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right)} \leq C\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}}, \tag{10}$$

for all $t \geq 0$ *and all* $U_0 \in \mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}$.

Proof. From Definition 3.6, it is easy to see that

$$\begin{aligned}
\|G(t)U_0\||_{L^{\frac{2}{1\pm\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda\pm\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right)} &= \sum_{j\in\mathbb{Z}} 2^{(\lambda\pm\alpha)jq} \left\|\mathcal{F}\left[G(t)\dot{\Delta}_j U_0\right]\right\|^q_{L^{\frac{2}{1\pm\alpha}}(0,T;M^\lambda_1)}\Big)^{\frac{1}{q}} \\
&= \sum_{j\in\mathbb{Z}} 2^{(\lambda\pm\alpha)jq} \left\|e^{-t\mathcal{A}(\xi)}\mathcal{F}\left[\dot{\Delta}_j U_0\right]\right\|^q_{L^{\frac{2}{1\pm\alpha}}(0,T;M^\lambda_1)}\Big)^{\frac{1}{q}} \\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{(\lambda\pm\alpha)jq} \left\| e^{-t2^{2j}} \left\|\mathcal{F}\left[\dot{\Delta}_j U_0\right]\right\|_{M^\lambda_1}\right\|^q_{L^{\frac{2}{1\pm\alpha}}(0,T)}\Big)^{\frac{1}{q}} \qquad \square \\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{(\lambda\pm\alpha)jq} 2^{-(1\pm\alpha)jq} \left\|\mathcal{F}\left[\dot{\Delta}_j U_0\right]\right\|^q_{M^\lambda_1}\Big)^{\frac{1}{q}} \\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{(\lambda-1)jq} \left\|\mathcal{F}\left[\dot{\Delta}_j U_0\right]\right\|^q_{M^\lambda_1}\Big)^{\frac{1}{q}} \\
&\lesssim \|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}}.
\end{aligned}$$

This completes the proof of Lemma 4.3 .

4.2 Bilinear estimates and product laws

Lemma 4.4. *Let $T > 0$, $s \in \mathbb{R}$ and $p,q,\gamma \in [1;+\infty]$ and $0 < \lambda < 3$. Then there exists a positive constant C such that*

$$\left\|\int_0^t G(t-\tau)f(\tau)d\tau\right\|_{\mathcal{L}^\gamma(0;T;\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})} \leq C\|f\|_{\mathcal{L}^1(0;T;\mathcal{F}\dot{\mathcal{N}}^{s-\frac{2}{\gamma}}_{p,\lambda,q})}$$

for all $f \in \mathcal{L}^1(0;T;\mathcal{F}\dot{\mathcal{N}}^{s-\frac{2}{\gamma}}_{p,\lambda,q})$.

Proof. By Young's inequality, we obtain

$$\begin{aligned}
&\left\|\int_0^t G(t-\tau)f(\tau)d\tau\right\|_{\mathcal{L}^\gamma(0;T;\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})} \\
&= \sum_{j\in\mathbb{Z}} 2^{jqs}\left\|\int_0^t e^{-(t-\tau)A(\xi)}\mathcal{F}\left[\dot{\Delta}_j f\right](\tau)d\tau\right\|^q_{L^\gamma\left(0,T;\mathrm{M}^\lambda_p\right)}\Bigg)^{\frac{1}{q}} \\
&\leq C\ \sum_{j\in\mathbb{Z}} 2^{jqs}\left\|\int_0^t e^{-(t-\tau)2^{2j}}\left\|\mathcal{F}\left[\dot{\Delta}_j f\right](\tau)\right\|_{\mathrm{M}^\lambda_p} d\tau\right\|^q_{L^\gamma(0,T)}\Bigg)^{\frac{1}{q}} \\
&\leq C\ \sum_{j\in\mathbb{Z}} 2^{jq\left(s-\frac{2}{\gamma}\right)}\left\|\mathcal{F}\left[\dot{\Delta}_j f\right](\tau)\right\|^q_{L^1\left(0,T;\mathrm{M}^\lambda_p\right)}\Bigg)^{\frac{1}{q}} \\
&\leq C\|f\|_{\mathcal{L}^1\ \ 0,T;\mathcal{F}\dot{\mathcal{N}}^{s-\frac{2}{\gamma}}_{p,\lambda,q}\big)}.
\end{aligned}$$

Which finish the proof. ∎

In the framework of homogeneous Fourier-Besov-Morrey spaces, we now gather an essential multiplication estimates.

Lemma 4.5. *Let $p,q \in [1.+\infty]$, $0 < \lambda < 3$, $T \in (0,+\infty]$ and $\alpha \in (0,1)$. Then there exists a positive constant C such that*

$$\begin{aligned}
\|uv\|_{\mathcal{L}^1(0,T;\mathcal{F}\dot{\mathcal{N}}^\lambda_{1,\lambda,q})} \lesssim &\| u \|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q})}\| v \|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q})} \\
&+ \| v \|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q})}\| u \|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q})}.
\end{aligned}$$

Proof. We introduce some notations about the standard localization operators. We set

$$u_j = \dot{\Delta}_j u,\quad \dot{S}_j u = \sum_{k\leq j-1}\dot{\Delta}_k u,\quad \tilde{\dot{\Delta}}_j u = \sum_{|k-j|\leq 1}\dot{\Delta}_k u,\quad j \in \mathbb{Z}.$$

Bony's decomposition for $\dot{\Delta}_j(uv)$ reads

$$\begin{aligned}
\dot{\Delta}_j(uv) = &\sum_{|k-j|\leq 4}\dot{\Delta}_j\left(\dot{S}_{k-1}u\dot{\Delta}_k v\right) + \sum_{|k-j|\leq 4}\dot{\Delta}_j\left(\dot{S}_{k-1}v\dot{\Delta}_k u\right) + \sum_{k\geq j-3}\dot{\Delta}_j(\dot{\Delta}_k u\tilde{\dot{\Delta}}_j v) \\
:= &I_1 + I_2 + I_3.
\end{aligned}$$

Then by the triangle inequalities, we have

$$\|uv\|_{L^1\left(0,T:\mathcal{F}\dot{\mathcal{N}}^{\lambda}_{1,\lambda,q}\right)} \leq \left(\sum_{j\in\mathbb{Z}} 2^{j\lambda q}\left\|\widehat{I_1}\right\|^q_{L^1\left(0,T:\mathrm{M}^{\lambda}_1\right)}\right)^{\frac{1}{q}} + \left(\sum_{j\in\mathbb{Z}} 2^{j\lambda q}\left\|\widehat{I_2}\right\|^q_{L^1\left(0,T:\mathrm{M}^{\lambda}_1\right)}\right)^{\frac{1}{q}} \tag{11}$$
$$+ \left(\sum_{j\in\mathbb{Z}} 2^{j\lambda q}\left\|\widehat{I_3}\right\|^q_{L^1\left(0,T:\mathrm{M}^{\lambda}_1\right)}\right)^{\frac{1}{q}}$$
$$:= J_1 + J_2 + J_3.$$

The terms I_1 and I_2 are symmetrical. Using Young's inequality and Hölder's inequality we have

$$\begin{aligned}
2^{j\lambda}\left\|\widehat{I_1}\right\|_{L^1\left(0,T,M^{\lambda}_1\right)} &\leq 2^{j\lambda}\sum_{|k-j|\leq 4}\left\|\left(\widehat{\dot{S}_{k-1}u\dot{\Delta}_k v}\right)\right\|_{L^1\left(0,T,M^{\lambda}_1\right)}\\
&\leq 2^{j\lambda}\sum_{|k-j|\leq 4}\|\widehat{v_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)}\sum_{l\leq k-2}\|\widehat{u_l}\|_{L^{\frac{2}{1-\alpha}}\left(0,T,L^1\right)}\\
&\leq 2^{j\lambda}\sum_{|k-j|\leq 4}\|\widehat{v_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)}\sum_{l\leq k-2}2^{\lambda l}\|\widehat{u_l}\|_{L^{\frac{2}{1-\alpha}}\left(0,T,M^{\lambda}_1\right)}
\end{aligned}$$

$$\begin{aligned}
&\leq 2^{j\lambda}\sum_{|k-j|\leq 4}\|\widehat{v_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)}\left(\sum_{l\leq k-2}2^{(\lambda-\alpha)lq}\|\widehat{u_l}\|_{L^{\frac{2}{1-\alpha}}\left(0,T,M^{\lambda}_1\right)}\right)^{\frac{1}{q}}\left(\sum_{l\leq k-2}2^{l\alpha q'}\right)^{\frac{1}{q'}}\\
&\leq 2^{j\lambda}\sum_{|k-j|\leq 4}2^{\alpha k}\|\widehat{v_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)}\|u\|_{L^{\frac{2}{1-\alpha}}\left(0;T;\mathcal{F}\dot{\mathcal{N}}^{-\alpha+\lambda}_{1,\lambda,q}\right)}\\
&\leq \sum_{|k-j|\leq 4}2^{(\alpha+\lambda)k}2^{(j-k)\lambda}\|\widehat{v_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)}\|u\|_{L^{\frac{2}{1-\alpha}}\left(0;T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}\right)}.
\end{aligned}$$

Taking ℓ^q-norm we get

$$J_1 \leq \| v \|_{L^{\frac{2}{1+\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}\right)}\| u \|_{L^{\frac{2}{1-\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}\right)},$$

and in a similar way we obtain

$$J_2 \leq \| u \|_{L^{\frac{2}{1+\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}\right)}\| v \|_{L^{\frac{2}{1-\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}\right)}.$$

To estimate I_3 , let

$$I^k_3 = \dot{\Delta}_j\Big(\sum_{|k'-k|}\dot{\Delta}_{k'}v\dot{\Delta}_k u\Big) = \sum_{k'=-1}^{1}\dot{\Delta}_k u\dot{\Delta}_{k+k'}v.$$

First we use Young's inequality (2) in Morrey spaces, and Lemma 3.3 with $|\gamma|= 0$, to obtain

$$\begin{aligned}
2^{j\lambda}\left\|\widehat{I_3}\right\|_{L^1\left(0,T,M^{\lambda}_1\right)} &\leq 2^{j\lambda}\sum_{k\geq j-3}\left\|\widehat{I^k_3}\right\|_{L^1\left(0,T,M^{\lambda}_1\right)}\\
&\leq 2^{j\lambda}\sum_{k\geq j-3}\sum_{|k'-k|\leq 1}\|\widehat{u_k}\|_{L^{\frac{2}{1-\alpha}}\left(0,T,M^{\lambda}_1\right)}\|\widehat{v_{k'}}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,L^1\right)}\\
&\leq 2^{j\lambda}\sum_{k\geq j-3}\sum_{|k'-k|\leq 1}\|\widehat{u_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)}2^{\lambda k'}\|\widehat{v_{k'}}\|_{L^{\frac{2}{1-\alpha}}\left(0,T,M^{\lambda}_1\right)} \qquad (12)\\
&\leq 2^{j\lambda}\sum_{k\geq j-3}\left(\sum_{|k'-k|\leq 1}2^{k'(\lambda-\alpha)q}\|\widehat{v_{k'}}\|_{L^{\frac{2}{1-\alpha}}\left(0,T,M^{\lambda}_1\right)}\right)^{\frac{1}{q}}2^{\alpha k}\|\widehat{u_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)}\\
&\leq C\|v\|_{L^{\frac{2}{1-\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}\right)}\sum_{k\geq j-3}2^{(\lambda+\alpha)k}2^{\lambda(j-k)}\|\widehat{u_k}\|_{L^{\frac{2}{1+\alpha}}\left(0,T,M^{\lambda}_1\right)},
\end{aligned}$$

taking ℓ^q-norm on both sides in the above estimate, we get

$$J_3 \leq \| v \|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q})} \| u \|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q})}.$$

Which gives the result. □

Below, we shall prove our result.

4.3 Proof of Theorem 3.8

1) Let $T > 0$ and $\alpha \in (0;1)$ we define the space X^{α}_T as

$$X^{\alpha}_T = \left\{ U : U \in \mathcal{L}^{\frac{2}{1+\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right) \cap \mathcal{L}^{\frac{2}{1-\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right)\right\}$$

and equipped with the following norm:

$$\| U \|_{X^{\alpha}_T} = \| U \|_{\mathcal{L}^{\frac{2}{1+\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right)} + \| U \|_{\mathcal{L}^{\frac{2}{1-\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right)}.$$

For all $U \in X^{\alpha}_T$ we define $\phi(U)$ as follows

$$\phi(U) = G(t)U_0 - \int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U\widetilde{\otimes}U)(\tau)d\tau. \tag{13}$$

Our goal is to show that U is a fixed point of ϕ.
Considering

$$B(U_1,U_2) = \int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U_1\widetilde{\otimes}U_2)(\tau)d\tau. \tag{14}$$

Then, by Lemma 4.4, Lemma 4.5 and the embedding $\mathcal{F}\dot{\mathcal{N}}^{\lambda}_{1,\lambda,q} \hookrightarrow \mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}$, we have

$$\begin{aligned}
\| B(U_1,U_2) \|_{X^{\alpha}_T} &= \| \int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U_1\widetilde{\otimes}U_2)(\tau)d\tau\|_{\mathcal{L}^{\frac{2}{1+\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right)} \\
&\quad + \| \int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U_1\widetilde{\otimes}U_2)(\tau)d\tau\|_{\mathcal{L}^{\frac{2}{1-\alpha}}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}(\mathbb{R}^3)\right)} \\
&\lesssim \|\widetilde{\mathbf{P}}\nabla\cdot(U_1\widetilde{\otimes}U_2)\|_{\mathcal{L}^1(0;T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q})} \\
&\lesssim \|\widetilde{\mathbf{P}}\nabla\cdot(U_1\widetilde{\otimes}U_2)\|_{\mathcal{L}^1(0;T;\mathcal{F}\dot{\mathcal{N}}^{\lambda}_{1,\lambda,q})} \\
&\lesssim \| U_1 \|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q})} \| U_2 \|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q})} + \| U_2 \|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q})} \| U_1 \|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q})} \\
&\leq C_1\|U_1\|_{X^{\alpha}_T}\|U_2\|_{X^{\alpha}_T}.
\end{aligned} \tag{15}$$

Then, by (13) and (15), one concludes

$$\|\phi(U)\|_{X^{\alpha}_T} \leq \|G(t)U_0\|_{X^{\alpha}_T} + C_1\|U_1\|_{X^{\alpha}_T}\|U_2\|_{X^{\alpha}_T}.$$

By Lemma 4.3 , we get

$$\|G(t)U_0\|_{X^{\alpha}_T} \leq C_2 \|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}}. \tag{16}$$

Now, consider the norm $\|G(t)U_0\|_{X_T^\alpha}$. Using the given expression for $G(t)$ and the definition of X_T^α, we can write:

$$\|G(t)U_0\|_{X_T^\alpha} = \|G(t)U_0\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}(\mathbb{R}^3))} + \|G(t)U_0\|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda-\alpha}_{1,\lambda,q}(\mathbb{R}^3))}$$

Now, let's analyze each term separately. First, note that $G(t)$ is a linear operator, so we can factor out U_0 from the norm:

$$\|G(t)U_0\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}(\mathbb{R}^3))} \leq \|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda+\alpha}_{1,\lambda,q}(\mathbb{R}^3))} \|G(t)\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T)}.$$

Similarly for the second term. Now, for sufficiently small T, $e^{-\mathcal{A}(\xi)t}$ tends to I (the identity operator) as t approaches 0. Therefore, for small T, $\|G(t)\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T)}$ tends to 0.

Then $\|G(t)U_0\|_{X_T^\alpha} \to 0$ as $T \to 0$, hence $\alpha \neq 1$, and there exists $T > 0$ such that $\|G(t)U_0\|_{X_T^\alpha} < \frac{1}{4C_1}$. Using Lemma 3.7, system (1) admits a unique mild solution $U \in X_T^\alpha$ with $\|U\|_{X_T^\alpha} < \frac{1}{2C_1}$.

For **2)** we replace X_T^α by X_∞^α and we get

$$\| B(U_1,U_2) \|_{X_\infty^\alpha} \leq C_1 \|U_1\|_{X_\infty^\alpha} \|U_2\|_{X_\infty^\alpha}. \quad (17)$$

Then, by (16) and (17), one obtaines

$$\|\phi(U)\|_{X_\infty^\alpha} \leq C_2 \|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}} + C_1 \|U_1\|_{X_\infty^\alpha} \|U_2\|_{X_\infty^\alpha}.$$

Then, by applying Lemma 3.7 , with $\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{\lambda-1}_{1,\lambda,q}} < \frac{1}{4C_1C_2}$. Then, system (1) admits a unique global mild solution $U \in X_\infty^\alpha$ with $\|U\|_{X_\infty^\alpha} < \frac{1}{2C_1}$. This completes the proof of Theorem3.8 (2).

REFERENCES

1. A. Azanzal, A. Abbassi, C. Allalou, On the Cauchy problem for the fractional drift-diffusion system in critical Fourier-Besov-Morrey spaces. International Journal On Optimization and Applications, 1(2021), p-28.
2. A. Azanzal, A. Abbassi, C. Allalou, Existence of Solutions for the Debye-Hückel System with Low Regularity Initial Data in Critical Fourier-Besov-Morrey Spaces. Nonlinear Dynamics and Systems Theory, 21, 367-380 (2021).
3. A. Azanzal, A. Abbassi, C. Allalou, S. Melliani, Well-posedness and blow-up of solutions for the 2D dissipative quasi-geostrophic equation in critical Fourier-Besov-Morrey spaces. J Elliptic Parabol Equ. https://doi.org/10.1007/s41808-021-00140-x (2021).
4. A. Azanzal, C. Allalou, A. Abbassi, Well-posedness and analyticity for generalized Navier-Stokes equations in critical Fourier-Besov-Morrey spaces. J. Nonlinear Funct. Anal. 2021(2021), Article ID 24.
5. Q. Chen, C. Miao, Global well-posedness for the micropolar fluid system in critical Besov spaces, J. Differential Equations 252 (3) (2012) 2698-2724.
6. B. Dong, J. Li, J.Wu, Global well-posedness and large-time decay for the 2D micropolar equations, J. Differential Equations 262 (2017) 3488-3523
7. A. El Baraka, M. Toumlilin, Uniform well-Posedness and stability for fractional Navier-Stokes Equations with Coriolis force in critical Fourier-Besov-Morrey Spaces, Open J. Math. Anal. 2019, 3(1), 70-89.
8. A.C. Eringen, Theory of micropolar fluids, J. Math. Mech. 16 (1966) 1-18.
9. L.C. Ferreira, E.J. Villamizar-Roa, Micropolar fluid system in a space of distributions and large time behavior, J. Math. Anal. Appl. 332 (2) (2007) 1425-1445.

10. L. C. Ferreira, L. S. Lima; Self-similar solutions for active scalar equations in Fourier-BesovMorrey spaces, Monatsh. Math., 175.4 (2014), 491-509.
11. GP. Galdi, S. Rionero, A note on the existence and uniqueness of solutions of the micropolar fuid equations. International Journal of Engineering Science 1977; 15:105-108.
12. H. Inoue, Matsuura M. Ôtani, Strong solutions of magneto-micropolar fuid equation. Discrete and Continuous Dynamical Systems 2003; 439-448; Dynamical systems and diferential equations (Wilmington, NC, 2002).
13. Z. Lei, F. Lin, Global mild solutions of Navier-Stokes equations, Comm. Pure Appl. Math., 64.9 (2011), 1297-1304.
14. P.G. LemariÃ©-Rieusset, The role of Morrey spaces in the study of Navier-Stokes and Euler equations. Eurasian Math. J. 3, 62-93 (2012).
15. G. Lukaszewicz On nonstationary flows of asymmetric fuids. Rendiconti Accademia Nazionale delle Scienze della dei XL Serie V. Memorie di Matematica 1988; 12:83-97.
16. F. Ouidirne, H. Srhiri, C. Allalou, and M. Oukessou, Global existence for the 3-D generalized micropolar fluid system in critical Fourier-Besov spaces with variable exponents. Nonlinear J. Dynamics and Systems Theory. 23(3):338-347
17. H. Srhiri, F. Ouidirne, C. Allalou, and K. Hilal, Well-posedness and stability of solutions for the 3-D generalized micropolar system in Fourier-BesovMorrey spaces. Journal of Elliptic and Parabolic Equations, 1-31, (2023).
18. J. Sun, S. Cui, Sharp well-posedness and ill-posedness in Fourier-Besov spaces for the viscous primitive equations of geophysics, arXiv:1510.0713v1, (2015).
19. H. Triebel, Theory of Function Spaces, in: Monogr. Math., vol. 78, Birkhäauser Verlag, Basel, 1983.
20. E.J. Villamizar-Roa, M.A. Rodríguez-Bellido, Global existence and exponential stability for the micropolar fluid system, Z. Angew. Math. Phys. 59 (5) (2008) 790-809.
21. W. Zhu, J. Zhao, Existence and regularizing rate estimates of solutions to the 3-D generalized micropolar system in Fourier- Besov spaces. Math. Methods Appl. Sci. 41 (4) (2018) 1703- 1722.
22. W. Zhu, Sharp well-posedness and ill-posedness for the 3-D micropolar fluid system in FourierBesov spaces. Nonlinear Analysis: Real World Applications 46 (2019) 335-351.