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This article established a fact on the order of difference of integral powers of all sets of natural numbers. The analysis was proof by use of established property of difference operator and principle of mathematical induction. The result proved conclusively that "if the elements of an arithmetic progression of set of natural numbers with positive common difference are raised to positive power k, then the k^{th} difference is equal to the product of the common difference raised to power k (d^K) and k factorial (k!).

Keywords: finite difference, integral order, positive powers, mathematical induction, arithmetic progression.

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I. INTRODUCTION

A set of positive integers is a set of natural numbers. A set is a collection of well-defined elements. A set of natural number is a sequence, defined with respect to a constant value called a common difference (d). The terms of a sequence are defined with respect to three parameters, the first term (a), the numbers of terms in the sequence (n) and the common difference (d). The sequence of natural numbers expressed mathematically as:

$$T_n = a + (n - 1)\, d \text{ .... I}$$

For all a ≥ 1, d > 1, n ϵ (o, ∞)

Any sequence generated with the above formular is called an ***arithmetic progression.***

A perfect power is a rational root chase.[1] An integral perfect power is the irrational root that is an integer. A finite difference is a mathematical expression of the form, f(x + b) − f(x + a) and a forward difference is of the form $\Delta h[f](x) = f(x + h) = f(x)$[2]

This article investigates all Arithmetic Progression of set of natural numbers with positive common difference. Further investigation reveals that there is a significant relationship between the difference of powers of set of natural numbers and factorial, which is represented symbolically as:

$$\Delta^k\ [a + (n-1)d]^k = \ d^k K! \text{ .... II}$$

For all a ≥ 1, d ≥ 1, k > 1 and

For some $\cap \epsilon (o, \infty)$

The implication of equation (II) is on the coefficient of the kth factorial, which sows the value of k is the power of the common difference and at the same time is the power of the sequence under consideration for which the k^{th} difference of the set is equal to the product of the common difference raised to power k and k!.

An examination of any set of natural numbers having common difference greater than 2 e.g. {2, 5, 8, 11, 14, 17, 20, 23, 26, 29} reveals that the pattern is sustained (See Table I and Appendix A).

The set of perfect squares of the set is {4, 25, 64, 121, 196, 289, 400, 529, 676, 841}. The actual set of the first difference is {21, 39, 57, 75, 93, 111, 129, 147, 165}. Furthermore, the set of the second difference is {18, 18, 18, 18, 18, 18, 18, 18} = {18}, a singleton set. Clearly, the set of differences of the above second difference set is {0}. The emerging pattern motives the investigation of whether the pattern will persist for all set of natural numbers with positive common difference d > 2. The set of the second difference of the set of perfect squares clearly shows that the element of the singleton set $18 = 3^2(2!)$.

This shows that for d = 3, k = 2.

$$\Delta^2 [a + (n - 1)d)]^2 = d^k . K!, \text{ where}$$

$$a = 2, d = 3, k = 2, \text{ and } n \epsilon (0,9)$$

$$\Rightarrow \Delta^2 [a + (n - 1)d)]^2 = 18 = 3^2(2!)$$

In their contribution in Exponential Diophantine Equations, Shorey and Tijdeman investigated perfect powers at integral values of a polynomial with rational integer coefficients and obtain in particular the following. Let f(x) be a polynomial with rational coefficients and with at least two simple rational zeros.

Suppose $b \neq 0$, $m \geq 0$, x and y with /Y/ > 1 are rational integers. Then the equation $f(x) = by^m$ implies that m is bounded by a computable number depending only on b and f. Also in their contribution Ladan, Tanko, Aliyu, Ahmad and Kabiru[4], they investigated integral powers of polynomials with Binomial coefficients. The result of their investigation shows that "the disposition of powers of polynomials with binomial coefficients generates even positive factorial (2k)!

Also in a research work by Ladan, Aliyu, Tanko, Ahmad and Kabiru[5], on the location of points on the plane and the order of disposition of sum of powers of cardinal coordinates. The result of their work proved conclusively that "the sum of the powers of cardinal points is equal to the coefficients of the Binomial expansion with respect to the Pascal triangle pattern and entries". In their contribution in the difference of perfect powers of integers, Ladan, Ukwu and Apine[6] investigated order of integral perfect powers and proved that "if any number of consecutive integers are raised to a positive integral power k, then the k^{th} difference is equal to k! Based on the generalization of the theorem in this article, it shows that Ladan, Ukwu and Apine work has common difference d = 1, for all k > 1.

Similarly, in a research conducted by Ladan, Emmanuel and Tanko[7], investigated order of product of perfect powers and proved conclusively that if any number of consecutive integer are raised to a positive power k, then the $(2k)^{th}$ difference of the product of the k^{th} power of two consecutive integers is equal to (2k)!. This established a relation between difference of powers of natural numbers and factorial. For more definitions, see [8, 9, 10 ……].

There was no discussion on the relationship between the difference of powers of set of all natural numbers and factorial with respect to the common difference which is pertinent to this article. Review of literature shows that no such investigation has been undertaken. Thus, this article adds to the existing body of knowledge.

II. METHODS

2.1 Preliminary Definitions

In what follows, the difference of finite order will be defined.

2.1.1 Differences of Order One (1)

Given a sequence $\{f_j\}_1^\infty$, defined the difference of order one at j with respect to the sequence by:

$\Delta(f_j) = f_{j+1} - f_j$, for every j

2.1.2 Higher Order Differences

Higher order differences can be defined recursively by:

$\Delta^k(f_j) = \Delta(\Delta^{k-1}(f_j)) = \Delta^{k-1}(\Delta(f_j)) = \Delta^{k-1}[f_{j+1} - f_j]$ for ≥ 2

III. RESULTS AND DISCUSSION

3.1 Preliminary Theorem

Suppose $f_j = j$ for all j belong to set of natural numbers.

Then:

(i) Δf_j^2 is natural number

(ii) $\Delta^2(f_j^2) = 18 = 3^2.(2!)$

(iii) $\Delta^k(f_j^2) = 0,\ k \in \{3, 4 \ldots\}$

Proof

Let j = {2, 5, 8, 11, 14, 17, 20, 23, 26, 29}

Then Table I below yields value of $\Delta^k(f_j^p)$ for selected values of k and p in the set {0, 1, 2, 3} and {1, 2} respectively, when $\Delta^0(f_j^p) = f_j^p$

Table I: Difference Order Table for Selected Elements of Set of Natural Numbers

f_j	f_j^2	$\Delta(f_j^2)$	$\Delta^2(f_j^2)$	$\Delta^3(f_j^2)$
2	4	21	18	0
5	25	39	18	0
8	64	57	18	0
11	121	75	18	0
14	196	93	18	0
17	289	111	18	0
20	400	129	18	0
23	529	147	18	0
26	676	165	18	
29	841			

It is clear that $\Delta^k(f_j^k)$ = d^kK! Where d = 3, k = 2, and j ϵ {2, 5, . . . 26, 29}.

That is $\Delta^2(f_j^2) = 3^2(2!) = 18$

Obviously, the theorem is valid for j ϵ {2, 5, 8, 11, 14, 17, 20, 23, 26, 29} as observed from column 4. The proofs of (i), (ii) and (iii) are direct.

(i) $\Delta(f_j^2) = f_{j+1}^2 - f_j^2 = \{j + 1)^2 - j2 = 2j + 1$, which is natural number for all j, proving (i).

(ii) $\Delta(\Delta f_j^2) = 3^2\,\Delta(2j + 1) = 3^2(\Delta 2j + \Delta 1) = 3^2(2\Delta j) = 3^2[2(j + 1 - j)] = 3^2 2[1] = 3^2(2)$

$= 3^2.2 = 18$, for d = 3, proving (ii).

The principle of mathematical induction is needed in the proof of (iii).

For k = 3, $\Delta^3(f_j^2) = \Delta(\Delta^2(f_j^2)) = \Delta\,(18) = 18 - 18 = 0$

Consequently $\Delta^k(f_j^2) = \Delta^{k-3}(\Delta^3(f_j^2)) = \Delta^{k-3}(0) = 0$

For all k ≥ 4, proving $\Delta^k(f_j^2) = 0$, for all positive integer k ≥ 3. This established the proof of (iii).

3.2 Main Theorem

Let $f_j = j$

Let $\{j\}_1^\infty$ be a sequence defined by $j_n = a + (n-1)d$, for all n ≥ 1, d ≥ 1 and a ≥ 1.

Let $[j_n]^k = [a + (n-1)d]^k$, for k = {1, 2, 3, ...}.

We shall consider the case for the power of k = {1, 2, 3} to establish the proof of the theorem.

Case (1), k = 1, n = {1, 2, 3, ...}

$[j_n]^k = [a + (n-1)d]^k$, k = 1 are constant.

$\Rightarrow [j_n]^1 = [a + (n-1)d]^1$ is the sequence.

The first difference is given as:

$\Delta^1[j_n]^1 = \Delta^1[a + (n-1)d]^1$,

$\Delta[j_n] = \Delta(a + dn - d)$

$= \Delta a + \Delta dn - \Delta d$

$= \Delta dn$

$= d[(n+1) - n]$

$= d(1)$

$= d$

$= d^1(1!)$

Therefore, for k = 1

$\Delta^1[j_n]^1 = \Delta^1[a + (n-1)d]^1 = d^1(1!)$

Case (2), k = 2, n = {1, 2, 3, ...}

$[j_n]^2 = [a + (n-1)d]^2$ is set of perfect squares.

$\Delta^2[j_n]^2 = \Delta^2[a + (n-1)d]^2$,

$\Delta[\Delta j_n^2] = \Delta^2[a + dn - d]$

$= \Delta^2[(a + dn - d)(a + dn - d)]$

$= \Delta^2[a + adn - ad + adn + d^2n^2 - d^2n - ad - d^2n + d^2]$

$= \Delta^2[a^2 + 2adn - 2ad + d^2n^2 - 2d^2n + d^2]$

$= \Delta[\Delta(a^2 + 2adn - 2ad + d^2n^2 - 2d^2n + d^2)]$

$= \Delta[\Delta a^2 + 2ad\Delta n - \Delta 2ad + d^2\Delta(n^2) - 2d^2\Delta(n) - \Delta d^2]$

$= \Delta[2ad((n+1) - (n) + d^2((n+1)^2 - (n)^2) - 2d^2((n+1) - (n)]$

$= \Delta[2ad(1) - d^2((n^2 + 2n - 1) - (n)^2 - 2d^2(1)]$

$= \Delta[2ad + d^2(2n - 1) - 2d^2)$

$= \Delta 2ad + \Delta 2d^2n - \Delta 3d^2$

$= \Delta 2d^2n$

$= 2d^2\Delta(n)$

$= 2d^2(n + 1 - n)$

$= 2d^2(1) = d^2 2!$

Therefore,

$\Delta^2[j_n]^2 \quad = \Delta^2[a + (n-1)d]^2 = d^2(2!)$, satisfied.

Case (3), k = 3, n = {1, 2, 3, ...}

$\Delta^3[j_n]^3 \quad = \Delta^3[a + (n-1)d]^3$

$= \Delta^3[(a + dn - d)^3]$

$= \Delta^3[(a + dn - d)(a + dn - d)^2]$

$= \Delta^3[(a + dn - d)(a^2 + 2adn - 2ad - 2d^2n + d^2n^2 + d^2)]$

$= \Delta^3[a^3 + 2a^2dn - 2a^2d - 2ad^2n + ad^2n^2 + ad^2 + a^2dn + 2ad^2n^2 - 2ad^2n^2 - 2d^3n^2 + d^3n^3 + d^3n - a^2d - 2ad^2n + 2ad^2 + 2d^3n - d^3n^2 - d^3]$

$= \Delta^3[a^3 + 3a^2dn - 3a^2d - 6ad^2n + 3ad^2n^2 + 3ad^2 - 3d^3n^2 + d^3n^3 + 3d^3n - d^3]$

$= \Delta^2[\Delta(a^3 + 3a^2dn - 3a^2d - 6ad^2n + 3ad^2n^2 + 3ad^2 - 3d^3n^2 + d^3n^3 + 3d^3n - d^3)]$

$= \Delta^2[\Delta(a^3 + \Delta 3a^2dn - \Delta 3a^2d - \Delta 6ad^2n + \Delta ad^2n^2 + \Delta 3ad^2 - \Delta 3d^3n^2 + \Delta d^3n^3 + \Delta 3d^3n - \Delta d^3)]$

$= \Delta^2[3a^2d\Delta n - 6ad^2\Delta n + ad^2\Delta n^2 + 3d^3\Delta n^2 + d^3\Delta n^3 + 3d^3\Delta n]$

$= \Delta^2[3a^2d(n + 1 - n) - 6ad^2(n + 1 - n) + ad^2((n+1)^2 - n^2) - 3d^3((n+1)^2 - n^2) + d^3((n+1)^3 - n^3) + 3d^3(n + 1 - n)\,]$

$= \Delta^2[3a^2d - 6ad^2(n^2 + 2n + 1 - n^2) - 3d^3n^2 + 2n + 1 - n^2 + d^3([n+1)(n^2 + 2n + 1 - n^3 + 3d^3]$

$= \Delta^2[3a^2d - 6ad^2 + ad^2(2n + 1) - 3d^3(2n + 1) + d^3(n^3 + 3n^2 + 3n + 1) - n^3 + 3d^3]$

$= \Delta^2[3a^2d - 5ad^2 + 2ad^2n - 3d^3n - 2d^3 + 3d^3n^2 + 3d^3]$

$= \Delta[\Delta(3a^2d - 5ad^2 + 2ad^2n - 3d^3n - 2d^3 + 3d^3n^2 + d^3]$

$= \Delta[\Delta 3a^2d - \Delta 5ad^2 + \Delta 2ad^2n - \Delta 3d^3n + \Delta 3d^3n^2 + \Delta d^3]$

$= \Delta[2ad^2\Delta n - 3d^3\Delta n + 3d^3\Delta n^2]$

$= \Delta[2ad^2(n + 1 - n) - 3d^3(n + 1 - n) + 3d^3(n^2 + 2n + 1 - n^2)]$

$= \Delta[2ad^3 - 3d^3 + 3d^3(2n + 1)]$

$= \Delta[2ad^3 - 3d^3 + 6d^2 + 3d^3]$

$= \Delta 2ad^3 - \Delta 3d^3 + \Delta 6d^3 + \Delta 3d^3]$

$= \Delta 6d^3n$

$= 6d^3\Delta n$

$= 6d^3$

$= d^3(6)$

$= d^3 3!$

Therefore,

$\Delta^3[\mathrm{j_n}]^3 \qquad = \Delta^3[a + (n - 1)d]^3 = d^3(3!)$

Thus, the proof of the theorem is established for the values integral power of k = {1, 2, 3}. This clearly reveals that the pattern is sustained for the values of k belong to natural numbers for all d > 2 and n ϵ $(1, \infty)$.

Assume the validity of the theorem of k = {4, 5, ... m} for some natural number m ≥ 4.

Inductively, the theorem holds for k = 1, see case (1) $\Delta[\mathrm{j_n}] = \Delta[a + (n - 1)d] = d.1!$

The truth of theorem for k = 1, implies is true for all positive k > 1, that is $\Delta^{\mathrm{k}}[\mathrm{j_n}]^{\mathrm{k}} = \Delta^{\mathrm{k}}[a + (n - 1)d]^k = d^k.k!$, for some k > 1. (See Case 2 and 3).

Thus by induction hypothesis, the truth of the theorem for k implies the validity of the theorem for k + 1.

Therefore, $\quad \Delta^{\mathrm{k+1}}[\mathrm{j_n}]^{\mathrm{k+1}} = \Delta^{\mathrm{k+1}}[a + (n - 1)d]^{k+1}$

$$\Delta^{\mathrm{k+1}}[a + (n - 1)d]^{k+1} = \mathrm{d}^{\mathrm{k+1}}(k + 1)!$$

This completes the proof, that is the pattern is sustained for all powers of set of natural numbers, for all k ≥ 1 and d ≥ 1, a ≥ 1, n ϵ $\{1, 2, 3, \ldots m\}, m \geq 1$.

The theorem is established.

3.3 Corollary

$\Delta^k[j_n]^k = d^k.k!$ for every arithmetic progression of set of natural numbers j_n and for any positive integer k, for all d ≥ 2. In other words, for any positive integer k, the k^{th} order difference of the k^{th} power of a sequence of natural numbers is equal to $d^k.k!$ where d is the common difference, d ≥ 1.

By implication, the theorem states that "if the elements of an arithmetic progression of the set of natural numbers with positive common difference are raised to a positive power k, then the k^{th} difference is equal to the product of the common difference and k factorial $[d^k.k!]$.

3.4 Remarks

The following strong relationship exist between the difference operator Δ and D operator (differential operator).

$D^k(K^p) = 0 \; if \; k > p$

$D^k x^k = k!$

The coefficient of x^{p-k}

$D^k(K^p) = \frac{P!}{(p-k)} x^{p-k}$ for k ≥ 2, k < p is even.

IV. CONCLUSION

This article established the structures of finite difference orders with respect to the powers of set of natural numbers. The result reveals a startling relationship between the common difference of the sequence and k!, as reflected in the theorem.

Competing Interests: Authors have declared that no competing interest exist.

REFERENCES

1. Chase, P.E. The elements of arithmetic... in which decimal and integral arithmetic are combined and taught inductively. U. Hunt and Son; 1984.
2. Finite difference, 2015. Retrieved October 2015. Available at: https://www.en.wikipedia.org/wiki/finite-difference
3. Cambridge Tracts in Mathematics, 87. Exponential Diophantine Equations. Cambridge University Press 1986, 169-183.
4. Ladan, Tanko, Aliyu, Ahmad and Kabiru. An Exposition of Integral Powers of Polynomials with Binomial Coefficient. Ladan et al./KJSET: Vol 01, No. 02, (Dec 2022) 18-25, ISSN: 1958-0641. https://www.kjset.ku.ac.ug/
5. Ladan, Aliyu, Tanko, Ahmad and Kabiru. Location of Points on Plane and the Order of Disposition of Sum of Powers of Cardinal Coordinates. Ladan et al./KJSET, Vol. 01, No. 02 (Dec 2022) 18-25, ISSN: 1958-0641. https://www.kjset.ku.ac.ug/

6. Ladan Umaru Ibrahim, Ukwu Chukwunenye and Apine Elijah. An investigation into the Order of Integral Perfect Powers. Ibrahim et al. AJOMCOR, 10(1) 12:18:2016.
7. Ibrahim, L. U., Garba, E. J. D., & Ishaya, T. (2023). An investigation into the Order of Difference of Product of Consecutive Integral Perfect Powers. Journal of Advances in Mathematics and Computer Science, 38 (11), 47-56. https://doi.org/10.9734/james/2023/V38i111844
8. Saylor.org Academy Sequence defined by a Recursive Formula, using Factorial Notation.learn.Saylor.org>mod>book>view.
9. Robert Blitzer College Algebra Second Edition Miami-Dade Community College. College Algebra/Robert Blitzer, 2nd ed. PCM ISBN 0-13-08728-61, I Algebra I. Title QA15.2.B5842001.
10. Perfect Powers: 2015 Retrieved November 23, 2015. Available at: https://en.wikipedia.org/wiki/perfect-power
11. Andreescu, T. Andrica D. Number theory. Birkhauser Boston, a part of Springer Science + Business Media, LLC; 2009. DOI: 10:1007/b//856-2.
12. Tarlok Shorey and Robert Tijdeman: Perfect Powers in Products of terms in arithmetic progression III. January 1992, Compositio Mathematica 82 (2) DOI. 10-1090/COMM/210
13. Shorey T. N. Perfect Powers in Values of Certain Polynomials at Integer Points Maths. Proc. Camb. Phil. Soc. 1986; 99 195 – 207.

APPENDIX A

Table 1: d = 3, k = 2

j_n	j_n^2	Δj_n^2	$\Delta^2 j_n^2$	$\Delta^3 j_n^2$
2	4	21	18	0
5	25	39	18	0
8	64	57	18	0
11	121	75	18	0
14	196	93	18	0
17	289	111	18	0
20	400	129	18	0
23	529	147	18	0
26	676	165	18	
29	841			

We have that: $\Delta^2 j_n^2 = 18 = 3^2(2!)$

Table 2: d = 3, k = 2

j_n	j_n^3	Δj_n^3	$\Delta^2 j_n^3$	$\Delta^2 j_n^3$	$\Delta^4 j_n^3$
2	8	117	270	162	0
5	125	387	432	162	0
8	512	819	594	162	0
11	1331	1413	756	162	0
14	2744	2169	918	162	
17	4913	3087	1080		
20	8000	4167			
23	12167				

We have that: $\Delta^3 j_n^3 = 162 = 3^3(3!)$

Table 3: d = 3, k = 2

j_n	j_n^2	Δj_n^2	$\Delta^2 j_n^2$	$\Delta^3 j_n^2$
1	1	15	18	0
4	16	33	18	0
7	49	51	18	0
10	100	69	18	0
13	169	87	18	
16	256	105		
19	361			

We have that: $\Delta^2 j_n^2 = 18 = 3^2(2!)$

Table 4: d = 3, k = 4

j_n	j_n^4	Δj_n^4	$\Delta^2 j_n^4$	$\Delta^3 j_n^4$	$\Delta^4 j_n^4$	$\Delta^5 j_n^4$
2	16	609	2862	4212	1944	0
5	625	3471	7074	6156	1944	0
8	4096	10545	13230	8100	1944	0
11	14641	23775	21330	10044	1944	0
14	38416	45105	31374	11988	1944	
17	83521	76479	43362	13932	1944	
20	16000	119841	57294	15876		
23	279841	177135	73170			
26	456976	250305				
29	707281					

We have that: $\Delta^4 j_n^4 = 1944 = 3^4(4!)$

Table 5: d = 3, k += 5

j_n	j_n^5	Δj_n^5	$\Delta^2 j_n^5$	$\Delta^3 j_n^5$	$\Delta^4 j_n^5$	$\Delta^5 j_n^5$	$\Delta^6 j_n^5$
2	32	3093	26550	72090	77760	29160	0
5	3125	29643	98640	149850	106920	29160	0
8	32768	128283	248490	256770	136080	29160	0
11	161051	376773	505260	392850	165240	29160	0
14	537824	882033	898110	558090	194400	29160	
17	1419857	1780143	1456200	752490	223560		
20	3200000	3236343	2208690	976050			
23	6436343	5445033	3184740				
26	11881376	8629773					
29	20511149						

We have that: $\Delta^5 j_n^5 = 29160 = 3^5(5!)$

Table 6: d = 4, k = 2

j_n	j_n^2	Δj_n^2	$\Delta^2 j_n^2$	$\Delta^3 j_n^2$
2	4	32	32	0
6	36	64	32	0
10	100	96	32	0
14	196	128	32	
18	324	160		
22	484			

We have that: $\Delta^2 j_n^2 = 32 = 4^2(2!)$

Table 7: d = 4, k = 3

j_n	j_n^3	Δj_n^3	$\Delta^2 j_n^3$	$\Delta^3 j_n^3$	$\Delta^4 j_n^3$
2	8	208	576	384	0
6	216	784	960	384	0
10	1000	1744	1344	384	0
14	2744	3088	1728	384	0
18	5832	4816	2112	384	0
22	10648	6928	2496	384	
26	17576	9424	2880		
30	27000	2304			
38	39304				

We have that: $\Delta^4 j_n^4 = 4^3(3!)$

Table 8: d = 4, k = 4

j_n	j_n^4	Δj_n^4	$\Delta^2 j_n^4$	$\Delta^3 j_n^4$	$\Delta^4 j_n^4$	$\Delta^5 j_n^5$
2	16	1280	7424	12288	6144	0
6	1296	8704	19712	18432	6144	0
10	10000	28416	38144	24576	6144	0
14	38416	66560	62720	30720	6144	0
18	104976	129280	93440	36864	6144	0
22	234256	222720	130304	43008	6144	
26	456976	353024	173312	49152		
30	810000	526336	222464			
34	1336336	748800				
38	2085136					

We have that: $\Delta^4 j_n^4 = 6144 = 4^4(4!)$

Table 9: d = 4, k = 5

j_n	j_n^5	Δj_n^5	$\Delta^2 j_n^5$	$\Delta^3 j_n^5$	$\Delta^4 j_n^5$	$\Delta^5 j_n^5$	$\Delta^6 j_n^5$
1	1	3124	52800	203520	276480	122880	0
5	3125	55924	256320	480000	399360	122880	0
9	59049	312244	736320	879360	522240	122880	0
13	371293	1048654	1615680	1401600	645120	122880	
17	1419857	2664244	3017280	2046720	768000		
21	4084101	5681524	5064000	2814720			
25	9765625	10745524	7878720				
29	20511149	18624244					
33	39135393						

We have that: $\Delta^5 j_n^5 = 122880 = 4^5(5!)$

Table 10: d = 7, k = 2

j_n	j_n^5	Δj_n^5	$\Delta^2 j_n^5$
2	4	77	98
9	81	175	98
16	256	273	98
23	527	271	
30	900		

We have that: $\Delta^2 j_n^2 = 98 = 7^2(2!)$

Table 11: d = 7, k = 3

j_n	j_n^3	Δj_n^3	$\Delta^2 j_n^3$	$\Delta^3 j_n^3$	$\Delta^4 j_n^4$
2	8	721	2646	2058	0
9	729	3367	4704	2058	0
16	4096	8071	6762	2058	0
23	12167	14833	8820	2058	0
30	27000	23653	10878	2058	
37	50653	34531	12936		
44	85184	47467			
51	132651				

We have that: $\Delta^3 j_n^3 = 2058 = 7^3(3!)$

Table 12: d = 7, k = 4

j_n	j_n^4	Δj_n^4	$\Delta^2 j_n^4$	$\Delta^3 j_n^4$	$\Delta^4 j_n^4$	$\Delta^5 j_n^4$
2	16	6545	52430	102900	57624	0
9	6561	58975	155330	160524	57624	0
16	65536	214305	315854	218148	57624	0
23	279841	530159	534002	275772	57624	0
30	810000	1064161	809774	333396		
37	1874161	1873935	1143170			
44	3748096	3017105				
51	6765201					

We have that: $\Delta^4 j_n^4 = 57624 = 7^4(4!)$

Table 13: d = 19, k = 3

j_n	j_n^3	Δj_n^3	$\Delta^2 j_n^3$	$\Delta^3 j_n^3$	$\Delta^4 j_n^3$
2	8	9253	45486	41154	0
21	9261	54739	86640	41154	0
40	64000	141379	127794	41154	0
59	205379	269173	168948	41154	0
78	474552	438121	210102	41154	0
97	912673	648223	251256	41154	0
116	1560896	899479	292410	41154	
135	2460375	1191889	333564		
154	3652264	1525453			
173	5177717				

We have that: $\Delta^3 j_n^3 = 41154 = 19^3(3!)$

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