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Keywords: discrete maximum principle, geometric arguments, diffusion-adjection-reaction problems, nite volume schemes.

I. INTRODUCTION

Let Ω be a bounded connected open subset of $\mathbb{R}^2$ whose boundary denoted by Γ is the union of polygonal lines $\Gamma_k]_{k\in K}$ where K is a finite subset of $\mathbb{N}$ which denotes the set of positive integers (see Figure 1 below). Note that if K is a singleton then Ω is a polygon (and so simply connected). Given the scalar functions $D(\cdot)$, $\mu(\cdot)$ and $f(\cdot)$ together with a vector field $(\cdot)$, all being defined in Ω, we consider the elliptic problem that consists in finding a scalar function $u(.)$ in an adequate function space such that

$$- div\,[D(x) grad\, u] + div\,[u\,\psi] + \mu\, u = f \quad \text{in} \quad \Omega \tag{1.1}$$

with the following homogeneous Dirichlet boundary conditions :

$$u = 0 \qquad \text{on} \qquad \Gamma \tag{1.2}$$

Under reasonable assumptions on the previous data i.e.

$$0 < D^- \leq D(x) \leq D^+, \quad and \quad \mu(x) \geq 0 \quad a.e.\ in\ \Omega \tag{1.3}$$

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$$f(\cdot) \in L^2 \;(\Omega) \tag{1.4}$$

$$\psi(.) \in C^1(\overline{\Omega}, \mathbb{R}^2\;) \tag{1.5}$$

with

$$div\,[\psi] \;\geq\; 0 \quad a.e.\;\; in\;\; \Omega \tag{1.6}$$

it is easy to prove that (see [1] for instance): The second order elliptic problem (1.1)-(1.2) gets a unique weak solution in the sense that

$$(PV) \quad \begin{cases} \text{There exists one and only one } \; u \in H_0^1(\Omega) \text{ such that :} \\ \mathcal{B}(u,v) \;=\; L(v) \qquad \forall v \in H_0^1(\Omega) \end{cases} \tag{1.7}$$

where we have set:

$$\mathcal{B}(u,v) = \int_\Omega D(x) grad\, u.grad\, v\, dx \;+\; \int_\Omega v\, div\,[u\,\psi]\, dx \;+\; \int_\Omega \mu\, u\, v\, dx \tag{1.8}$$

and

$$L(v) = \int_\Omega f\, v\, dx. \tag{1.9}$$

Following [2] one can prove that if the given function f is positive almost everywhere in Ω then the weak solution of the system (1.1)-(1.2) is also positive almost everywhere in Ω. That is the weak form of the Maximum Principle. Several works on construction of positivity-preserving numerical methods for diffusion, diffusion-convection, diffusion-reaction and diffusion-convection-reaction problems are available in the literature (see for instance [5, 6, 11, 13]). Such numerical methods are sometimes called monotone schemes.

The main objective of this work is to expose geometrical arguments for proving the well-known discrete version of the Maximum Principle satisfied by the conventional finite volume solution to the system (1.1)-(1.2).

II. PRELIMINARY TOOLS

Definition 2.1 (Partition of Ω). Let $\overline{\Omega}$ be the closure of Ω in the sense of the standard topology of $\mathbb{R}^2$ and let J be a finite subset of $\mathbb{N}$ which is the set of positive integers. A family $\{\Omega_j\}_{j\in J}$ made up of subsets of $\overline{\Omega}$ defines a partition of $\overline{\Omega}$ if the following conditions are satisfied:

$$\begin{cases} (i) \quad Int(\Omega_j) \neq \emptyset \qquad \forall j \in J \\ (ii) \quad \overline{\Omega} = \underset{j\in J}{\cup} \overline{\Omega}_j \\ (iii) \quad \forall j', j'' \in J, \qquad j' \neq j'' \implies Int(\Omega_{j'}) \cap Int(\Omega_{j''}) \;=\; \emptyset \end{cases} \tag{2.1}$$

where $Int(\diamond)$ denotes the interior of $\diamond$ in the sense of standard topology of $\mathbb{R}^2$.

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Let us consider a *partition* $\mathcal{P}$ over $\overline{\Omega}$ consisting in a finite family of closed convex polygons (named also polygonal elements) generically denoted by T. These polygonal elements are the so-called control volumes in the language of Finite Volume theory. The control volumes from the partition $\mathcal{P}$ defines a *conforming Finite Volume mesh* over $\overline{\Omega}$ if (in addition to conditions (i)-(iii) from Definition 2.1) the following conditions are satisfied:

$$\begin{cases} \forall T', T'' \in \mathcal{P}, & T' \neq T'' \text{ implies that :} \\ \circ \quad \text{either} \quad T' \cap T'' = \emptyset \\ \circ \quad \text{or} \quad T' \cap T'' = \text{ common vertex} \\ \circ \quad \text{or} \quad T' \cap T'' = \text{ common edge,} \end{cases} \tag{2.2}$$

where $\emptyset$ denotes the empty set. Let us denote by $\partial\mathcal{P}$ the set of boundary edges (viewed as degenerate control volumes) and we briefly define the conventional finite volume mesh $\mathcal{T}$ as it follows: $\mathcal{T} = \{\mathcal{P}, \partial\mathcal{P}\}$.

We should use intensively in what follows a notion of characteristic function slightly different from the usual one and defined as follows.

Definition 2.2 Let T be a control volume either from $\mathcal{P}$ or from $\partial\mathcal{P}$. We call in this work the characteristic function of T denoted by $\mathbf{1}_T$ the function defined almost everywhere either in Ω (with respect to Lebesgue measure in 2-D) or on $\Gamma = \underset{k\in K}{\cup}\Gamma_k$ (with respect to Lebesgue measure in 1-D) by :

$$\mathbf{1}_T(x) = \begin{cases} 1 & \text{if} \quad x \in Int(T) \\ 0 & \text{if} \quad x \in Ext(T) \end{cases} \tag{2.3}$$

where $Ext(\diamond)$ denotes the exterior of a subset $\diamond$ from $\mathbb{R}^2$ (with respect to the natural topology of $\mathbb{R}^2$). Recall that $Int(\diamond)$ stands for the interior of $\diamond$ from $\mathbb{R}^2$. $\square$

Let us introduce the following discrete function spaces that play a key-role in the sequel.

Definition 2.3 We set :

$$\mathbf{S}^{\mathcal{P}} = \Big\{ v_{\mathcal{P}} : \Omega \longrightarrow \mathbb{R}\,;\, v_{\mathcal{P}}(x) = \sum_{T\in\mathcal{P}} v_T \mathbf{1}_T(x), \text{ with } v_T \in \mathbb{R} \;\; \forall T \in \mathcal{P} \Big\}, \tag{2.4}$$

$$\mathbf{S}^{\partial\mathcal{P}} = \Big\{ v_{\partial\mathcal{P}} : \Gamma \longrightarrow \mathbb{R}\,;\, v_{\partial\mathcal{P}}(s) = \sum_{L\in\partial\mathcal{P}} v_L \mathbf{1}_L(s), \text{ with } v_L \in \mathbb{R} \;\; \forall L \in \partial\mathcal{P} \Big\}, \tag{2.5}$$

and

$$\mathbf{S}^{\mathcal{T}} = \mathbf{S}^{\mathcal{P}} \times \mathbf{S}^{\partial\mathcal{P}}, \qquad \mathbf{S}_0^{\mathcal{T}} = \mathbf{S}^{\mathcal{P}} \times \Big\{ 0_{\mathbf{S}^{\partial\mathcal{P}}} \Big\} \tag{2.6}$$

where $0_{\mathbf{S}^{\partial\mathcal{P}}}$ is the zero-function (denoted simply 0 if there is no risk of confusion) from the discrete function space $\mathbf{S}^{\partial\mathcal{P}}$. $\square$

The Finite Volume method is based on the fundamental idea that the exact solution u could be approximated inside any control-volume T with a constant U_T corresponding to either the mean-value of u or its approximation at a given point located inside T, with Cartesian coordinates x_T. In the context of conventional Finite Volumes the choice of that point is not arbitrary as we will be seeing in assumption $(\mathcal{A}_3)$ below. Let us denote by Γ_T the boundary of any control-volume T. We need to specify the following assumptions that make the conventional Finite Volumes very attractive and realistic for certain engineering problems as subsurface flow problems (notice that [3, 4, 15] are among distinguished references on fluid flow in porous media):

$(\mathcal{A}_1)$ The diffusion coefficient $D(\cdot)$ is a piecewise constant function i.e.

$$\exists\, S \subseteq \mathbb{N}, \text{ with } S \text{ finite, such that:} \qquad D(x) = \sum_{s\in S} D_s \mathbf{1}_{\Omega_s}(x). \tag{3.1}$$

where $\{\Omega_s\}_{s\in S}$ defines a partition $\mathcal{P}$ of the domain $\overline{\Omega}$ in the sense of Definition 2.1.

Denote by $\mathcal{T}$ the Finite Volume mesh corresponding to the partition $\mathcal{P}$. Let us make the following assumption on $\mathcal{T}$.

$(\mathcal{A}_2)$ $\mathcal{T}$ is compatible with the discontinuities of $D(\cdot)$ in the sense that the discontinuity points of $D(\cdot)$ belong to the mesh interfaces $\Gamma^{\mathcal{T}} = \underset{T\in\overline{\mathcal{P}}}{\cup} \Gamma_T$, where we have set $\overline{\mathcal{P}} = \mathcal{P} \cup \partial\mathcal{P}$. In other words any discontinuity point of the function $D(\cdot)$ is located in a control volume boundary.

$(\mathcal{A}_3)$ For all $(T', T'') \in \overline{\mathcal{P}} \times \mathcal{P}$ such that T' and T'' are adjacent (that is $\Gamma_{T'} \cap \Gamma_{T''}$ is a common edge for control volumes T' and T''), the vector $x_{T'} - x_{T''}$ is orthogonal to the common edge. This is the so-called orthogonality condition required for conventional Finite Volume meshes (see [5, 6]).

An immediate consequence of the assumption (1.3) is that $D(\cdot)$ is a nonnegative constant function in each control volume T. We denote by D^T the constant value of $D(\cdot)$ in the control volume T.

Let us give a brief description of the different steps for getting a conventional finite volume scheme. We start with introducing some useful notations: $\mathcal{E}$ is the set of all mesh edges, $\mathcal{E}^{int}$ is the subset of $\mathcal{E}$ made of interior mesh edges and $\mathcal{E}^{ext}$ is the subset of $\mathcal{E}$ made of exterior mesh edges i.e. mesh edges lying on the domain boundary.

Step 1: Integrate the two sides of the balance equation (1.1) in each control volume T from the family $\mathcal{P}$. So we get what follows (thanks to Ostrogradski's theorem):

$$-\int_{\Gamma_T} D^T grad\, u \,.\, \nu_T ds + \int_{\Gamma_T} u\,\psi \,.\, \nu_T ds + \int_T \mu(x)u(x)dx = \int_T f(x)dx \qquad \forall\, T \in \mathcal{P}. \tag{3.2}$$

where ν_T stands for outward unit vector normal to the control-volume boundary Γ_T.

Step 2 : Re-write the first two integral terms from the left-hand side of (3.2) as follows for all $T \in \mathcal{P}$:

$$\sum_{\sigma\in\mathcal{E}_T} -\int_\sigma D^T grad\, u \,.\, \nu_{\sigma,T} ds \;+\; \sum_{\sigma\in\mathcal{E}_T}\int_\sigma u\,\psi\,.\,\nu_{\sigma,T} ds \;+\; \int_T \mu(x)u(x)dx \;=\; \int_T f(x)dx \tag{3.3}$$

where $\mathcal{E}_T$ is the set of mesh edges σ lying in Γ_T and where $\nu_{\sigma,T}$ stands for outward unit vector normal to the portion σ of the control-volume boundary Γ_T, called again mesh edge associated with Γ_T. Integrals from the first summation are diffusion fluxes while integrals from the second summation are convection fluxes (called sometimes advection fluxes).

Step 3: Perform the approximation of the unknown function u in the control-volume T with the unknown real constant $u(x_T)$. So one could set what follows concerning approximation of the reaction term :

$$\int_T \mu(x)u(x)dx \;\approx\; u(x_T)\, I_T(\mu) \qquad \forall T \in \mathcal{P} \tag{3.4}$$

where $I_T(\diamond)$ is the integral of a function $\diamond$ defined in the control volume T.

Step 4 : Look for reasonable approximations of flux integral terms from the left-hand side of (3.3). What should one understand by reasonable approximations ? We mean that the flux approximations should take account of the following constraints :

• Perform the upwind approximation of the convection flux in view to ensure the stability of the global finite volume scheme. For that purpose, let us start with setting :

Definition 3.1

$$\psi_{\sigma,T} \stackrel{def}{=} \int_\sigma \psi\,.\,\nu_{\sigma,T} ds \tag{3.5}$$

Definition 3.2 (Upwind approximation of the convection flux over $\sigma \in \mathcal{E}^{int}$)
Let σ in $\mathcal{E}_T \cap \mathcal{E}_L$, with T and L from the set $\mathcal{P}$. We set:

$$\int_\sigma u\,\psi\,.\,\nu_{\sigma,T} ds \;\approx\; \begin{cases} u(x_T)\;\psi_{\sigma,T} & \text{if} \quad {}_{\sigma,T} \;\geq\; 0 \\ u(x_L)\;\psi_{\sigma,T} & \text{if} \quad {}_{\sigma,T} \;<\; 0. \end{cases} \tag{3.6}$$

In other words the upwind approximation of the convective flux across the interior edge σ in $\mathcal{E}_T \cap \mathcal{E}_L$ could be defined as follows :

$$\int_\sigma u\,\psi\,.\,\nu_{\sigma,T} ds \approx u(x_T)\,\max\{\psi_{\sigma,T}, 0\} \;-\; u(x_L)\,\max\{-\psi_{\sigma,T}, 0\}. \tag{3.7}$$

Since (according to the flux continuity principle over grid-block interfaces)

$$\psi_{\sigma,T} \;+\; \psi_{\sigma,L} \;=\; 0$$

the preceding approximation of the convective flux is equivalent to the following one

$$\int_\sigma u\,\psi\,.\,\nu_{\sigma,T} ds \;\approx\; u(x_T)\,\max\{\psi_{\sigma,T},0\} \;-\; u(x_L)\,\max\{\psi_{\sigma,L},0\}. \tag{3.8}$$

• The flux continuity across interior edges σ, i.e. $\sigma \in \mathcal{E}^{int}$, is a fundamental physical principle to be met. So we have necessarily for all $\sigma \in \mathcal{E}^{int}$

$$[-D^T grad\, u\,.\,\nu_{\sigma,T} \,+\, u\,\psi\,.\,\nu_{\sigma,T}] \;+$$

$$+\;\; [-D^L grad\, u\,.\,\nu_{\sigma,L} \,+\, u\,\psi\,.\,\nu_{\sigma,L}] = 0 \qquad on \quad \sigma \tag{3.9}$$

Integrating the right-hand and the left-hand sides of (3.9) over $\sigma \in \mathcal{E}^{int}$ leads to the following "weak formulation" of flux continuity :

$$[-\int_\sigma D^T grad\, u\,.\,\nu_{\sigma,T} ds \;+\; \int_\sigma u\,\psi\,.\,\nu_{\sigma,T} ds] +$$

$$+[-\int_\sigma D^L grad\, u\,.\,\nu_{\sigma,L} ds \;+\; \int_\sigma u\,\psi\,.\,\nu_{\sigma,L} ds] \;=\; 0 \qquad \forall\, \mathcal{E}^{int} \ni \sigma = \Gamma_T \cap \Gamma_L \tag{3.10}$$

Since the weak solution u of the system (1.1)-(1.2) lies in $H_0^1(\Omega)$, the trace $u_{|\sigma}$ exists (in $H^{\frac{1}{2}}(\sigma)$ for instance) in a unique manner. In consequence we naturally get what follows :

$$[\int_\sigma u\,\psi\,.\,\nu_{\sigma,T} ds] \;+\; [\int_\sigma u\,\psi\,.\,\nu_{\sigma,L} ds] \;=\; 0 \qquad \forall\, \mathcal{E}^{int} \ni \sigma = \Gamma_T \cap \Gamma_L \tag{3.11}$$

Thus, the previous "weak formulation" of flux continuity (3.10) is reduced to

$$[-\int_\sigma D^T grad\, u\,.\,\nu_{\sigma,T} ds] \;+\; [-\int_\sigma D^L grad\, u\,.\,\nu_{\sigma,L} ds] \;=\; 0 \qquad \forall\, \mathcal{E}^{int} \ni \sigma = \Gamma_T \cap \Gamma_L \tag{3.12}$$

In the context of conventional Finite Volumes the family $\mathcal{P}$ satisfies the so-called orthogonality condition (see assumption $(\mathcal{A}_3)$ above at the beginning of the current Section). So there exists a family of points $\{x_T;\; T \in \mathcal{P}\}$, such that for any pair $(T, L) \in \mathcal{P} \times \overline{\mathcal{P}}$, with T and L adjacent, the orthogonal projections of x_T and x_L on their common edge σ coincides and let call it x_σ. We make the following convention:

"If T is adjacent to the domain boundary we set : $L \overset{def}{=} \sigma$, where σ is the boundary edge associated with T, and x_L coincides with x_σ".

This being said, from the following diffusion flux approximation (assuming the exact solution restriction $u_{|_T}$ in $C^0(\overline{T})$ for any $T \in \mathcal{P}$; it is the case if $u_{|_T} \in H^2(T)$):

$$-\int_\sigma [D^T grad\, u \,.\, \nu_{\sigma,T}]ds \;\approx\; \frac{D^T\, mes(\sigma)}{dist(x_T, x_\sigma)}[u(x_T) - u(x_{\sigma,T})] \qquad \forall\, \sigma \in \mathcal{E}_T \tag{3.13}$$

where $mes(\,.\,)$ stands for Lebesgue measure in one-space dimension, $dist(.,.)$ represents the Euclidean distance and where $x_{\sigma,T}$ is in fact the point x_σ seen as from the boundary of T by an observer standing inside T. The principle of continuity of u on grid-block interfaces is expressed at the discrete level by the relation :

$$u(x_{\sigma,T}) \;=\; u(x_{\sigma,L}) \qquad \forall\, \sigma \in \mathcal{E}_T \cap \mathcal{E}_L \qquad \forall\, T, L \in (\mathcal{P} \times \mathcal{P})_{adj}$$

where $(\mathcal{P} \times \mathcal{P})_{adj}$ is the subset of $\mathcal{P} \times \mathcal{P}$ made of (T, L) such that T and L are adjacent. So it is reasonable to set:

$$u(x_\sigma) \;\stackrel{def}{=}\; u(x_{\sigma,T}) \qquad \forall\, T \in \mathcal{P} \qquad \forall\, \sigma \in \mathcal{E}_T.$$

With the above notation the diffusion flux approximation could read as follows

$$-\int_\sigma [D^T grad\, u \,.\, \nu_{\sigma,T}]ds \;\approx\; \frac{D^T\, mes(\sigma)}{dist(x_T, x_\sigma)}[u(x_T) - u(x_\sigma)] \qquad \forall\, \sigma \in \mathcal{E}_T \tag{3.14}$$

Writing down the *discrete analogue* of the ”weak formulation” (3.12) of continuity of the diffusion flux (across any interior edge $\sigma \in \mathcal{E}_T \cap \mathcal{E}_L$) yields

$$\frac{D^T\, mes(\sigma)}{dist(x_T, x_\sigma)}[u(x_T) - u(x_\sigma)] \;+\; \frac{D^L\, mes(\sigma)}{dist(x_L, x_\sigma)}[u(x_L) - u(x_\sigma)] = 0 \qquad \forall\, \sigma \in \mathcal{E}_T \cap \mathcal{E}_L. \tag{3.15}$$

This relation could be viewed as a linear equation with only discrete unknown $u(x_\sigma)$. This unknown can be obviously determined as a function of discrete unknowns $u(x_T)$ and $u(x_L)$ as indicated hereafter. Indeed elementary operations on (3.15) leads to

$$u(x_\sigma) \;=\; \frac{\lambda_{T,\sigma}\, u(x_T) \;+\; \lambda_{L,\sigma}\, u(x_L)}{\lambda_{T,\sigma} \;+\; \lambda_{L,\sigma}} \qquad \forall\, \sigma \in \mathcal{E}_T \cap \mathcal{E}_L \tag{3.16}$$

where we have set

$$\lambda_{K,\sigma} \;\stackrel{def}{=}\; \frac{D^K}{dist(x_K, x_\sigma)} \qquad \forall\, K \in \mathcal{P} \qquad \forall\, \sigma \in \mathcal{E}_K. \tag{3.17}$$

Substituting the right-hand side of (3.16) to $u(x_\sigma)$ in the diffusion flux approximation given by (3.14) leads to what follows for any $\sigma \in \mathcal{E}_T \cap \mathcal{E}_L$:

$$-\int_\sigma [D^T grad\, u \,.\, \nu_{\sigma,T}]ds \;\approx\; \frac{D^T D^L mes(\sigma)}{D^T dist(x_L, x_\sigma) + D^L dist(x_T, x_\sigma)}[u(x_T) - u(x_L)]\,. \tag{3.18}$$

Remark 3.3 (Important to notice)

First of all the diffusion flux approximation (3.18) has been established for interior edges i.e. $\sigma \in \mathcal{E}^{int}$. Let us explain why the convention consisting to consider boundary edges σ as also degenerate control-volumes L allows to recover (3.14) from the relation (3.18). Indeed if $\sigma \in \mathcal{E}_T \cap \mathcal{E}^{ext}$ then $x_\sigma = x_L$, and it follows that $dist(x_L, x_\sigma) = 0$ in (3.18).

The following Conventional Finite Volume scheme is obtained from preceding approximations of different terms of the left-hand side of the balance equation (3.3): see relations (3.4), (3.8) and (3.18). One could learn more on this topic with [5, 6] for instance.

Definition 3.4 (Conventional Finite Volume Scheme)

The conventional Finite Volume approximation of the system (1.1)-(1.2) consists in what follows :

Find

$$U_{\mathcal{T}} = \Big(\sum_{K\in\mathcal{P}} U_K \mathbf{1}_K \,,\, 0_{\mathbf{S}^{\partial\mathcal{P}}}\Big) \in \mathbf{S}_0^{\mathcal{T}}$$

such that:

$$\sum_{L\in\overline{\mathcal{P}}, L\neq T} \frac{D^T D^L mes(\Gamma_T \cap \Gamma_L)}{D^T dist(x_L, T) + D^L dist(x_T, L)} [U_T - U_L] +$$

$$+ \sum_{L\in\overline{\mathcal{P}}, L\neq T} [U_T \max\{\psi_{\sigma,T}, 0\} - U_L \max\{\psi_{\sigma,L}, 0\}] + U_T I_T(\mu) = I_T(f) \quad \forall T \in \mathcal{P} \tag{3.19}$$

where $\sigma \in \mathcal{E}_T \cap \mathcal{E}_L$. Recall that $I_T(\diamond)$ is the integral of a function $\diamond$ defined in the control volume T. □

The Finite Volume and Mimetic Finite Difference approximations of solutions to isotropic or anisotropic diffusion problems on distorted grids have been intensively developed in the literature and are today considered as classical topics (see for instance [6, 7, 8, 9, 10, 14]. Some extensions of Finite Volume Methods have been designed and known under the name of Gradient Discretization Methods (see [12] for learning more) and many other extensions are underdevelopment (see [11] for instance). Let us state the following well-known Discrete Maximum Principle followed by a proof based upon a Geometrical Technique that seems new in this context to the best of our knowledge.

Theorem 3.5 (Discrete Maximum Principle) Let us suppose that Ω is a bounded open subset of $\mathbb{R}^2$, connected by polygonal arcs. Let its boundary Γ be the union of polygonal lines $\Gamma_k]_{k\in\mathbf{K}}$, where $\mathbf{K}$ is a finite subset of $\mathbb{N}$ (see Figure 1 below). The linear system (3.19) gets a unique solution that satisfies the following positivity property:

- If $I_T(f) \geq 0$ for all $T \in \mathcal{P}$ then

$$U_T \geq 0 \qquad \forall T \in \mathcal{T}. \tag{3.20}$$

Moreover the following discrete maximum principle holds :

- If there exists a control volume $\overline{T}$ from $\mathcal{P}$ such that

$$U_{\overline{T}} = 0 \equiv \min\{U_B;\ B \in \partial\mathcal{P}\}$$

then

$$U_T = 0 \qquad \forall T \in \mathcal{P}. \qquad \square \tag{3.21}$$

The originality of this work relies up on the technique exposed hereafter to prove that the solution to (3.19) meets the discrete Maximum Principle. This technique has been successfully applied to a new finite volume method introduced recently by A. Njifenjou, A. Toudna and S. Moussa in [11]. To the best of our knowledge the technique widely exposed in the literature (for proving the discrete maximum principle) is based up on algebraic arguments (see for instance [6,10]). We are going to develop geometric arguments for proving the discrete Maximum Principle stated above in Theorem 3.5.

IV. GEOMETRICAL TECHNIQUE FOR PROVING (3.20) AND (3.21)

◇ We have to first prove (3.20), that is:

If

$$\sum_{L\in\overline{\mathcal{P}},L\neq T} \frac{D^T D^L mes(\Gamma_T \cap \Gamma_L)}{D^T dist(x_L,T) + D^L dist(x_T,L)}[U_T - U_L] +$$

$$+ \sum_{L\in\overline{\mathcal{P}},L\neq T} [U_T \max\{\psi_{\sigma,T},0\} - U_L \max\{\psi_{\sigma,L},0\}] + U_T I_T(\mu) \geq 0 \quad \forall T \in \mathcal{P} \tag{4.1}$$

with

$$U_T = 0 \qquad \forall T \in \partial\mathcal{P} \tag{4.2}$$

then

$$U_T \geq 0 \qquad \forall T \in \mathcal{P}. \tag{4.3}$$

Let us set for all $(T, L) \in \mathcal{P} \times \overline{\mathcal{P}}$:

$$\alpha_{TL} \stackrel{def}{=} \frac{D^T D^L mes(\Gamma_T \cap \Gamma_L)}{D^T dist(x_L,T) + D^L dist(x_T,L)} \tag{4.4}$$

. Then notice that if T and L are adjacent control volumes i.e. Γ_T and Γ_L get a common edge, we have

$$\alpha_{TL} \succ 0. \tag{4.5}$$

and otherwise we have

$$\alpha_{TL} = 0. \tag{4.6}$$

In the sequel $\mathcal{V}_E$ denotes the set of control volumes from $\overline{\mathcal{P}}$ adjacent to a given control volume E from $\mathcal{P}$.

Let us start the proof with assuming that we have (4.1) and (4.2). We should deduce that (4.3) holds. Now let us set:

$$\begin{cases} U_{min}^{\overline{\mathcal{P}}} \overset{def}{=} \min\{U_T;\, T\, browsing\ the\ set\, \overline{\mathcal{P}}\} \\ and \\ \overline{\mathcal{P}}^{min} \overset{def}{=} \{T \in \overline{\mathcal{P}} \;/\; U_T = U_{min}^{\overline{\mathcal{P}}}\}. \end{cases} \tag{4.7}$$

First of all we should notice that $U_{min}^{\overline{\mathcal{P}}}$ exists as $\{U_T;\, T\, browsing\ the\ set\, \overline{\mathcal{P}}\}$ is a finite subset of $\mathbb{R}$. Therefore $\overline{\mathcal{P}}^{min}$ is not an emptyset.

- If $\overline{\mathcal{P}}^{min} \cap \partial\mathcal{P} \neq \emptyset$, it is clear that the discrete Maximum Principle is satisfied. Indeed, denote by L a (degenerate) control-volume belonging to $\overline{\mathcal{P}}^{min} \cap \partial\mathcal{P}$. So we have

$$U_L = 0 \quad (\text{since } L \in \partial\mathcal{P}) \qquad and \qquad U_L = U_{min}^{\overline{\mathcal{P}}} \quad (\text{since } L \in \overline{\mathcal{P}}^{min}). \tag{4.8}$$

Hence

$$U_T \geq 0 \qquad \forall T \in \mathcal{P}. \tag{4.9}$$

- We are going to geometrically prove that $\overline{\mathcal{P}}^{min} \cap \partial\mathcal{P} = \emptyset$ is *impossible*. Reasoning by the absurd let us suppose that:

$$\overline{\mathcal{P}}^{min} \cap \partial\mathcal{P} = \emptyset. \tag{4.10}$$

This assumption necessarily ensures that: $\overline{\mathcal{P}}^{min} \subset \mathcal{P}$ and $\overline{\mathcal{P}}^{min}$ is not empty. Let us *arbitrarily* consider a control volume $\overline{T}$ from $\overline{\mathcal{P}}^{min}$ (notice that $\overline{T}$ is not the closure of T). Since $\overline{T}$ necessarily belongs to $\mathcal{P}$, the assumption (4.1) applies for $T = \overline{T}$ and, thanks to definition (4.4) and relation (4.5), we get (with $\sigma \in \mathcal{E}_{\overline{T}} \cap \mathcal{E}_L$, if $\overline{T}$ and L adjacent):

$$0 \leq \sum_{L \in \mathcal{V}_{\overline{T}}} \underbrace{\alpha_{\overline{T}L}}_{\succ 0} \overbrace{[U_{\overline{T}} - U_L]}^{\leq 0} + \underbrace{I_{\overline{T}}(\mu)}_{\geq 0} \overbrace{U_{\overline{T}}}^{\leq 0} +$$

$$+ \sum_{L \in \overline{\mathcal{P}}, L \neq \overline{T}} [U_{\overline{T}} \max\{\psi_{\sigma,\overline{T}}, 0\} \; - \; U_L \max\{\psi_{\sigma,L}, 0\}] \tag{4.11}$$

Let us prove the following lemma stating that the last summation in the right-hand side of the preceding inequality is in fact less than or equal to zero.

Lemma 4.1

$$\sum_{L \in \overline{\mathcal{P}}, L \neq \overline{T}} [U_{\overline{T}} \max\{\psi_{\sigma,\overline{T}}, 0\} \; - \; U_L \max\{\psi_{\sigma,L}, 0\}] \quad \leq \quad 0 \tag{4.12}$$

where $\sigma \, = \, \Gamma_{\overline{T}} \cap \Gamma_L$.

Notice that if $\Gamma_{\overline{T}} \cap \Gamma_L = \emptyset$ the expression $[U_{\overline{T}} \max\{\psi_{\sigma,\overline{T}}, 0\} \; - \; U_L \max\{\psi_{\sigma,L}, 0\}]$ is zero.

Proof. The following equality is obvious :

$$\sum_{L \in \overline{\mathcal{P}}, L \neq \overline{T}} [U_{\overline{T}} \max\{\psi_{\sigma,\overline{T}}, 0\} - U_L \max\{\psi_{\sigma,L}, 0\}] \; = \; \sum_{\sigma \in \mathcal{E}_{\overline{T}}} \overbrace{[U_{\overline{T}} \; - \; U_L]}^{\leq 0} \underbrace{\max\{\;\; _{\sigma,L}, 0\}}_{\geq 0} +$$

$$+ \sum_{\sigma \in \mathcal{E}_{\overline{T}}} U_{\overline{T}} \, [\max\{\psi_{\sigma,\overline{T}}, 0\} \; - \; \max\{\psi_{\sigma,L}, 0\}] \, . \tag{4.13}$$

The proof is ended if we show that the second summation in the right-hand side of the preceding equality is less than or equal to zero. That is

$$\sum_{\sigma \in \mathcal{E}_{\overline{T}}} U_{\overline{T}} \, [\max\{\psi_{\sigma,\overline{T}}, 0\} \; - \; \max\{\psi_{\sigma,L}, 0\}] \; \leq \; 0 \, .$$

This assertion is true. Indeed we have (since for all $\sigma \, = \, \Gamma_{\overline{T}} \cap \Gamma_L, \psi_{\sigma,\overline{T}} \, + \, \psi_{\sigma,L} \, = \, 0$ holds in virtue of the convection flux continuity):

$$\sum_{\sigma \in \mathcal{E}_{\overline{T}}} U_{\overline{T}} \, [\max\{\psi_{\sigma,\overline{T}}, 0\} \; - \; \max\{\psi_{\sigma,L}, 0\}] \; =$$

$$= \sum_{\sigma \in \mathcal{E}_{\overline{T}}} U_{\overline{T}} \, [\max\{\psi_{\sigma,\overline{T}}, 0\} - \max\{- \, \psi_{\sigma,\overline{T}}, 0\}] \; =$$

$$= \sum_{\sigma \in \mathcal{E}_{\overline{T}}} U_{\overline{T}} \, [\max\{\psi_{\sigma,\overline{T}}, 0\} \; + \; \min\{\psi_{\sigma,\overline{T}}, 0\}]$$

Therefore we get

$$\sum_{\sigma \in \mathcal{E}_{\overline{T}}} U_{\overline{T}} \, [\max\{\psi_{\sigma,\overline{T}}, 0\} \; - \; \max\{\psi_{\sigma,L}, 0\}] \; = \; \sum_{\sigma \in \mathcal{E}_{\overline{T}}} U_{\overline{T}} \; \psi_{\sigma,\overline{T}}$$

i.e.

$$\sum_{\sigma\in\mathcal{E}_{\overline{T}}} U_{\overline{T}}\,[\max\{\psi_{\sigma,\overline{T}},0\} \;-\; \max\{\psi_{\sigma,L},0\}] \;=\; U_{\overline{T}} \sum_{\sigma\in\mathcal{E}_{\overline{T}}} \psi_{\sigma,\overline{T}}$$

In virtue of definition (3.5) it is clear that

$$\sum_{\sigma\in\mathcal{E}_{\overline{T}}} U_{\overline{T}}\,[\max\{\psi_{\sigma,\overline{T}},0\} \;-\; \max\{\psi_{\sigma,L},0\}] \;=\; U_{\overline{T}} \sum_{\sigma\in\mathcal{E}_{\overline{T}}} \int_\sigma \psi\cdot\,\nu_{\sigma,\overline{T}}\,ds$$

It follows from Ostrogradski's theorem (called some times Divergence theorem) that

$$\sum_{\sigma\in\mathcal{E}_{\overline{T}}} U_{\overline{T}}\,[\max\{\psi_{\sigma,\overline{T}},0\} \;-\; \max\{\psi_{\sigma,L},0\}] \;=\; U_{\overline{T}} \int_{\overline{T}} \mathbf{div}(\psi)\,dx$$

Thanks to the assumption (1.6) and since $U_{\overline{T}} \leq 0$, it becomes obvious that

$$\sum_{\sigma\in\mathcal{E}_{\overline{T}}} U_{\overline{T}}\,[\max\{\psi_{\sigma,\overline{T}},0\} \;-\; \max\{\psi_{\sigma,L},0\}] \;\leq\; 0\,.$$

This ends the proof of the Lemma. ∎

It obviously follows from inequalities (4.11) and the preceding Lemma as well that :

$$U_L \;=\; U_{\overline{T}} \qquad \forall\, L\in\mathcal{V}_{\overline{T}}\,. \tag{4.14}$$

For any pair of points from $\mathbb{R}^2$, with Cartesian coordinates x and y, define the subset $[x,y]$ of $\mathbb{R}^2$ in the following way :

$$[x,y] \;=\; \Big\{z\in\mathbb{R}^2 \;/\; \exists 0\leq\theta\leq 1 \text{ such that } z=\theta x+(1-\theta)y\Big\}. \tag{4.15}$$

Let us set:

$$\begin{cases} \mathcal{F}_{\overline{T}} \;=\; \Big\{x_\Gamma\in\Gamma \;/\; \exists\, x_{\overline{T}}\in\overline{T} \;\text{ such that }\; [x_{\overline{T}},x_\Gamma]\subset\overline{\Omega}\Big\} \\ \text{and} \\ \mathcal{S}_{\overline{T}} \;=\; \Big\{[x_{\overline{T}},x_\Gamma] \;/\; x_{\overline{T}}\in\overline{T} \text{ and } x_\Gamma\in\mathcal{F}_{\overline{T}}\Big\}. \end{cases} \tag{4.16}$$

Remark that $\mathcal{F}_{\overline{T}}$ is an infinite set and there is an obvious bijective mapping from $\mathcal{S}_{\overline{T}}$ onto $\overline{T}\times\mathcal{F}_{\overline{T}}$. So $\mathcal{S}_{\overline{T}}$ is also an infinite set. The set $\mathcal{S}_{\overline{T}}$ contains a finite subset $\mathcal{A}_{\overline{T}}$ made up of segments that pass through a mesh vertex or a mesh edge. So its complement $\mathcal{A}^C_{\overline{T}}$ in $\mathcal{S}_{\overline{T}}$ is also infinite. Thus there exists (at least) a segment $\Delta(\widetilde{x}_{\overline{T}},\widetilde{x}_\Gamma)$ from $\mathcal{A}^C_{\overline{T}}$, with extremities $\widetilde{x}_{\overline{T}}\in\overline{T}$ and $\widetilde{x}_\Gamma\in\Gamma$. In the sequel $\Delta(x_{\overline{T}},\widetilde{x}_\Gamma)$ is simply denoted by Δ since there is no risk of confusion. Let us set:

$$\overline{\mathcal{P}}_\Delta \;=\; \Big\{T\in\overline{\mathcal{P}} \;/\; T\cap\Delta\,\neq\,\emptyset\Big\}. \tag{4.17}$$

∘ The first important remark is that $\overline{\mathcal{P}}_\Delta$ contains at least two control volumes namely the control volume $\overline{T}$ belonging to $\mathcal{P}$ and a degenerate control volume T_Γ (belonging to $\partial\mathcal{P}$ of course) such that $\widetilde{x}_\Gamma\in T_\Gamma$.

- ◦ The second important remark straightly coming from (4.14) is that :

$$U_L = U_{\overline{T}} \quad \forall L \in \overline{\mathcal{P}}_\Delta. \tag{4.18}$$

From these two remarks we see that

$$U_{T_\Gamma} = U_{\overline{T}}, \quad \text{with} \quad T_\Gamma \in \partial\mathcal{P}.$$

Therefore we have the following result:

$$T_\Gamma \in \overline{\mathcal{P}}^{min} \cap \partial\mathcal{P}$$

which is in contradiction with the assumption (4.10). The proof of the Positivity Property (3.20) ends here.

◇ We have now to prove (3.21). For this purpose let us assume that there exists a control volume $\overline{T}$ from $\mathcal{P}$ such that

$$U_{\overline{T}} = 0 \equiv \min\{U_B;\ B \in \partial\mathcal{P}\}.$$

We shall deduce that

$$U_T = 0 \qquad \forall T \in \mathcal{P}. \tag{4.19}$$

Let us recall that a subset A of $\mathbb{R}^2$ is connected by polygonal arcs if and only if for any pair of points from A there exists a polygonal line inside $\overline{\Omega}$ joining these two points.

Let T be an *arbitrarily* chosen non degenerate control volumes i.e. $T \in \mathcal{P}$ and let $\mathcal{C}(\overline{T}, T)$ be the set of polygonal lines inside $\overline{\Omega}$ joining $\overline{T}$ to T. It is clear that $\mathcal{C}(\overline{T}, T)$ is an infinite set. Likewise it is clear that the subset of $\mathcal{C}(\overline{T}, T)$ denoted by $\mathcal{D}(\overline{T}, T)$ and made up of polygonal lines passing through a mesh vertex or involving a mesh edge is a finite set. So the complement $\mathcal{D}^C(\overline{T}, T)$ of $\mathcal{D}(\overline{T}, T)$ in $\mathcal{C}(\overline{T}, T)$ is an infinite set. Notice that any polygonal line from $\mathcal{D}^C(\overline{T}, T)$ is associated with a finite family of nondegenerate control volumes. Let us denote by $\Pi(\overline{T}, T)$ a polygonal line from $\mathcal{D}^C(\overline{T}, T)$. So there exists a finite sequence of nondegenerate control volumes $\{T_n\}_{n=1}^N$ associated with $\Pi(\overline{T}, T)$, where the numbering is such that for all $T \in \mathcal{P}$:

$$\begin{cases} T_1 = \overline{T}, \quad T_N = T \\ \text{and} \\ \forall\, 2 \le n \le N-1, \quad T_n \ \text{is adjacent to}\ T_{n-1} \ \text{and}\ T_{n+1}. \end{cases} \tag{4.20}$$

We know from the previous development of this proof that (see (4.14) above):

$$U_{T_n} = U_{T_{n+1}} \qquad \forall\, 1 \le n \le N-1$$

Thus, by transitivity of the equality relation we get what follows:

$$U_{\overline{T}} = U_T \qquad \forall T \in \mathcal{P}.$$

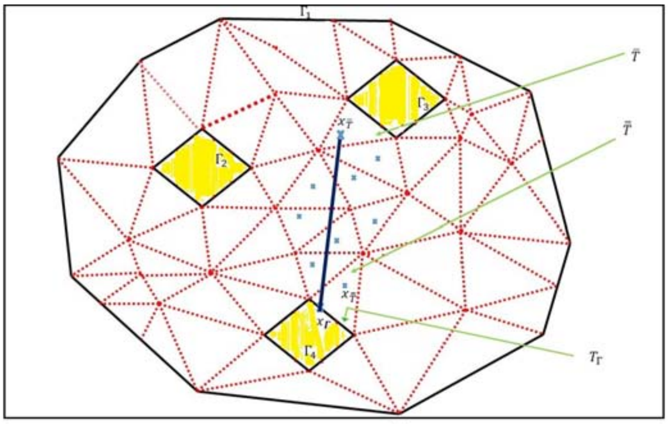

Figure: 1 Illustration of gridding defined over an open bounded subset Ω of $\mathbb{R}^2$, connected by polygonal arcs, with borders $\Gamma_k]_{k\in K}$ surrounding hollows represented by yellow quadrilaterals.

Declaration: We have no con ict of interests to declare.

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