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Nonlinear Analysis as a Calculus

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Abstract

Index terms—

There are two universal methods for local study of nonlinear equations and systems of different kinds (algebraic, ordinary and partial differential): (a) normal form and (b) truncated equations.

(a) Equations with linear parts can be reduced to their normal forms by local changes of coordinates. For algebraic equation, it is Implicit Function Theorem. For systems of ordinary differential equations (ODE), I completed the theory of normal forms, began by **??**oincaré (1879) [Poincaré, 1928] and **??**ulac (1912) **??**Dulac, 1912] for general systems **??**Bruno, 1964; **??**971] and began by **??**irkhoff (1929) [Birkhoff, 1966] for Hamiltonian systems **??**Bruno, 1972;1994].

(b) Equations without linear part: I proposed to study properties of solutions to equations (algebraic, ordinary differential and partial differential) by studying sets of vector power exponents of terms of these equations. Namely, to select more simple ("truncated") equations **??**Bruno, 1962;1989;2000] by means of generalization to polyhedrons the **??**ewton (1678) [Newton, 1964] and the **??**adamard (1893) **??**Hadamard, 1893] polygons.

By means of power transformations **??**Bruno, 1962;1989; **??**022b] the normal forms and the truncated equations can be strongly simplified and often solved. Solutions to the truncated equations are asymptotically the first approximations of the solutions to the full equations. Continuing that process, we can obtain then **??**2.1)

1 II. SINGLE ALGEBRAIC EQUATION

2 The implicit function theorem:

London Journal of Research in Science: Natural and Formal Theorem 2.1. Let f (X, ?, T) = ?a Q,r (T)X Q ? r , where 0 ? Q ? Z n , 0 ? r ? Z, the sum is finite and a Q,r (T) are some functions of T = (t 1 , . . . , t m), besides a 00 (T) ? 0, a 01 (T) ? ? 0. Then the solution to the equation f (X, ?, T) = 0 has the form? = ?b R (T)X R def = b(T, X),

where 0 ? R ? Z n , 0 < ?R?, the coefficients b R (T) are functions on T that are polynomials from a Q,r (T) with ?Q? + r ? ?R? divided by a

2?R?-1 01

. The expansion b(T, X) is unique. Let g(X, ?, T) = f (X, ? + b(T, X), T), **??**2.2) then g(X, 0, T) ? 0.

This is a generalization of Theorem 1.1 of **??**Bruno, 2000, Ch. II] on the implicit function and simultaneously a theorem on reducing the algebraic equation **??**2.1) to its normal form **??**2.2) when the linear part a 01 (T) ? ? 0 is nondegenerate. In it, we must exclude the values of T near the zeros of the function a 01 (T).

Let the point X 0 = 0 be singular. Write the polynomial in the formf (X) = ?a Q X Q , where a Q = const ? R, or C. Let S(f) = {Q : a Q ? = 0}.

The set S is called the support of the polynomial f (X). Let it consist of points Q 1 , . . . , Q k . The convex hull of the support S(f) is the set **??**2.3) which is called Newton's polyhedron.

Its boundary ?Î?"(f) consists of generalized faces Î?" (d) j , where d is its dimension of 0 ? d ? n -1 and j is the number. Each (generalized) face Î?" (d) j corresponds to its:

? boundary subset S (d)j = S ? Î?" (d) j , ? truncated polynomial f (d) j (X) = ?a Q X Q over Q ? S (d) j ,

? and normal coneU (d) j = P : ?P, Q ? ? = ?P, Q ?? ? > ?P, Q ??? ?, Q ? , Q ?? ? S (d) j , Q ??? ? S\S (d) j

, **??**2.4) where P = (p 1 , . . . , p n) ? R n Let X = (x 1 , . . . , x n) ? R n or C n , and f (X) be a polynomial. A point X = X 0 , f (X 0) = 0 is called simple if the vector (?f /?x 1 , . . . , ?f /?x n) in it is non-zero. Otherwise, the point X = X 0 is called singular or critical. By shifting X = X 0 + Y we move the point X 0 to the origin Y = 0. If at this point the derivative ?f /?x n ? = 0, then near X 0 all solutions to the equation f (X) = 0 have the form y n = ?b q 1 ,...q n-1 y q1 1 ? ? ? y q n-1 n-1 , that is, lie in (n -1)-dimensional space. Î?”(f) = Q = k j=1 µ j Q j , µ j ? 0, k j=1 µ j = 1 ,

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At X ? 0 solutions to the full equation f (X) = 0 tend to non-trivial solutions of those truncated equations f (d) j (X) = 0 whose normal cone U (d) j intersects with the negative orthant P ? 0 in R n * .

Remark 1. If in the sum **??**2.1) all Q belong to a forward cone C:?Q, K i ? > c i , i = 1, . . . , m,

then in the solution (2.2) of Theorem 2.1 all R belong to the same cone C: **??**Bruno, 1989, Part I, Chapter 1, § 3].?Q, K i ? > c i , i = 1, . . . , m,

Let ln X def =(ln x 1 , . . . , ln x n). The linear transformation of the logarithms of the coordinates(ln y 1 , . . . , ln y n) def = ln Y = (ln X)?,

(2.5) **??**Bruno, 1962;2000: where ? is a nondegenerate square n-matrix, is called power transformation.

3 Power transformations

By the power transformation (2.5), the monomial X Q tranforms into the monomial Y R , where R = Q (? *) -1 and the asterisk indicates a transposition.

A matrix ? is called unimodular if all its elements are integers and det ? = ±1. For an unimodular matrix ?, its inverse ? -1 and transpose ? * are also unimodular.

Theorem 2.2. For the face Î?” (d) j there exists a power transformation (2.5) with the unimodular matrix ? which reduces the truncated sumf (d) j (X) to the sum from d coordinates, that is, f (d) j (X) = Y S ?(d) j (Y), where ?(d) j (Y) = ?(d) j (y 1 , . . . , y d

) is a polynomial. Here S ? Z n . The additional coordinates y d+1 , . . . , y n are local (small).

The article **??**Bruno, Azimov, 2023] specifies an algorithm for computing the unimodular matrix ? of Theorem 2.2.

4 Let Î?” (d) j

be a face of the Newton polyhedron Î?”(f). Let the full equation f (X) = 0 is changed into the equation g(Y) = 0 after the power transformation of Theorem 2.2. Thus ?(d) j (y 1 , . . . , y d) = g(y 1 , . . . , y d , 0, . . . , 0).

5 Parametric expansion of solutions:

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Let the polynomial ?j be the product of several irreducible polynomials?(d) j = m k=1 h l k k (y 1 , . . . , y d),**(2.6)**

where 0 < l k ? Z. Let the polynomial h k be one of them. Three cases are possible:

Case 1. The equation h k = 0 has a polynomial solution y d = ?(y 1 , . . . , y d-1). Then in the full polynomial g(Y) let us substitute the coordinatesy d = ? + z d ,

for the resulting polynomial h(y 1 , . . . , y d-1 , z d , y d+1 . . . , y n) again construct the Newton polyhedron, separate the truncated polynomials, etc. Such calculations were made in [Bruno, Batkhin, 2012] and were shown in **??**Bruno, 2000, Introduction].

Case 2. The equation h k = 0 has no polynomial solution, but has a parametrization of solutionsy j = ? j (T), j = 1, . . . , d, T = (t 1 , . . . , t d-1).

Then in the full polynomial g (Y) we substitute the coordinatesy j = ? i (T) + ? j ?, j = 1, . . . , d,**(2.7)**

where ? j = const, ? |? j | ? = 0, and from the full polynomial g(Y) we get the polynomialh = ?a Q ?? ,r (T)Y ?? Q ?? ? r ,**(2.8)**

whereY ?? = (y d+1 , . . . , y n), 0 ? Q ?? = (q d+1 , . . . , q n) ? Z n-d , 0 ? r ? Z. Thus a 00 (T) ? 0, a 01 (T) = d j=1 ? j ?? (d) j /?y j (T).

If in the expansion (5.7) l k = 1, then a 01 ? ? 0. By Theorem 2.1, all solutions to the equation h = 0 have the form i.e., according to (2.7) the solutions to the equation g = 0 have the form? = ?b Q ?? (T)Y ?? Q ?? , London Journal ofy j = ? j (T) + ? j ?b Q ?? (T)Y ?? Q ?? , j = 1, . . . , d.

Such calculations were proposed in [Bruno, 2018a].

If in (5.7) l k > 1, then in (2.8) a 01 (T) ? 0 and for the polynomial (2.8) from Y ?? , ? we construct a Newton polyhedron by support S(h) = {Q ?? , r : a Q ?? ,r (T) ? ? 0}, separate the truncations and so on.

Case 3. The equation h k = 0 has neither a polynomial solution nor a parametric one. Then, using Hadamard's polyhedron [Bruno, 2018a; **??**019a], one can compute a piece-wise approximate parametric solution to the equation h k = 0 and look for an approximate parametric expansion.

Similarly, one can study the position of an algebraic manifold in infinity.

Here we consider an ordinary differential equation of the formf x, y, y ? , . . . , y (n) = 0,**(3.1)**

where x is independent variable, y is the dependent variable, y ? = dy/dx and f is a polynomial of its arguments. Near x 0 = 0 or ? we look for solutions of equation **??**3.1) in the form of asymptotic seriesy = ? k=1 b k x s k ,**(3.2)**

III. SINGLE ODE [BRUNO, 2004] 3.1. Setting of the problem:

where b k are functions of log x and ?s k > ?s k+1 with? = -1, if x 0 = 0, 1, if x 0 = ?. (3.3)

We set X = (x, y). By a differential monomial a(x, y) we mean the product of an ordinary monomial To every differential monomial a (X) one assigns its (vector) exponent Q(a) = (q 1 , q 2) ? R 2 by the following rules. For a monomial of the form **??**3.4) let **??** 1 , **??** 2); for a derivative of the form (3.5) let Q d l y/dx l = (-l, 1).cx r 1 y r 2 def = cX R ,(**3.4**Q cX R = R, that is, Q (cx r 1 y r 2) = (

When differential monomials are multiplied, their exponents are summed as vectors:Q (a 1 a 2) = Q (a 1) + Q (a 2) .

The set S(f) of exponents Q (a i) of all the differential monomials a 2 (X) in a differential sum of the form **??**3.6) is called the support of the sum f (X). Obviously, S(f) ? R 2 . The closure Î?”(f) of the convex hull of the support S(f) is referred to as the polygon of the sum f (X). The boundary ?Î?”(f) of the polygon Î?”(f) consists of vertices Î?” f (d) j (X) = a i (X) over Q (a i) ? S (d) j . (3.7)

Let R 2 * be the plane conjugate to the plane R 2 so that the inner (scalar) product?P, Q? def = p 1 q 1 + p 2 q 2

is defined for anyP = (p 1 , p 2) ? R 2 * and Q = (q 1 , q 2) ? R 2 . Corresponding to any face Î?” (d) j are its normal cone, U (d) j = P : ?P, Q? = ?P, Q ? ? , Q, Q ? ? S (d) j ?P, Q? > ?P, Q ?? ? , Q ?? ? S(f)\S (d) j

and the truncated sum (3.7). All these constructions are applicable to equation **??**3.1), where f is a differential sum.

Let x ? 0 or x ? ? and suppose that a solution of the equation (3.1) has the formy = c r x r + o |x| r+? ,(**3.8)**

where c r is a coefficient, c r = const ? C, c r ? = 0, the exponents r and ? are in R, and ?? < 0. Then we say that the expressiony = c r x r , c r ? = 0 (3.9)

gives the power-law asymptotic form of the solution (3.8).

Thus, corresponding to any faceÎ?” (d) j are the normal cone U (d) j in R 2

* and the truncated equation f (d) j (X) = 0. (**3**= 0. We set g(X) def = X -Q f (0) j (X).

Then the solution (3.7), (3.10) satisfies the equation

6 Solution of the truncated equation:

London Journal of Research in Science: Natural and Formal g(X) = 0

Substituting y = cx r into g(X), we see that g (x, cx r) does not depend on x, c and is a polynomial in r, that is,g (x, cx r) ? ?(r),

where ?(r) is the characteristic polynomial of the differential sum f (0) j (X). Hence, in a solution (3.9) of the equation (3.10) the exponent r is a root of the characteristic equation **??**3.11) and the coefficient c r is arbitrary. Among the roots r i of the equation **??**3.11), one must single out only those for which one of the vectors ? **??**1, r), where ? = ±1, belongs to the normal cone U (0)?(r) def = g (x, x r) = 0,j of the vertex Î?” (0) j .

In this case the value of ? uniquely determined. The corresponding expressions of the sum with an arbitrary constant c r are candidates for the role of truncated solutions of the equation **??**3.1). Moreover, by **??**3.3), if ? = -1, then x ? 0, and if ? = 1, then x ? ?.

Complex roots r to characteristic equation **??**3.11) may bring to exotic expansions of solutions **??**3.2), where coefficients b k are power series in x ?i with real ? ? R and i 2 = -1.

7 Corresponding to an edge Î?”

(1)j is a truncated equation (3.10) with d = 1 whose normal cone U (1) j is a ray {?N j , ? > 0}. If ?(1, r) ? U (1)

j , this condition uniquely determines the exponent r of the truncated solution (3.9) and the value ? = ±1 in **??**3.3). To find the coefficient c r , one must substitute the expression (3.9) into the truncated equation (3.10). After cancelling a factor which is a power of x, we obtain an algebraic defining equation for the coefficient c r ,f (c r) def = x -s f (1) j (x, c r x r) = 0 Corresponding to every root c r = c (i)

r ? = 0 of this equation is an expression of the form (3.9) which is a candidate for the role of a truncated solution of the equation **??**3.1). Moreover, by **??**3.3), if in the normal cone U

(1) j one has p 1 < 0, then x ? 0, and if p 1 > 0, then x ? ?. From the polygon Î?” of the initial equation (3.1) we take a vertex or an edge Î?” (d) j . Then we found a power solution y = b 1 x P 1 of the truncated equation f (d) j (X) = 0, as it was described above, puty = b 1 x P 1 + z and obtain new equation g(x, z) = 0.

We construct the polygon 1 Î?” for the new equation, take a vertex or an edge 1 Î?” (e) k , solve the truncated equation?(e) k (x, z) = 0,

and obtain the second term b 2 x P 2 of expansion (3.2) and so on.

8 Computation of solution to equation (3.1) as expansion (3.2)

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We construct the polygon 1 Î?” for the new equation, take a vertex or an edge 1 Î?” (e)

k , solve the truncated equation?(e) k (x, z) = 0,

and obtain the second term b 2 x P 2 of expansion (3.2) and so on.

In [Bruno, 2004] there are some properties, that simplify computation. Thus, we can obtain the 4 types of expansions (3. **??**Bruno, 2006; **??**018b]; 4. Exotic, when all b k are power series in x i? [Bruno, 2007].

Except expansions (3.2) of solutions y(x) of equation **??**3.1), there are exponential expansionsy = ? k=1 b k (x) exp [k?(x)] ,

where b k (x) and ?(x) are power series in x **??**Bruno, 2012a,b].

Also there are solutions in the form of transseries [Bruno, 2019b]. These results were applied to 6 Painlevè equations [Bruno, 2015; **??**018b,c; **??**runo, Goruchkina, 2010]. Written as differential sums they are: Equation P 5 :Equation P 1 : f (x, y) def = -y ?? + 3y 2 + x = 0. Equation P 2 : f (x,f (z, w) def = -z 2 w(w -1)w ?? + z 2 3 2 w - 1 2 (w ?) 2 -zw(w -1)w ? + + (w -1) 3 (?w 2 + ?) + ?zw 2 (w -1) + ?z 2 w 2 (w + 1) = 0. Equation P 6 : f (x, y) def = 2y ?? x 2 (x -1) 2 y(y -1)(y -x) -(y ?) 2 [x 2 (x -1) 2 (y -1)(y -x)+ + x 2 (x -1) 2 y(y -x) + x 2 (x -1) 2 y(y -1)]+ + 2y ? [x(x -1) 2 y(y -1)(y -x) + x 2 (x -1)y(y -1)(y -x)+ + x 2 (x -1) 2 y(y -1)] -[2?y 2 (y -1) 2 (y -x) 2 + 2?x(y -1) 2 (y -x) 2 + + 2?(x -1)y 2 (y -x) 2 + 2?x(x -1)y 2 (y -1) 2] = 0.

Here a, b, c, d and ?, ?, ?, ? are complex parameters. If all they are nonzero, then polygons for these equations are shown in Figures 1,2,3.

9 Supports and polygons for equations

q 2 q 1 0 1 1 P 1 q 2 q 1 0 1 1 P 2

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Then there exists such power series ?(x) with integral increasing exponents, that after substitutiony = z + ?(x) (3.13) the transformed differential sum g(x, z) = f (x, z + ?(x)) (3.14) for z = z ? = ? ? ? = z (n) = 0 (3.15)

has only resonant terms b m x m , wherem = v + ? k ? Z (3.16)

and m ? ?.

So here the eigenvalue? k is resonant if ? -v ? ? k ? Z.

3) truncated differential sum f (0) 1 (X) have eigenvalues ? 1 , . . . , ? l , 0 ? l ? n; 4) the most left point of the support S(f) in the axis q 2 = 0 be (?, 0). Evidently ? ? Z.

Supports and polygons for equations P 3 (left), P 4 (right). -1 0 1 q 1 q 2 P 3 q 2 q 1 0 1 1 P 4**2**

Nonlinear Analysis as a Calculus

Theorem 3.3. Let 1) f x, y, y ? , . . . , y (n) be a polynomial in x, y, y ? , . . . , y (n) ; 2) its Newton polygon Î?"(f) have a vertex Î?" (0) 1 = (v, 1) at the right side of its boundary ?Î?";

3) truncated differential sum f (0) j (X) have eigenvalues ? 1 , . . . , ? l , 0 ? l ? n; 4) the most right point of the support S(f) in the axis q 2 = 0 be (?, 0). Evidently ? ? Z.

Then there exists such power series ?(x) with integral decreasing exponents, that after substitution (3.13), the differential sum (3.14) for identities **??**3.15) has only resonant terms b m x m , where equality **??**3.16) is true, and m ? ?. f (0) j (X) has no integral eigenvalue ? k ? ?-v (for Theorem 3.2) or ? k ? ? -v (for Theorem 3.3), then the initial equation f (X) = 0 has formal solution y = ?(x). If the truncated sum f (0) j (X) contains the derivation y (n) , then the series ?(x) converges according to Theorem 3.4 in [Bruno, 2004].

10 So here the eigenvalue ?

k is resonant if ? -v ? ? k ? Z. Equations g(x, z) = 0 for (3.

Remark 2. If the truncated sum f (0) j (X) has integral eigenvalue ? k ? ?v (for Theorem 3.2) or ? k ? ? -v (for Theorem 3.3), then the initial equation f (X) = 0 Supports and polygons for equations P 5 (left), P 6 (right). q 2 q 1 0 1 1 P 5 q 2 q 1 0 1 1 P 6

Nonlinear Analysis as a Calculus

We will consider such a generalization of the power function cx r which preserves their main properties. The real numberp ? (?(x)) = ? lim x ? ?? log |?(x)| ? log |x| ,

where arg x = const ? [0, 2?), is called the order of the function ?(x) on the ray when x ? 0 or x ? ?. The order p ? (?) is not defined on the ray arg x = const, where the limit point x = 0 or x = ? is a point of accumulation of poles of the function ?(x).

In Subsections 3.2-3.4 it was shown that as x ? 0 (? = -1) or as x ? ? (? = 1) solutions y = ?(x) to the ODE f (x, y) = 0, where f (x, y) is a differential sum, can be found by means of algorithms of Plane PG, ifp ? (?(x)) -l = p ? d l ?/dx l , l = 1, . . . , n,

where n is the maximal order of derivatives in f (x, y). Here we introduce algorithms, which allow calculate solutions y = ?(x) with the propertyp ? (?(x)) + l? ? = p ? d l ?/dx l , l = 1, . . . , n, where ? ? ? R, ? = ±1. Lemma 3.3.1. If p ? (?(x)) = -? ? + p ? (? ? (x)) = -2? ? + p ? (? ?? (x)), then ? + ?? ? ? 0.

Note, that in Plane PG we had ? ? = -1, i. e. ? +?? ? = 0. So, new interesting possibilities correspond to ? + ?? ? > 0.

We consider the ODEf (x, y) = i a i (x, y) = 0,

where f (x, y) is a differential sum. To each differential monomial a i (x, y), we assign its (vector) power exponent Q(a i) = (q 1 , q 2 , q 3) ? R 3 by the following rules: power exponent of the product of differential monomials is the sum of power exponents of factors:Q(a 1 a 2) = Q(a 1) + Q(a 2).

The set S(f) of power exponents Q(a i) of all differential monomials a i (x, y) presented in the differential sum f (x, y) is called the space support of the sum f (x, y). Obviously, S(f) ? R 3 . The convex hull Î?"(f) of the support S(f) is called the polyhedron of the sum f (x, y). The boundary ?Î?”(f) of the polyhedron Î?”(f) consists of the vertices Î?” (0) j , the edges Î?” f (d) j (x, y) = a i (x, y) over Q(a i) ? Î?” (d) j ? S(f).

Support and polyhedron for equation P 1 . The approach allows to obtain solutions with expansions **??**3.2), where coefficients b k (x) are all periodic or all elliptic functions **??**Bruno, 2012c,d; **??**runo, Parusnikova, 2012].

Expansions of solutions to more complicated equations such as hierarchies Painlevé see in **??**Anoshin, Beketova, (et al.), 2023; Bruno, For P 1 -P 5 with all parameters nonzero, their polyhedrons are shown in **??**igures 4, **??**, **??**, **??**, **??** correspondingly.

Here we consider the system ?i = f i (X), i = 1, . . . , n, **??**4.1) where?= d/d t, X = (x 1 , . . . , x n) ? C n or R n , all f i (X) are polynomials from X. A point X = X 0 = const is called singular if all f i (X 0) = 0, i = 1, . . . , n.

Let the point X 0 = 0 be a singular point. Then the system (4.1) has the linear part? = XA,

where A is a square n-matrix. Let ? = (? 1 , . . . , ? n) be a vector of its eigenvalues.

Theorem **??**.1 ([Bruno, 1964; **??**971 **??** 1972]). There exists an invertible formal change of coordinatesx i = ? i (Y), i = 1, . . . , n,

where ? i (Y) are power series from Y = (y 1 , . . . , y n) without free terms, which reduces the system (4.1) to normal form?i = y i g i (Y) = y i Y Q , i = 1, . . . , n,**(4.2)**

IV. AUTONOMOUS ODE SYSTEM

11 Normal form:

Support and polyhedron for equation P 2 . Here y i g i (Y) are power series on Y without free terms.Let N i = {Q ? Z n : q j ? 0, j ? = i, q i ? -1} , i = 1, . . . , n,

andN = N 1 ? N 2 ? ? ? ? ? N n .

Then the number k of linearly independent Q ? N satisfying the equation (4.3) is called multiplicity of resonance.

Theorem 4.2. Let k be the multiplicity of resonance of the system (4.1). Then there exists a power transformationln Z = (ln Y) ?

with unimodular matrix ? which reduces the normal form (4.2), (4.3) to the system(ln z i) = h i (y 1 , . . . , y k), i = 1, . . . , n,

in which the first k coordinates form a closed subsystem without a linear part, and the remaining nk coordinates are expressed via them by means of integrals.

Thus, if ? ? = 0, then the original system (4.1) of order n can be reduced to a system of order k, but without the linear part. Support and polyhedron for equation P 3 .

12 Figure 6:

Let’s write the system (4.1) as **??**4.4) and put A Q = (a 1Q , . . . , a nQ).(ln x i) = a iQ X Q , i = 1, . . . , n,

The setS = {Q : A Q ? = 0}

is called the support of the system **??**4.4). Its convex hull Î?” (2.3) is its Newton’s polyhedron. Its boundary ?Î?” consists of generalized faces Î?” ? boundary subset S (d) j = Î?” (d) j ? S, ? truncated system (ln X) = Â(d) j (X) = A Q X Q over Q ? S (d) j , (4.5) ? normal cone U (d) j ? R n * (2.4) and ? tangent cone T (d) j .

According to **??**Bruno, 2000, Chapt. 1, §2] let d > 0 and Q be the interior point of a face Î?” (d) j , that is, Q does not lie in a face of smaller dimension. If d = 0, then 4.2: Newton’s polyhedron **??**Bruno, 1962;2000]. Support and polyhedron for equation P 4 . Q = Î?” (0) j . The conic hull of the set S -Q T (d) j = Q = µ 1 Q 1 -Q + ? ? ? + µ k Q k -Q , µ 1 , . . . , µ k ? 0, Q 1 , . . . , Q k ? S is called the tangent cone of the face Î?” (d) j , 0 ? d ? n -1, T (d) j ? R n .d ? = X R d t,

R ? Z n , which reduce the system (4.4) to the formd (ln Y) /d ? = B(Y),**(4.6)**

where the systemd (ln Y) /d ? = B(d) j (Y) ? B(d) j (y 1 , . . . , y d) = B(y 1 , . . . , y d , 0, . . . , 0),**(4.7)**

corresponds to the truncated system (4.5). be singular for the truncated system (4.7). Near the point (4.8), the local coordinates arez i =y i -y 0 i , i = 1, . . . , d, z j =y j , j = d + 1, . . . , n.

Let at the point Z = (z 1 , . . . , z n) = 0 the eigenvalues of the matrix of the linear part of the system (4.7) are ? = ?1 , . . . , ?n , where ?1 , . . . , ?d are the eigenvalues of the subsystem of the first d equations.

Theorem 4.4. There exists an invertible formal change of coordinates **??**Bruno, 2022b]: where W = (w 1 , . . . , w n) which reduces the system (4.6) to the generalized normal formz i = ? i (W), i = 1, . . . , n,

13 Generalized normal form

?i = w i c i (W) = w i c iQ W Q , i = 1, . . . , n,**(4.9)**

whereQ, ? = 0 and Q ? T (d) j ? Z n .

(4.10)Here ? i = w i ? iQ W Q , i = 1, . . . , n, where Q ? T (d) j ? Z n .

The system (4.9), (4.10) is reduced to a system of lower order by the power transformation of Theorem 4.2 .

Let X = X 0 be a singular point of the sys tem **??**4.1). Two cases are possible: Case 1. ? ? = 0, then by Theorem 4.1 we reduce the system to a normal form, then by Theorem 4.2 we reduce the normal form to a subsystem of order k < n without linear part and obtain the problem of studying its singular points.

Case 2. ? = 0, then we compute the Newton polyhedron and separate truncated systems in which the normal cone U (d) j intersects the negative orthant of P ? 0. Each of them is reduced to the form (4.6), (4.7) by the transformation of Theorem 4.3. For each singular point (4.8), we apply Theorem 4.4 and obtain a subsystem of smaller order.

Continuing this branching process, after a finite number of resolution of singularities we come to an explicitly solvable system from which we can understand the nature of solutions of the original system. But Theorem 4.3 can be applied to the original system (4.1), i.e. to each of the generalized faces Î?" (d) j of its Newton polyhedron Î?". Then to each singular point (4.8) we apply Theorems 4.4, 4.2 and reduce the order of the system. Here also through a finite number of steps of the singularity resolution we come to an explicitly solvable system. This allows us to study the singularities of the original system in infinity. This is the basis of the integrability criterion in [Bruno, Enderal, 2009;Bruno, Enderal, Romanovski, 2017].

The normal form can be computed in the neighborhood of a periodic solution or invariant torus **??**Bruno, 1972, II, §11], **??**Bruno, 2022a].

See **??**Bruno, Batkhin, 2023] for similar computations for a system of partial differential equations.

14 Analysis of singularities: tem

and is defined by one Hamiltonian function H(x, y), where x = (x 1 , . . . , x n), y = (y 1 , . . . , y m). Here the normal form of the system (4.11) corresponds to the normal form of one Hamiltonian function. See details in [Bruno, Batkhin, 2021].

Let X = (x 1 , . . . , x n) ? C n or R n independent variables and y ? C or R be a dependent one. Consider Z = (z 1 , . . . , z n , z n+1) = (x 1 , . . . , x n , y).

Differential monomial a(Z) is called a product of an ordinary monomialcZ R = cz r 1 1 ? ? ? z r n+1 n+1 ,

where c = const, and a finite number of derivatives of the following form? l y ?x l 1 1 ? ? ? ? l n x n def = ? l y ?X L , 0 ? l j ? Z, n j=1 l j = l, L = (l 1 , . . . , l n) .

Vector power exponent Q(a) ? R n+1 corresponds to the differential monomial a(Z), it is constructed according to the following rules:Q(c) = 0, if c ? = 0, Q Z R = R, Q ? l y j /?X L = (-L, 1).

The product of monomials corresponds to the sum of their vector power exponents:Q(ab) = Q(a) + Q(b).

Differential sum is the sum of differential monomials V. ONE PARTIAL DIFFERENTIAL EQUATION 5.1. Support [Bruno, 2000 Ch. be 6-8]:f (Z) = a k (Z). (**5**

Let the support S(f) of the differential sum (5.1) consists of one point E n+1 = (0, . . . , 0, 1). Then the substitution y = cX P , P = (p 1 , . . . , p n) ? R n

(5.2) in the differential sum f (Z) gives the monomial c? P (P)X P

where ? P (P) is a polynomial of P which coefficients depend on P .

Monomial (5.2) will be called resonant for f (Z) if for it ? P (P) = 0.

Let µ k be the maximal order of the derivative over x k in f (Z), k = 1, . . . , n.If in P = (p 1 , . . . , p n) p k ? µ k , k = 1, . . . , n, (5.3) then f (Z) = c?(P)X P ,

where ?(P) is the characteristic polynomial of the sum of f (Z) and its coefficients do not depend on P . But if the inequalities (5.3) are not satisfied, then ? P (P) ? = ?(P).Example. Let n = 2, f (Z) = x 1 ?y ?x 1 + x 2 2 ? 2 y ?x 2 2 . If P = (1, 1), then f (x 1 , x 2 , cx 1 x 2) = cx 1 x 2 . If P = (1, 2), then f (x 1 , x 2 , cx 1 x 2 2) = c x 1 x 2 2 + x 1 ? x 2 2 ? 2 = c ? 3x 1 x 2 2 . Generally here for p 1 ? 1, p 2 ? 2 we have f (x 1 , x 2 , cx 1 x 2) = c[p 1 + p 2 (p 2 - 1)]x p 1 1 x p 2 2 and ?(P) = p 1 + p 2 (p 2 -1).

For a differential sum f (Z) we denote by f k (Z) the sum of all differential monomials of f (Z) which have n + 1 coordinate q n+1 of vector power exponents Q = (q 1 , . . . , q n , q n+1) equal to k:q n+1 = k. Denote Z n + = {P : 0 ? P ? Z n }.

Consider the PDE f (Z) = 0.

(5.4)

15 Normal form:

5.2. Resonant monomials:

1. f 0 (Z) = ?(X) is a power series from X without a free term, 2. f 1 (Z) = a(Z)+b(Z), where S(a) = E n+1 = (0, . . . , 0, 1), S(b) ? Z n+1 + \0 × {q n+1 = 1}.

Then there exists a substitution y = ? + (X), where (X) is a power series from X without a free term, which transforms the equation (5.4) to the normal form g(X, ?) = 0,

(5.5)

where g 0 (X) = c P X P is a power series without a free term, P ? Z n + containing only resonant monomials c P X P for sum a(Z). is the formal solution to the equation (5.4).

If in equation (5.4) differential sum does not contain derivatives, thena(Z) = const ? z n+1 = const ? y.

Closure of a convex hull where the space R n+1 * is conjugate to the space R n+1 , ??, ?? is the scalar product, and truncated sumÎ?"(f) = Q = ? j Q j , Q j ? S, ? j ? 0, ? j = 1 of the support S(f) is called the polyhedron of sum f (Z). The boundary ?Î?" of the polyhedron Î?"(f) consists of generalized faces Î?" (d) j , where d = dim Î?" (d) j . Each face Î?" (d) j corresponds to normal cone U (d) j = P ? R n+1 * : ?P, Q ? ? = ?P, Q ? ? > ?P, Q ? ?, where Q, Q ? ? Î?" (d) j , Q ? ? Î?"\Î?"f (d) j (Z) = a k (Z)by Q(a k) ? Î?" (d) j S.

Consider the equationf (Z) = 0, (5.6)

where f is the differential sum. In the solution of equation (5.6)y = ?(X), (5.7)

where ? is a series on the powers of x k and their logarithms, the series ? corresponds to its support, polyhedron, normal cones u i and truncations. The logarithm ln x i has a zero power exponent on x i . The truncated solution y = ? corresponds to the normal coneu ? R n+1 * .

Theorem 5.2. If the normal cone u intersects with the normal cone (5.2), then the truncation y = ?(X) of the solution (5.3) satisfies the truncated equationf (d) j (Z) = 0.

(5.8)

To simplify the truncated equation (5.8), it is convenient to use a power transformation. Let ? be a square real nondegenerate block matrix of dimension n + 1 of the form? = ? 11 ? 12 0 ? 22 ,

where ? 11 and ? 22 are square matrices of dimensions n and 1, respectively. We denote ln Z = (ln z 1 , . . . , ln z n+1), and by the asterisk * we denote transposition.

Variable change. ln W = (ln Z) ? (5.9) is called the power transformation.

Theorem 5. **??** ([Bruno, 2000]). The power transformation (5.5) reduces a differential monomial a(Z) with a power exponent Q(a) into a differential sum b(W) with a power exponent Q(b):R = Q(b) = Q(a)? -1 * .

16 Power transformations:

Corollary 5.3.1. The power transformation (5.9) reduces the differential sum (2.1) with support S(f) to the differential sum g(W) with support S(g) = S(f)? -1 * , i.e.

17 S(f) = S(g)? *

Theorem 5.4. For the truncated equationf (d) j (Z) = 0

there is a power transformation (5.9) and monomial Z T that translates the equation above into the equationg(W) = Z T fj (Z) = 0,

where g(W) is a differential sum whose support has n + 1d zero coordinates.

Let z j be one of the coordinates x k or y. Transformation ? j = ln z j is called logarithmic.

Theorem 5.5. Let f (Z) be a differential sum such that all its monomials have a jth component q j of the vector exponent of degree Q = (q 1 , . . . , q m+n) equal to zero, then the logarithmic transformation (5.1) reduces the differential sum f (Z) into a differential sum from z 1 , . . . , ? j , . . . , z n .

18 Logarithmic transformation:

A truncated equation 5.7. Calculating asymptotic forms of solutions: f (n) j (Z) = 0 is taken. If it cannot be solved, then a power transformation of the Theorem 5.4 and then a logarithmic transformation of the Theorem 5.5 should be performed. Then a simpler equation is obtained. In case it is not solvable again, the above procedure is repeated until we get a solvable equation. Having its solutions, we can return to the original coordinates by doing inverse coordinate transformations. So the solutions written in original coordinates are the asymptotic forms of solutions to the original equation (5.2).

In **??**Bruno, Batkhin, 2023] method of selecting truncated equations was applied to systems of PDE.

Traditional approach to PDE see in [Oleinik, Samokhin, 1999;Polyanin, Zhurov, 2021].

Here we provide a list of some applications in complicated problems of (c) Mathematics, (d) Mechanics, (e) Celestial Mechanics and (f) Hydromechanics.

(c) In Mathematics: together with my students I found all asymptotic expansions of five types of solutions to the six Painlevé equations **??**1906) **??**Bruno, 2018c; **??**runo, Goruchkina, 2010] and also gave very effective method of determination of integrability of ODE system [Bruno, Enderal, 2009;Bruno, Enderal, Romanovski, 2017].

(d) In Mechanics: I computed with high precision influence of small mutation oscillations on velocity of precession of a gyroscope [Bruno, 1989] and also studied values of parameters of a centrifuge, ensuring stability of its rotation [Batkhin, Bruno, (et al.), 2012].

(e) In Celestial Mechanics: together with my students I studied periodic solutions of the Beletsky equation (**??**956) [Bruno, 2002;Bruno, Varin, 2004], describing motion of satellite around its mass center, moving along an elliptic orbit. I found new families of periodic solutions, which are important for passive orientation of the satellite [Bruno, 1989], including cases with big values of the eccentricity of the orbit, inducing a singularity. Besides, simultaneously with [Hénon, 1997], I found all regular and singular generating families of periodic solutions of the restricted three-body problem and studied bifurcations of generated families. It allowed to explain some singularities of motions of small bodies of the Solar System [Bruno, Varin, 2007]. In particular, I found orbits of periodic flies round planets with close approach to the Earth **??**Bruno, 1981].

(f) In Hydromechanics: I studied small surface waves on a water **??**Bruno, 2000, Chapter 5], a boundary layer

6

Figure 1:) 6 ©

on a needle **??**Bruno, Shadrina, 2007], where equations of a flow have a singularity, and an one-dimensional model of turbulence bursts **??**Bruno, Batkhin, 2023]. [1] [2]

Figure 2: j

Figure 3:

Figure 4:

Figure 5:

Figure 6:

1

Figure 7: Figure 1 :

2

Figure 8: Figure 2 :

Figure 9:

3

Figure 10: Figure 3 :

Figure 11:

Figure 12:

4

Figure 13: Figure 4 :

5

Figure 14: Figure 5 :

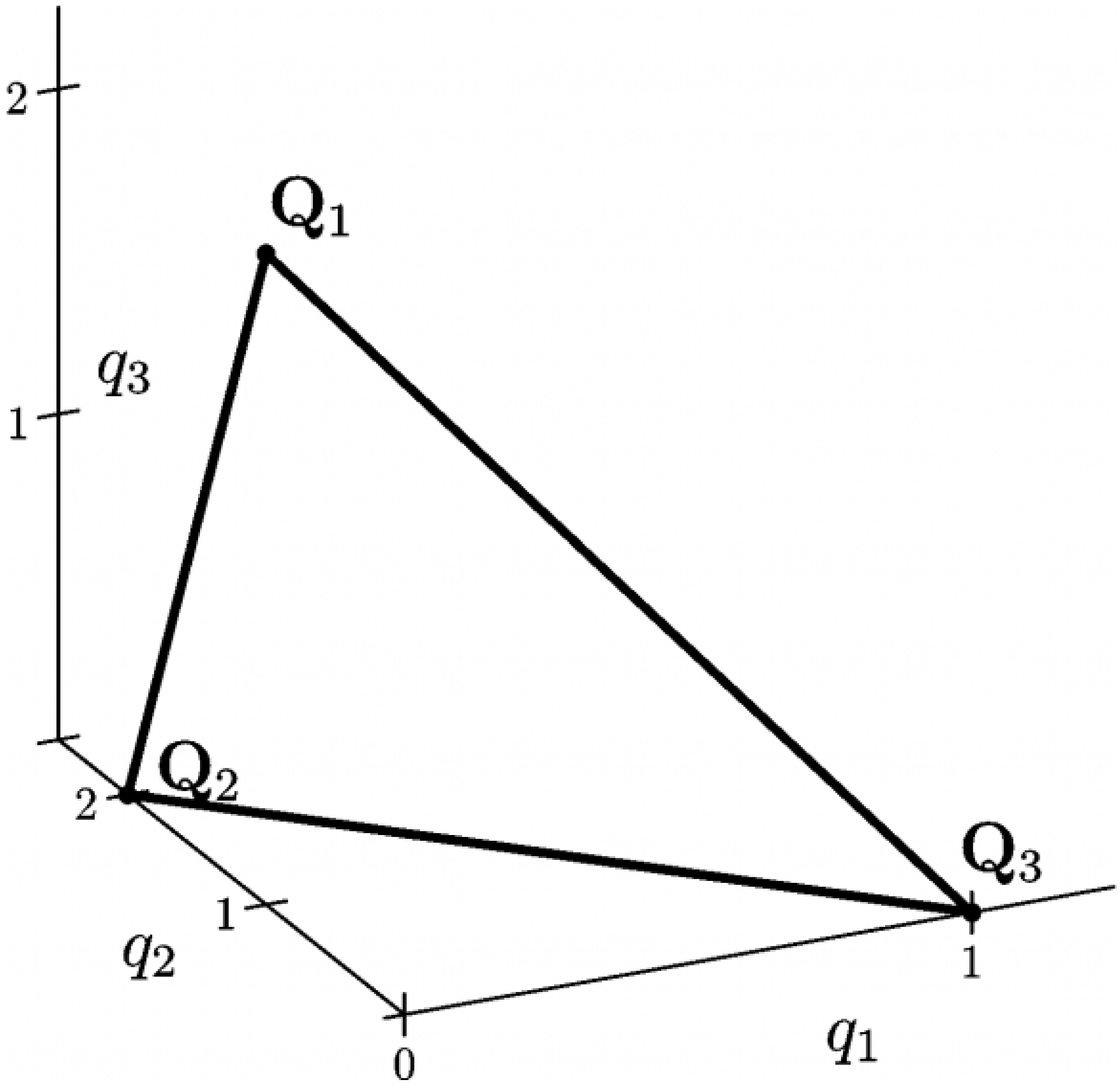

Figure 15: j

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Figure 16: Theorem 4 . 3 .

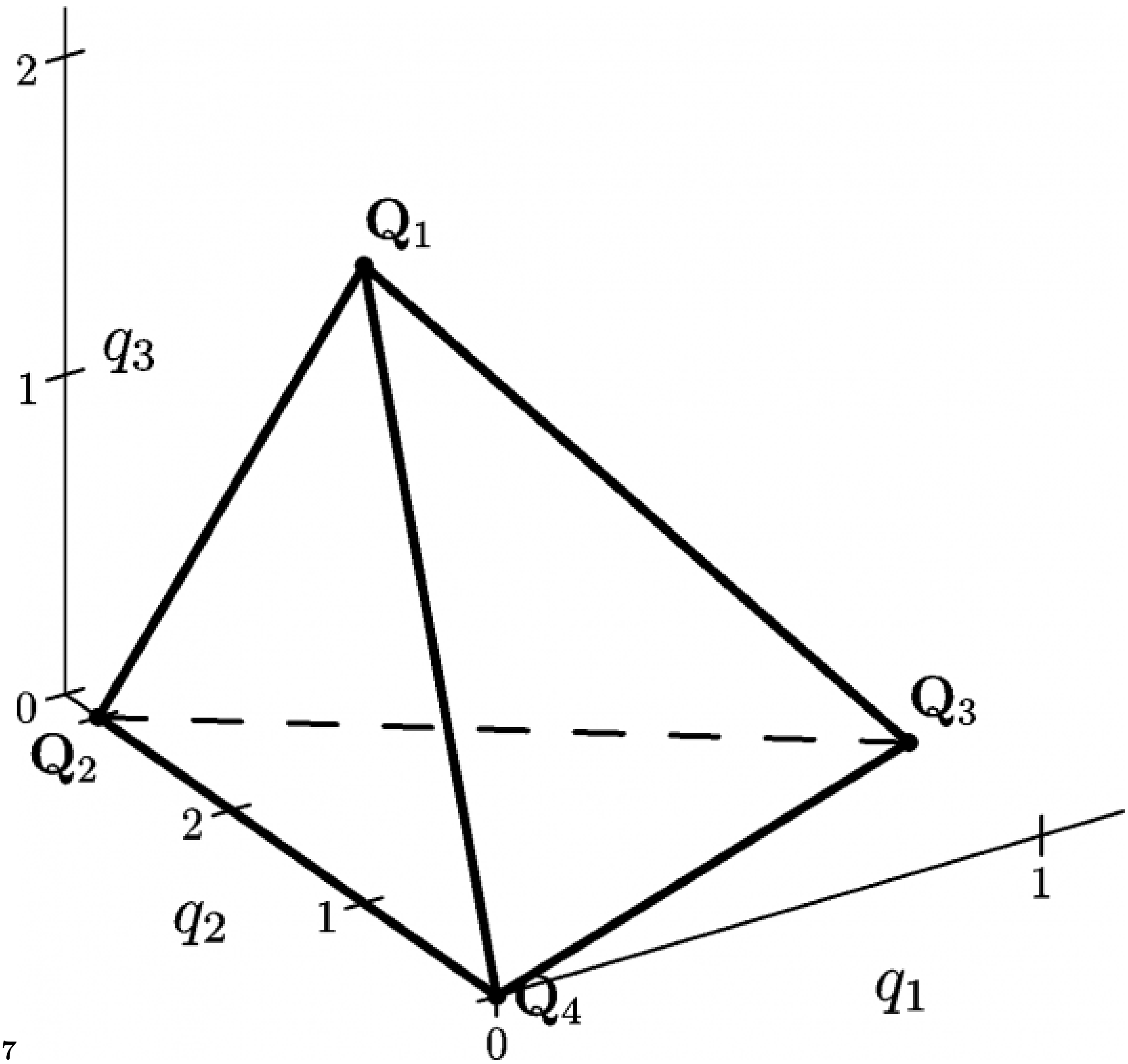

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Figure 17: Figure 7 :

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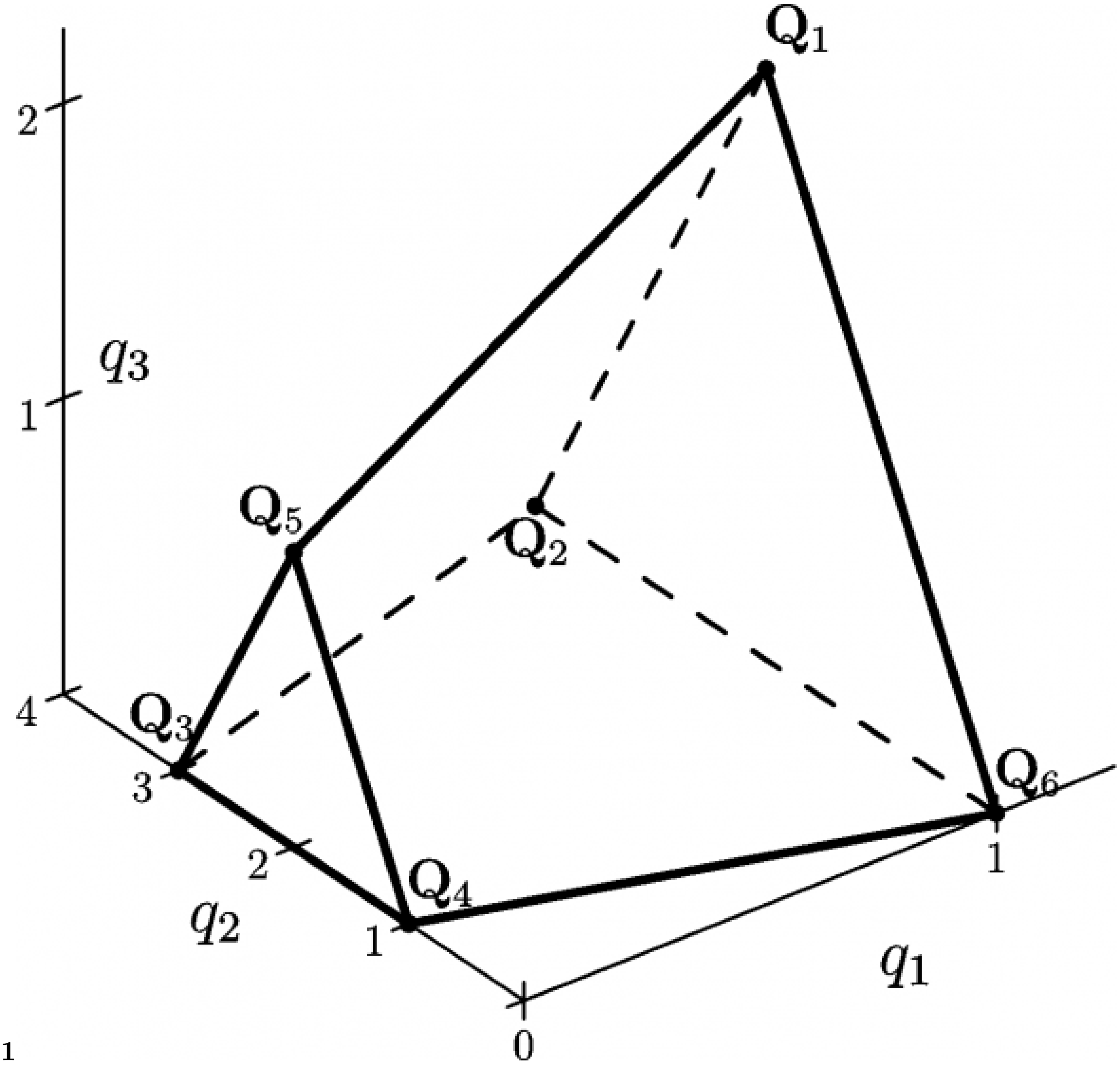
Figure 18: Figure 8 :



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$\mathbf{Q}_6$
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Figure 19: . 1)

Figure 20:

4

Figure 21: 4 .

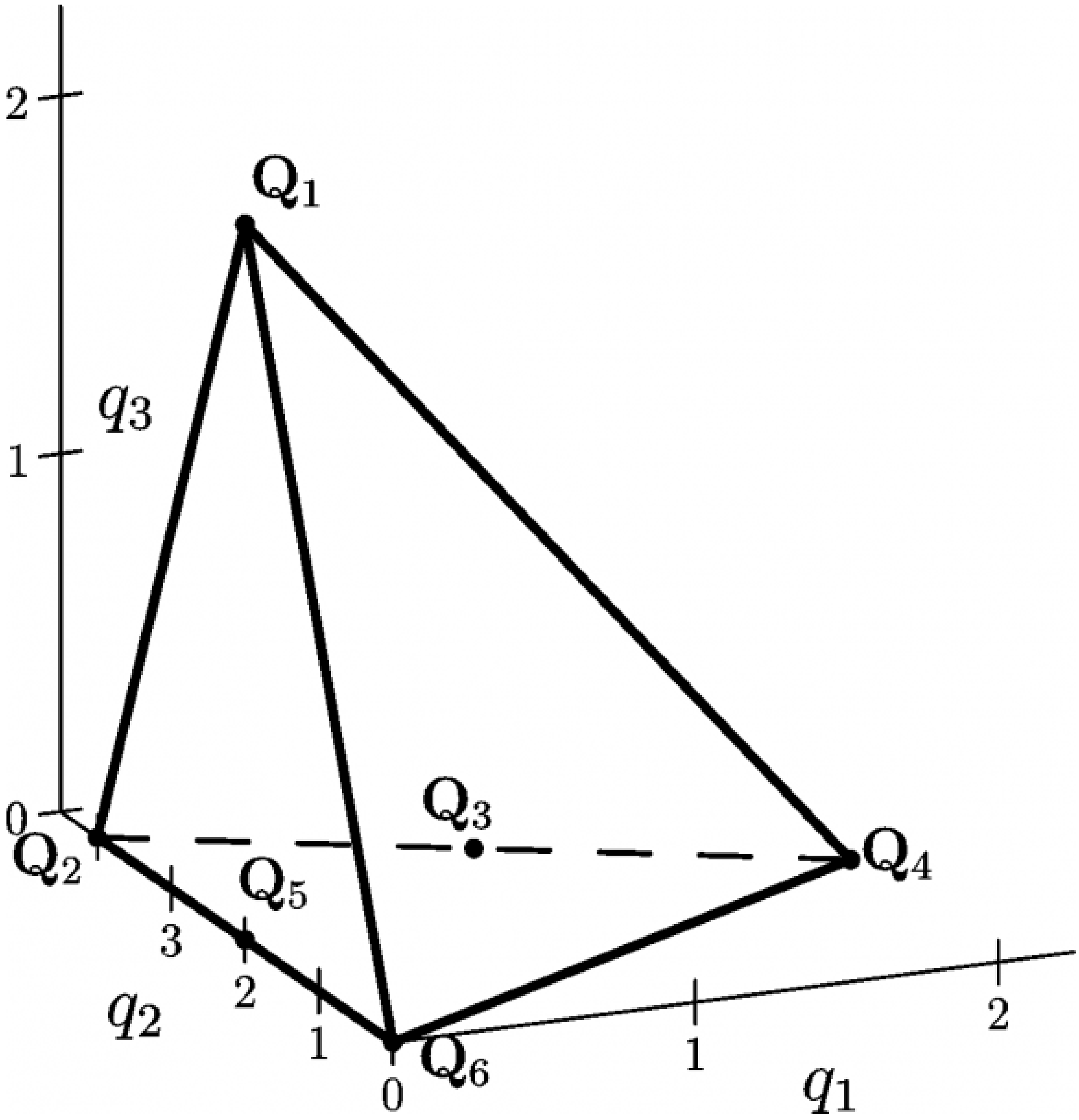

Figure 22: London

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