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Author: Keldysh Institute of Applied Mathematics of RAS, Moscow, Russia. email: abruno@keldysh.ru

I. INTRODUCTION

There are two universal methods for local study of nonlinear equations and systems of different kinds (algebraic, ordinary and partial differential): **(a)** normal form and **(b)** truncated equations.

(a) Equations with linear parts can be reduced to their normal forms by local changes of coordinates. For algebraic equation, it is Implicit Function Theorem. For systems of ordinary differential equations (ODE), I completed the theory of normal forms, began by Poincaré (1879) [Poincaré, 1928] and Dulac (1912) [Dulac, 1912] for general systems [Bruno, 1964; 1971] and began by Birkhoff (1929) [Birkhoff, 1966] for Hamiltonian systems [Bruno, 1972; 1994].

(b) Equations without linear part: I proposed to study properties of solutions to equations (algebraic, ordinary differential and partial differential) by studying sets of vector power exponents of terms of these equations. Namely, to select more simple ("truncated") equations [Bruno, 1962; 1989; 2000] by means of generalization to polyhedrons the Newton (1678) [Newton, 1964] and the Hadamard (1893) [Hadamard, 1893] polygons.

By means of power transformations [Bruno, 1962; 1989; 2022b] the normal forms and the truncated equations can be strongly simplified and often solved. Solutions to the truncated equations are asymptotically the first approximations of the solutions to the full equations. Continuing that process, we can obtain

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approximations of any precision to solutions of initial equations. Basing on the developed Asymptotic and Local Nonlinear Analysis, I proposed algorithms for solutions of a wide set of singular problems. In particular, for computation of six different types of asymptotic expansions of solutions to ODE [Bruno, 2004; 2018b; Bruno, Goruchkina, 2010], including expansions into trans-series [Bruno, 2019b].

In this article it is shown for a single algebraic equation in Section 2, for a single ordinary differential equation in Section 3, for an autonomous system of ODE's in Section 4, for Hamiltonian system in Subsection 4.5, for a single partial differential equation in Section 5. A survey of some applications is in Section 6.

From that point of view, the usual Classical Analysis is linear one, because it considers only linear approximations of problems near known solutions.

II. SINGLE ALGEBRAIC EQUATION

2.1. The implicit function theorem: Let $X = (x_1, \dots, x_n)$, $Q = (q_1, \dots, q_n)$, then

$$X^Q = x_1^{q_1} \cdots x_n^{q_n}, \quad \|Q\| = q_1 + \cdots + q_n.$$

Theorem 2.1. *Let*

$$f(X, \varepsilon, T) = \Sigma a_{Q,r}(T) X^Q \varepsilon^r, \tag{2.1}$$

where $0 \leqslant Q \in \mathbb{Z}^n$, $0 \leqslant r \in \mathbb{Z}$, *the sum is finite and* $a_{Q,r}(T)$ *are some functions of* $T = (t_1, \dots, t_m)$, *besides* $a_{00}(T) \equiv 0$, $a_{01}(T) \not\equiv 0$. *Then the solution to the equation* $f(X, \varepsilon, T) = 0$ *has the form*

$$\varepsilon = \Sigma b_R(T) X^R \stackrel{\text{def}}{=} b(T, X),$$

where $0 \leqslant R \in \mathbb{Z}^n$, $0 < \|R\|$, *the coefficients* $b_R(T)$ *are functions on* T *that are polynomials from* $a_{Q,r}(T)$ *with* $\|Q\| + r \leqslant \|R\|$ *divided by* $a_{01}^{2\|R\|-1}$. *The expansion* $b(T, X)$ *is unique. Let*

$$g(X, \delta, T) = f(X, \delta + b(T, X), T), \tag{2.2}$$

then $g(X, 0, T) \equiv 0$.

This is a generalization of Theorem 1.1 of [Bruno, 2000, Ch. II] on the implicit function and simultaneously a theorem on reducing the algebraic equation (2.1) to its normal form (2.2) when the linear part $a_{01}(T) \not\equiv 0$ is nondegenerate. In it, we must exclude the values of T near the zeros of the function $a_{01}(T)$.

Let $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ or $\mathbb{C}^n$, and $f(X)$ be a polynomial. A point $X = X^0$, $f(X^0) = 0$ is called *simple* if the vector $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ in it is non-zero. Otherwise, the point $X = X^0$ is called *singular* or *critical*. By shifting $X = X^0 + Y$ we move the point X^0 to the origin $Y = 0$. If at this point the derivative $\partial f/\partial x_n \neq 0$, then near X^0 all solutions to the equation $f(X) = 0$ have the form $y_n = \Sigma b_{q_1,\ldots q_{n-1}} y_1^{q_1} \cdots y_{n-1}^{q_{n-1}}$, that is, lie in $(n-1)$-dimensional space.

2.2. Newton's polyhedron: Let the point $X^0 = 0$ be singular. Write the polynomial in the form

$$f(X) = \Sigma a_Q X^Q,$$

where $a_Q = \text{const} \in \mathbb{R}$, or $\mathbb{C}$. Let $\mathbf{S}(f) = \{Q : a_Q \neq 0\}$.

The set $\mathbf{S}$ is called the *support* of the polynomial $f(X)$. Let it consist of points $Q_1, \ldots, Q_k$. The convex hull of the support $\mathbf{S}(f)$ is the set

$$\Gamma(f) = \left\{ Q = \sum_{j=1}^{k} \mu_j Q_j, \mu_j \geqslant 0, \sum_{j=1}^{k} \mu_j = 1 \right\}, \tag{2.3}$$

which is called *Newton's polyhedron.*

Its boundary $\partial\Gamma(f)$ consists of generalized faces $\Gamma_j^{(d)}$, where d is its dimension of $0 \leqslant d \leqslant n-1$ and j is the number.

Each (generalized) face $\Gamma_j^{(d)}$ corresponds to its:

- *boundary subset*

$$\mathbf{S}_j^{(d)} = \mathbf{S} \cap \Gamma_j^{(d)},$$

- *truncated polynomial*

$$\hat{f}_j^{(d)}(X) = \Sigma a_Q X^Q \text{ over } Q \in \mathbf{S}_j^{(d)},$$

- and *normal cone*

$$\mathbf{U}_j^{(d)} = \left\{ P : \langle P, Q' \rangle = \langle P, Q'' \rangle > \langle P, Q''' \rangle, Q', Q'' \in \mathbf{S}_j^{(d)}, Q''' \in \mathbf{S} \backslash \mathbf{S}_j^{(d)} \right\}, \tag{2.4}$$

where $P = (p_1, \ldots, p_n) \in \mathbb{R}_*^n$, the space $\mathbb{R}_*^n$ is conjugate (dual) to the space $\mathbb{R}^n$ and $\langle P, Q \rangle = p_1 q_1 + \ldots + p_n q_n$ is the scalar product.

At $X \to 0$ solutions to the full equation $f(X) = 0$ tend to non-trivial solutions of those truncated equations $\hat{f}_j^{(d)}(X) = 0$ whose normal cone $\mathbf{U}_j^{(d)}$ intersects with the negative orthant $P \leqslant 0$ in $\mathbb{R}_*^n$.

Remark 1. If in the sum (2.1) all Q belong to a forward cone C:

$$\langle Q, K_i \rangle > c_i, \quad i = 1, \dots, m,$$

then in the solution (2.2) of Theorem 2.1 all R belong to the same cone C: $\langle Q, K_i \rangle > c_i$, $i = 1, \dots, m$, [Bruno, 1989, Part I, Chapter 1, § 3].

2.3. Power transformations [Bruno, 1962; 2000:] Let $\ln X \stackrel{\text{def}}{=} (\ln x_1, \dots, \ln x_n)$. The linear transformation of the logarithms of the coordinates

$$(\ln y_1, \dots, \ln y_n) \stackrel{\text{def}}{=} \ln Y = (\ln X)\alpha, \tag{2.5}$$

where α is a nondegenerate square n-matrix, is called *power transformation*.

By the power transformation (2.5), the monomial X^Q tranforms into the monomial Y^R, where $R = Q\,(\alpha^*)^{-1}$ and the asterisk indicates a transposition.

A matrix α is called *unimodular* if all its elements are integers and $\det \alpha = \pm 1$. For an unimodular matrix α, its inverse α^{-1} and transpose α^* are also unimodular.

Theorem 2.2. *For the face $\Gamma_j^{(d)}$ there exists a power transformation* (2.5) *with the unimodular matrix α which reduces the truncated sum $\hat{f}_j^{(d)}(X)$ to the sum from d coordinates, that is, $\hat{f}_j^{(d)}(X) = Y^S \hat{g}_j^{(d)}(Y)$, where $\hat{g}_j^{(d)}(Y) = \hat{g}_j^{(d)}(y_1, \dots, y_d)$ is a polynomial. Here $S \in \mathbb{Z}^n$. The additional coordinates $y_{d+1}, \dots, y_n$ are local (small).*

The article [Bruno, Azimov, 2023] specifies an algorithm for computing the unimodular matrix α of Theorem 2.2.

2.4. Parametric expansion of solutions: Let $\Gamma_j^{(d)}$ be a face of the Newton polyhedron $\Gamma(f)$. Let the full equation $f(X) = 0$ is changed into the equation $g(Y) = 0$ after the power transformation of Theorem 2.2. Thus $\hat{g}_j^{(d)}(y_1, \dots, y_d) = g(y_1, \dots, y_d, 0, \dots, 0)$.

Let the polynomial $\hat{g}_j$ be the product of several irreducible polynomials

$$\hat{g}_j^{(d)} = \prod_{k=1}^{m} h_k^{l_k}(y_1, \ldots, y_d), \tag{2.6}$$

where $0 < l_k \in \mathbb{Z}$. Let the polynomial h_k be one of them. Three cases are possible:

Case 1. The equation $h_k = 0$ has a polynomial solution $y_d = \varphi(y_1, \ldots, y_{d-1})$. Then in the full polynomial $g(Y)$ let us substitute the coordinates

$$y_d = \varphi + z_d,$$

for the resulting polynomial $h(y_1, \ldots, y_{d-1}, z_d, y_{d+1} \ldots, y_n)$ again construct the Newton polyhedron, separate the truncated polynomials, etc. Such calculations were made in [Bruno, Batkhin, 2012] and were shown in [Bruno, 2000, Introduction].

Case 2. The equation $h_k = 0$ has no polynomial solution, but has a parametrization of solutions

$$y_j = \varphi_j(T), j = 1, \ldots, d, \quad T = (t_1, \ldots, t_{d-1}).$$

Then in the full polynomial $g\ Y$ (we)substitute the coordinates

$$y_j = \varphi_i(T) + \beta_j \varepsilon, j = 1, \ldots, d, \tag{2.7}$$

where $\beta_j = \text{const}$, $\Sigma\,|\beta_j| \neq 0$, and from the full polynomial $g(Y)$ we get the polynomial

$$h = \Sigma a_{Q'',r}(T) Y''^{Q''} \varepsilon^r, \tag{2.8}$$

where $Y'' = (y_{d+1}, \ldots, y_n)$, $0 \leqslant Q'' = (q_{d+1}, \ldots, q_n) \in \mathbb{Z}^{n-d}$, $0 \leqslant r \in \mathbb{Z}$. Thus $a_{00}(T) \equiv 0$, $a_{01}(T) = \sum_{j=1}^{d} \beta_j \partial \hat{g}_j^{(d)} / \partial y_j(T)$.

If in the expansion (5.7) $l_k = 1$, then $a_{01} \not\equiv 0$. By Theorem 2.1, all solutions to the equation $h = 0$ have the form

$$\varepsilon = \Sigma b_{Q''}(T) Y''^{Q''},$$

i.e., according to (2.7) the solutions to the equation $g = 0$ have the form

$$y_j = \varphi_j(T) + \beta_j \Sigma b_{Q''}(T) Y''^{Q''}, j = 1, \ldots, d.$$

Such calculations were proposed in [Bruno, 2018a].

If in (5.7) $l_k > 1$, then in (2.8) $a_{01}(T) \equiv 0$ and for the polynomial (2.8) from Y'', ε we construct a Newton polyhedron by support $\mathbf{S}(h) = \{Q'', r : a_{Q'',r}(T) \not\equiv 0\}$, separate the truncations and so on.

Case 3. The equation $h_k = 0$ has neither a polynomial solution nor a parametric one. Then, using Hadamard's polyhedron [Bruno, 2018a; 2019a], one can compute a piece-wise approximate parametric solution to the equation $h_k = 0$ and look for an approximate parametric expansion.

Similarly, one can study the position of an algebraic manifold in infinity.

III. SINGLE ODE [BRUNO, 2004]

3.1. Setting of the problem: Here we consider an ordinary differential equation of the form

$$f\left(x, y, y', \ldots, y^{(n)}\right) = 0, \tag{3.1}$$

where x is independent variable, y is the dependent variable, $y' = dy/dx$ and f is a polynomial of its arguments. Near $x^0 = 0$ or ∞ we look for solutions of equation (3.1) in the form of asymptotic series

$$y = \sum_{k=1}^{\infty} b_k x^{s_k}, \tag{3.2}$$

where b_k are functions of $\log x$ and $\omega s_k > \omega s_{k+1}$ with

$$\omega = \begin{cases} -1, & \text{if } x^0 = 0, \\ \;\;1, & \text{if } x^0 = \infty. \end{cases} \tag{3.3}$$

We set $X = (x, y)$. By a differential monomial $a(x, y)$ we mean the product of an ordinary monomial

$$c x^{r_1} y^{r_2} \stackrel{\text{def}}{=} c X^R, \tag{3.4}$$

is called a *differential sum*. In equation (3.1) polynomial f is the differential sum.

To every differential monomial $a(X)$ one assigns its (vector) *exponent* $Q(a) = (q_1, q_2) \in \mathbb{R}^2$ by the following rules. For a monomial of the form (3.4) let

$$Q\left(cX^R\right) = R,$$

that is, $Q\left(cx^{r_1}y^{r_2}\right) = (r_1, r_2)$; for a derivative of the form (3.5) let

$$Q\left(d^l y/dx^l\right) = (-l, 1).$$

When differential monomials are multiplied, their exponents are summed as vectors:

$$Q\left(a_1 a_2\right) = Q\left(a_1\right) + Q\left(a_2\right).$$

The set $\mathbf{S}(f)$ of exponents $Q\left(a_i\right)$ of all the differential monomials $a_2(X)$ in a differential sum of the form (3.6) is called the *support* of the sum $f\,(X)$. Obviously, $\mathbf{S}(f) \in \mathbb{R}^2$. The closure $\Gamma(f)$ of the convex hull of the support $\mathbf{S}(f)$ is referred to as the *polygon of the sum* $f(X)$. The boundary $\partial\Gamma(f)$ of the polygon $\Gamma(f)$ consists of vertices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$. These objects are called (generalized) *faces* $\Gamma_j^{(d)}$, where the superscript indicates the dimension of the face and the subscript is the number of the face. Corresponding to any face $\Gamma_j^{(d)}$ are the related *boundary subset* $\mathbf{S}_j^{(d)} = \mathbf{S}(f) \cap \Gamma_j^{(d)}$ of the set $\mathbf{S}$ and the *truncated sum*

$$\hat{f}_j^{(d)}(X) = \sum a_i(X) \quad \text{over} \quad Q\left(a_i\right) \in \mathbf{S}_j^{(d)}. \tag{3.7}$$

Let $\mathbb{R}_*^2$ be the plane conjugate to the plane $\mathbb{R}^2$ so that the inner (scalar) product

$$\langle P, Q\rangle \stackrel{\text{def}}{=} p_1 q_1 + p_2 q_2$$

is defined for any $P = (p_1, p_2) \in \mathbb{R}_*^2$ and $Q = (q_1, q_2) \in \mathbb{R}^2$. Corresponding to any face $\Gamma_j^{(d)}$ are its *normal cone*,

$$\mathbf{U}_j^{(d)} = \left\{ P : \begin{array}{ll} \langle P, Q\rangle = \langle P, Q'\rangle, & Q, Q' \in \mathbf{S}_j^{(d)} \\ \langle P, Q\rangle > \langle P, Q''\rangle, & Q'' \in \mathbf{S}(f) \backslash \mathbf{S}_j^{(d)} \end{array} \right\}$$

and the truncated sum (3.7).

All these constructions are applicable to equation (3.1), where f is a differential sum.

Let $x \to 0$ or $x \to \infty$ and suppose that a solution of the equation (3.1) has the form

$$y = c_r x^r + o\left(|x|^{r+\varepsilon}\right), \tag{3.8}$$

where c_r is a coefficient, $c_r = \text{const} \in \mathbb{C}$, $c_r \neq 0$, the exponents r and ε are in $\mathbb{R}$, and $\varepsilon\omega < 0$. Then we say that the expression

$$y = c_r x^r, \quad c_r \neq 0 \tag{3.9}$$

gives the *power-law asymptotic form* of the solution (3.8).

Thus, corresponding to any face $\Gamma_j^{(d)}$ are the normal cone $\mathbf{U}_j^{(d)}$ in $\mathbb{R}_*^2$ and the truncated equation

$$\hat{f}_j^{(d)}(X) = 0. \tag{3.10}$$

Theorem 3.1 ([Bruno, 2000, Chap. VI, Theorem 1.1])**.** *If the equation* (3.1) *has a solution of the form* (3.8) *and if* $\omega(1,p) \in \mathrm{U}_j^{(d)}$, *then the truncation* (3.9) *of the solution* (3.8) *is a solution of the truncated equation* (3.7), (3.10).

Therefore, to find all truncated solutions (3.9) of the equation (3.1), one must calculate the support $\mathbf{S}(f)$, the polygon $\Gamma(f)$, all its faces $\Gamma_j^{(d)}$, the outward normals N_j to the edges $\Gamma_j^{(1)}$, the normal cones $\mathbf{U}_j^{(1)}$ of the edges, and the normal cones $\mathbf{U}_j^{(0)}$ of the vertices. For each truncated equation (3.7), (3.10) one must then find all its solutions of the form (3.9) such that the vector $(1; r)$ belongs to $\mathbf{U}_j^{(d)}$, and single out the solutions of this kind for which one of the vectors $\pm(1, r)$ belongs to the normal cone $\mathbf{U}_j^{(d)}$. If $d = 0$, then one of the vectors $\pm(1, r)$ belongs to $\mathbf{U}_j^{(d)}$. If $d = 1$, then this property always holds. Here the value of ω is also determined.

3.2. Solution of the truncated equation:

Here we consider separately two cases: a vertex $\Gamma_j^{(0)}$ and an edge $\Gamma_j^{(1)}$. Corresponding to a vertex $\Gamma_j^{(0)} = \{Q\}$ is a truncated equation (3.10) with one-point support Q and with $d = 0$. We set $g(X) \stackrel{\text{def}}{=} X^{-Q}\hat{f}_j^{(0)}(X)$. Then the solution (3.7), (3.10) satisfies the equation

$$g(X)=0$$

Substituting $y=cx^r$ into $g(X)$, we see that $g(x, cx^r)$ does not depend on x, c and is a polynomial in r, that is,

$$g(x, cx^r) \equiv \chi(r),$$

where $\chi(r)$ is the *characteristic polynomial* of the differential sum $\hat{f}_j^{(0)}(X)$. Hence, in a solution (3.9) of the equation (3.10) the exponent r is a root of the characteristic equation

$$\chi(r) \stackrel{\text{def}}{=} g(x, x^r)=0, \tag{3.11}$$

and the coefficient c_r is arbitrary. Among the roots r_i of the equation (3.11), one must single out only those for which one of the vectors $\omega(1, r)$, where $\omega=\pm 1$, belongs to the normal cone $\mathbf{U}_j^{(0)}$ of the vertex $\Gamma_j^{(0)}$. In this case the value of ω uniquely determined. The corresponding expressions of the sum with an arbitrary constant c_r are candidates for the role of truncated solutions of the equation (3.1). Moreover, by (3.3), if $\omega=-1$, then $x \rightarrow 0$, and if $\omega=1$, then $x \rightarrow \infty$.

Complex roots r to characteristic equation (3.11) may bring to *exotic expansions* of solutions (3.2), where coefficients b_k are power series in $x^{\alpha i}$ with real $\alpha \in \mathbb{R}$ and $i^2=-1$.

Corresponding to an edge $\Gamma_j^{(1)}$ is a truncated equation (3.10) with $d=1$ whose normal cone $\mathbf{U}_j^{(1)}$ is a ray $\{\lambda N_j, \lambda>0\}$. If $\omega(1, r) \in \mathbf{U}_j^{(1)}$, this condition uniquely determines the exponent r of the truncated solution (3.9) and the value $\omega=\pm 1$ in (3.3). To find the coefficient c_r, one must substitute the expression (3.9) into the truncated equation (3.10). After cancelling a factor which is a power of x, we obtain an algebraic *defining equation* for the coefficient c_r,

$$\tilde{\tilde{f}}(c_r) \stackrel{\text{def}}{=} x^{-s} \hat{f}_j^{(1)}(x, c_r x^r)=0$$

Corresponding to every root $c_r=c_r^{(i)} \neq 0$ of this equation is an expression of the form (3.9) which is a candidate for the role of a truncated solution of the equation (3.1). Moreover, by (3.3), if in the normal cone $\mathbf{U}_j^{(1)}$ one has $p_1<0$, then $x \rightarrow 0$, and if $p_1>0$, then $x \rightarrow \infty$.

Thus, every truncated equation (3.10) has several suitable solutions of the form (3.9). Let us combine these solutions into families that are continuous with respect to ω, r, c_r, and the parameters of the equation (3.1) and denote these families by $\mathcal{F}_i^{(d)}k$, where $k = 1, 2 \ldots$.

If in the truncated equation (3.10), we make the *power transformation*

$$y = x^P z$$

and the *logarithmic transformation*

$$\xi = \log x,$$

then we obtain ODE

$$\varphi(\xi, z) = 0, \tag{3.12}$$

where φ is a differential sum, i. e. it has the form (3.1). If the equation (3.12) has a solution in the form

$$z = \sum_{j=1}^{\infty} c_j \xi^{r_j}, \quad r_j > r_{j+1},$$

then in the expansion (3.2) coefficients b_k are functions from $\log x$. If $b_1 = \text{const}$, then it is the *power-logarithmic expansion*, where other b_k are polynomials in $\log x$. If b_1 depends of $\log x$, then all b_k are power series in $\log x$ and the expansion (3.2) is *complicated*.

3.3. Computation of solution to equation (3.1) as expansion (3.2)

From the polygon Γ of the initial equation (3.1) we take a vertex or an edge $\Gamma_j^{(d)}$. Then we found a power solution $y = b_1 x^{P_1}$ of the truncated equation $\hat{f}_j^{(d)}(X) = 0$, as it was described above, put

$$y = b_1 x^{P_1} + z$$

and obtain new equation

$$g(x, z) = 0.$$

We construct the polygon ${}_1\Gamma$ for the new equation, take a vertex or an edge ${}_1\Gamma_k^{(e)}$, solve the truncated equation

$$\hat{g}_k^{(e)}(x, z) = 0,$$

and obtain the second term $b_2 x^{P_2}$ of expansion (3.2) and so on.

We construct the polygon ${}_1\Gamma$ for the new equation, take a vertex or an edge ${}_1\Gamma_k^{(e)}$, solve the truncated equation

$$\hat{g}_k^{(e)}(x, z) = 0,$$

and obtain the second term $b_2 x^{P_2}$ of expansion (3.2) and so on.

In [Bruno, 2004] there are some properties, that simplify computation.

Thus, we can obtain the 4 types of expansions (3.2) of solutions to equation (3.1):

1. *Power*, when all $b_k = \text{const}$ [Ibid.];
2. *Power-logarithmic*, when $b_1 = \text{const}$ and other b_k are polynomial in $\log x$ [Ibid.];
3. *Complicated*, when all b_k are power series in $\log x$ [Bruno, 2006; 2018b];
4. *Exotic*, when all b_k are power series in $x^{i\alpha}$ [Bruno, 2007].

Except expansions (3.2) of solutions $y(x)$ of equation (3.1), there are *exponential expansions*

$$y = \sum_{k=1}^{\infty} b_k(x) \exp\left[k\varphi(x)\right],$$

where $b_k(x)$ and $\varphi(x)$ are power series in x [Bruno, 2012a,b].

Also there are solutions in the form of transseries [Bruno, 2019b].

These results were applied to 6 Painlevè equations [Bruno, 2015; 2018b,c; Bruno, Goruchkina, 2010].

Written as differential sums they are:

Equation P_1: $$f(x, y) \overset{\text{def}}{=} -y'' + 3y^2 + x = 0.$$

Equation P_2: $$f(x, y) \overset{\text{def}}{=} -y'' + 2y^3 + xy + a = 0.$$

Equation P_3: $$f(x, y) \overset{\text{def}}{=} -xyy'' + xy'^2 - yy' + ay^3 + by + cxy^4 + dx = 0.$$

Equation P_4: $$f(x, y) \overset{\text{def}}{=} -2yy'' + y'^2 + 3y^4 + 8xy^3 + 4(x^2 - a)y^2 + 2b = 0.$$

Equation P_5:

$$f(z,w) \stackrel{\text{def}}{=} -z^2w(w-1)w'' + z^2\left(\frac{3}{2}w - \frac{1}{2}\right)(w')^2 - zw(w-1)w' + \\ + (w-1)^3(\alpha w^2 + \beta) + \gamma z w^2(w-1) + \delta z^2 w^2 (w+1) = 0.$$

Equation P_6:

$$f(x,y) \stackrel{\text{def}}{=} 2y''x^2(x-1)^2y(y-1)(y-x) - (y')^2[\, x^2(x-1)^2(y-1)(y-x) + \\ + x^2(x-1)^2y(y-x) + x^2(x-1)^2y(y-1)\,] + \\ + 2y'[\, x(x-1)^2y(y-1)(y-x) + x^2(x-1)y(y-1)(y-x) + \\ + x^2(x-1)^2y(y-1)\,] - [\, 2\alpha y^2(y-1)^2(y-x)^2 + 2\beta x(y-1)^2(y-x)^2 + \\ + 2\gamma(x-1)y^2(y-x)^2 + 2\delta x(x-1)y^2(y-1)^2\,] = 0.$$

Here a, b, c, d and $\alpha, \beta, \gamma, \delta$ are complex parameters. If all they are nonzero, then polygons for these equations are shown in Figures 1, 2, 3.

Figure 1: Supports and polygons for equations P_1 (left), P_2 (right).

3.4. Normal form: Roots $r_j \stackrel{\text{def}}{=} \lambda_j$ of the equation (3.11) are called as *eigenvalues of the differential sum* $\hat{f}_j^{(0)}(X)$, corresponding to the vertex. If the differential sum $\hat{f}_j^{(0)}(X)$ has order l, then the characteristic equation (3.11) has l roots, $0 \le l \le n$.

Theorem 3.2. *Let*

1) $f(X)$ *be a polynomial in* $x, y, \ldots, y^{(n)}$;
2) *its polygon* $\Gamma(f)$ *have a vertex* $\Gamma_1^{(0)} = (v, 1)$ *at the left side of its boundary* $\partial\Gamma$;

3) *truncated differential sum* $\hat{f}_1^{(0)}(X)$ *have eigenvalues* $\lambda_1, \ldots, \lambda_l$, $0 \leq l \leq n$;
4) *the most left point of the support* $\mathbf{S}(f)$ *in the axis* $q_2 = 0$ *be* $(\alpha, 0)$. *Evidently* $\alpha \in \mathbb{Z}$.

Then there exists such power series $\varphi(x)$ *with integral increasing exponents, that after substitution*

$$y = z + \varphi(x) \tag{3.13}$$

the transformed differential sum

$$g(x, z) = f(x, z + \varphi(x)) \tag{3.14}$$

for

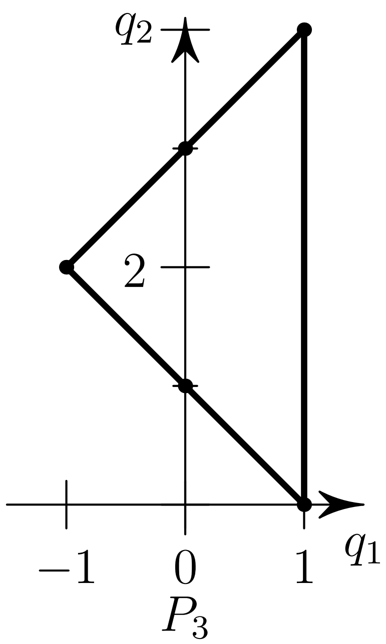
$$z = z' = \cdots = z^{(n)} = 0 \tag{3.15}$$

has only resonant terms $b_m x^m$, *where*

$$m = v + \lambda_k \in \mathbb{Z} \tag{3.16}$$

and $m \geq \alpha$.

So here the eigenvalue λ_k is resonant if $\alpha - v \leq \lambda_k \in \mathbb{Z}$.



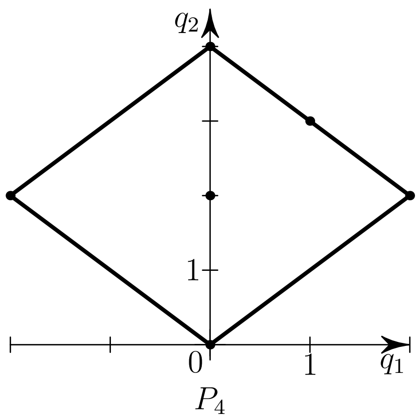


Figure 2: Supports and polygons for equations P_3 (left), P_4 (right).

Theorem 3.3. *Let*

1) $f\left(x, y, y', \ldots, y^{(n)}\right)$ *be a polynomial in* $x, y, y', \ldots, y^{(n)}$*;*

2) its Newton polygon $\Gamma(f)$ *have a vertex* $\Gamma_1^{(0)} = (v, 1)$ *at the right side of its boundary* $\partial\Gamma$*;*

3) truncated differential sum $\hat{f}_j^{(0)}(X)$ *have eigenvalues* $\lambda_1, \ldots, \lambda_l$*,* $0 \leq l \leq n$*;*

4) the most right point of the support $\mathbf{S}(f)$ *in the axis* $q_2 = 0$ *be* $(\beta, 0)$*. Evidently* $\beta \in \mathbb{Z}$*.*

Then there exists such power series $\varphi(x)$ *with integral decreasing exponents, that after substitution* (3.13)*, the differential sum* (3.14) *for identities* (3.15) *has only resonant terms* $b_m x^m$*, where equality* (3.16) *is true, and* $m \leq \beta$*.*

So here the eigenvalue λ_k is resonant if $\beta - v \geq \lambda_k \in \mathbb{Z}$. Equations $g(x, z) = 0$ for (3.14) in situations of Theorems 3.2 and 3.3 we will call *normal forms*.

Corollary 3.3.1. *If the truncated sum* $\hat{f}_j^{(0)}(X)$ *has no integral eigenvalue* $\lambda_k \geqslant \alpha - v$ *(for Theorem 3.2) or* $\lambda_k \leqslant \beta - v$ *(for Theorem 3.3), then the initial equation* $f(X) = 0$ *has formal solution* $y = \varphi(x)$*. If the truncated sum* $\hat{f}_j^{(0)}(X)$ *contains the derivation* $y^{(n)}$*, then the series* $\varphi(x)$ *converges according to Theorem 3.4 in [Bruno, 2004].*

Remark 2. If the truncated sum $\hat{f}_j^{(0)}(X)$ has integral eigenvalue $\lambda_k \geqslant \alpha - v$ (for Theorem 3.2) or $\lambda_k \leqslant \beta - v$ (for Theorem 3.3), then the initial equation $f(X) = 0$

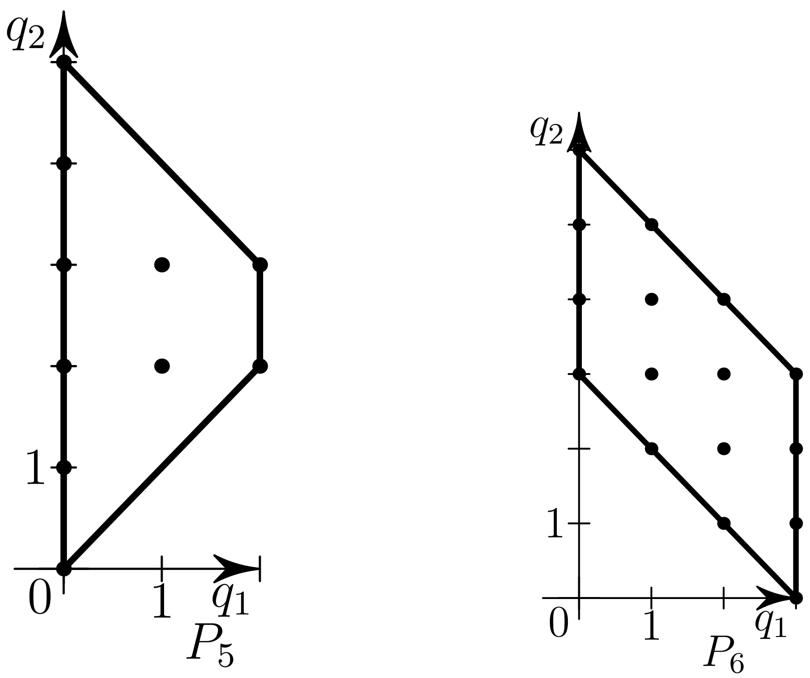


Figure 3: Supports and polygons for equations P_5 (left), P_6 (right).

3.5. Space Power Geometry: We will consider such a generalization of the power function cx^r which preserves their main properties. The real number

$$p_\omega(\varphi(x)) = \omega \varlimsup_{x^\omega \to \infty} \frac{\log|\varphi(x)|}{\omega \log|x|},$$

where $\arg x = \text{const} \in [0, 2\pi)$, is called the *order* of the function $\varphi(x)$ on the ray when $x \to 0$ or $x \to \infty$. The order $p_\omega(\varphi)$ is not defined on the ray $\arg x = \text{const}$, where the limit point $x = 0$ or $x = \infty$ is a point of accumulation of poles of the function $\varphi(x)$.

In Subsections 3.2–3.4 it was shown that as $x \to 0$ $(\omega = -1)$ or as $x \to \infty$ $(\omega = 1)$ solutions $y = \varphi(x)$ to the ODE $f(x, y) = 0$, where $f(x, y)$ is a differential sum, can be found by means of algorithms of Plane PG, if

$$p_\omega(\varphi(x)) - l = p_\omega\left(d^l\varphi/dx^l\right), \quad l = 1, \ldots, n,$$

where n is the maximal order of derivatives in $f(x, y)$. Here we introduce algorithms, which allow calculate solutions $y = \varphi(x)$ with the property

$$p_\omega(\varphi(x)) + l\gamma_\omega = p_\omega\left(d^l\varphi/dx^l\right), \quad l = 1, \ldots, n,$$

where $\gamma_\omega \in \mathbb{R}$, $\omega = \pm 1$.

Lemma 3.3.1. *If*

$$p_\omega(\varphi(x)) = -\gamma_\omega + p_\omega(\varphi'(x)) = -2\gamma_\omega + p_\omega(\varphi''(x)),$$

then $\omega + \omega\gamma_\omega \geqslant 0$.

Note, that in Plane PG we had $\gamma_\omega = -1$, i. e. $\omega + \omega\gamma_\omega = 0$. So, new interesting possibilities correspond to $\omega + \omega\gamma_\omega > 0$.

We consider the ODE

$$f(x,\, y) = \sum_i a_i(x, y) = 0,$$

where $f(x,\, y)$ is a differential sum. To each differential monomial $a_i(x, y)$, we assign its (vector) *power exponent* $\mathbf{Q}(a_i) = (q_1, q_2, q_3) \in \mathbb{R}^3$ by the following rules:

$$\mathbf{Q}\left(cx^{r_1}y^{r_2}\right) = (r_1,\; r_2,\; 0); \quad \mathbf{Q}\left(d^l y/dx^l\right) = (0,\; 1,\; l);$$

power exponent of the product of differential monomials is the sum of power exponents of factors: $\mathbf{Q}(a_1a_2) = \mathbf{Q}(a_1) + \mathbf{Q}(a_2)$.

The set $\tilde{\mathbf{S}}(f)$ of power exponents $\mathbf{Q}(a_i)$ of all differential monomials $a_i(x,y)$ presented in the differential sum $f(x,y)$ is called the *space support of the sum* $f(x,y)$. Obviously, $\tilde{\mathbf{S}}(f) \subset \mathbb{R}^3$. The convex hull $\mathbf{\Gamma}(f)$ of the support $\tilde{\mathbf{S}}(f)$ is called the *polyhedron of the sum* $f(x,y)$. The boundary $\partial\mathbf{\Gamma}(f)$ of the polyhedron $\mathbf{\Gamma}(f)$ consists of the vertices $\mathbf{\Gamma}_j^{(0)}$, the edges $\mathbf{\Gamma}_j^{(1)}$ and the faces $\mathbf{\Gamma}_j^{(2)}$. They are called (generalized) *faces* $\mathbf{\Gamma}_j^{(d)}$, where the upper index indicates the dimension of the face, and the lower one is its number. Each face $\mathbf{\Gamma}_j^{(d)}$ corresponds to the *space truncated sum*

$$\check{f}_j^{(d)}(x,y) = \sum a_i(x,y) \text{ over } \mathbf{Q}(a_i) \in \mathbf{\Gamma}_j^{(d)} \cap \tilde{\mathbf{S}}(f).$$

The approach allows to obtain solutions with expansions (3.2), where coefficients $b_k(x)$ are all periodic or all elliptic functions [Bruno, 2012c,d; Bruno, Parusnikova, 2012].

Expansions of solutions to more complicated equations such as hierarchies Painlevé see in [Anoshin, Beketova, (et al.), 2023; Bruno, Kudryashov, 2009].

For Painlevé equations $P_1 - P_5$ with all parameters nonzero, their polyhedrons are shown in Figures 4, 5, 6, 7, 8 correspondingly.

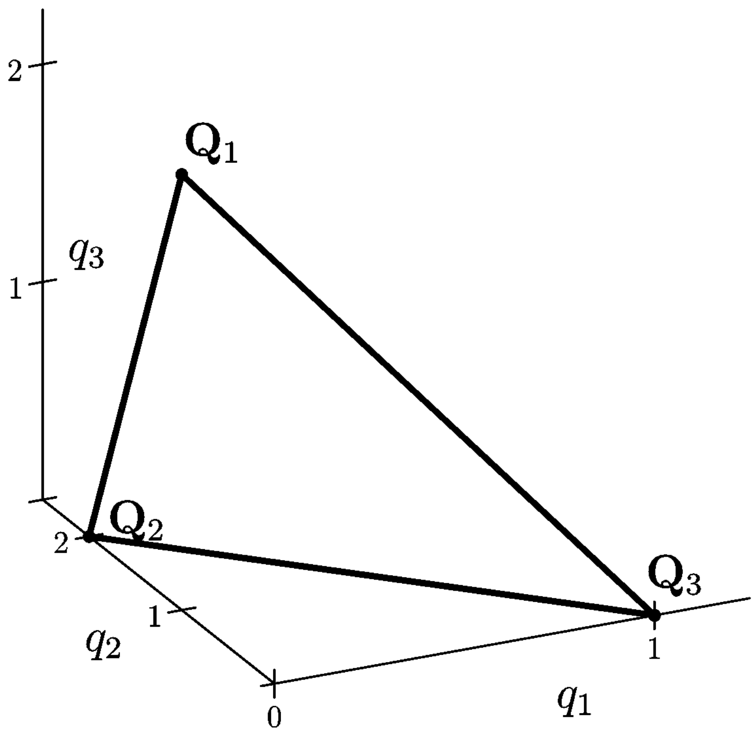


Figure 4: Support and polyhedron for equation P_1.

IV. AUTONOMOUS ODE SYSTEM

Here we consider the system

$$\dot{x}_i = f_i(X), \quad i = 1, \ldots, n, \tag{4.1}$$

where $\dot{}\ = d/dt$, $X = (x_1, \ldots, x_n) \in \mathbb{C}^n$ or $\mathbb{R}^n$, all $f_i(X)$ are polynomials from X. A point $X = X^0 =$ const is called *singular* if all $f_i(X^0) = 0$, $i = 1, \ldots, n$.

4.1. Normal form: Let the point $X^0 = 0$ be a singular point. Then the system (4.1) has the linear part

$$\dot{X} = XA,$$

where A is a square n-matrix. Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ be a vector of its eigenvalues.

Theorem 4.1 ([Bruno, 1964; 1971, 1972]). *There exists an invertible formal change of coordinates*

$$x_i = \varphi_i(Y), \quad i = 1, \ldots, n,$$

where $\varphi_i(Y)$ are power series from $Y = (y_1, \ldots, y_n)$ without free terms, which reduces the system (4.1) *to* ***normal form***

$$\dot{y}_i = y_i g_i(Y) = y_i \sum g_{iQ} Y^Q, \quad i = 1, \ldots, n, \tag{4.2}$$

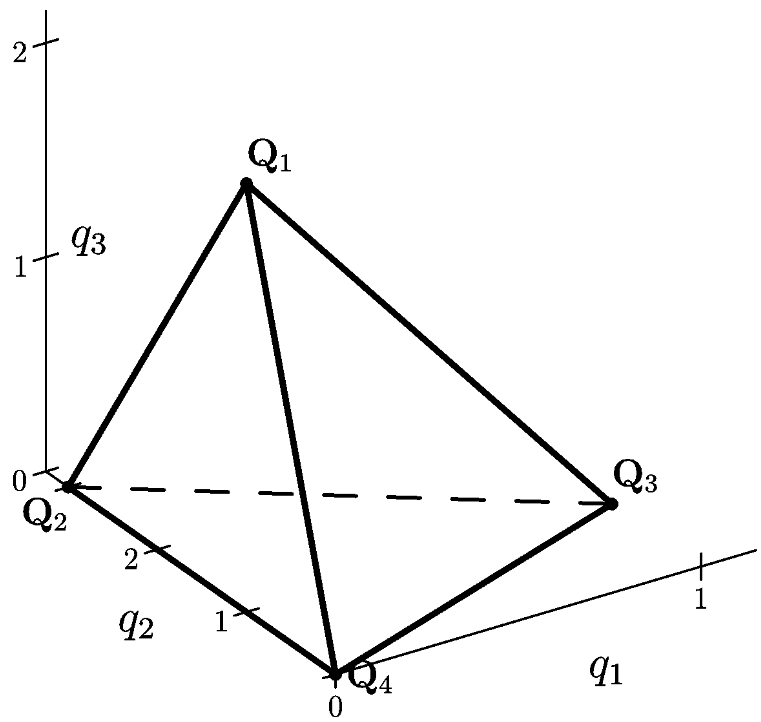


Figure 5: Support and polyhedron for equation P_2.

containing only resonant terms $y_i g_{iQ} Y^Q$, which have

$$\langle Q, \Lambda \rangle = 0. \tag{4.3}$$

Here $y_i g_i(Y)$ are power series on Y without free terms.

Let

$$N_i = \{Q \in \mathbb{Z}^n : q_j \geqslant 0, j \neq i, q_i \geqslant -1\}, i = 1, \dots, n,$$

and $N = N_1 \cup N_2 \cup \dots \cup N_n$. Then the number k of linearly independent $Q \in N$ satisfying the equation (4.3) is called *multiplicity of resonance.*

Theorem 4.2. *Let k be the multiplicity of resonance of the system* (4.1)*. Then there exists a power transformation*

$$\ln Z = (\ln Y)\,\alpha$$

with unimodular matrix α which reduces the normal form (4.2), (4.3) *to the system*

$$(\ln \dot{z}_i) = h_i(y_1, \dots, y_k), \quad i = 1, \dots, n,$$

in which the first k coordinates form a closed subsystem without a linear part, and the remaining $n - k$ coordinates are expressed via them by means of integrals.

Thus, if $\Lambda \neq 0$, then the original system (4.1) of order n can be reduced to a system of order k, but without the linear part.

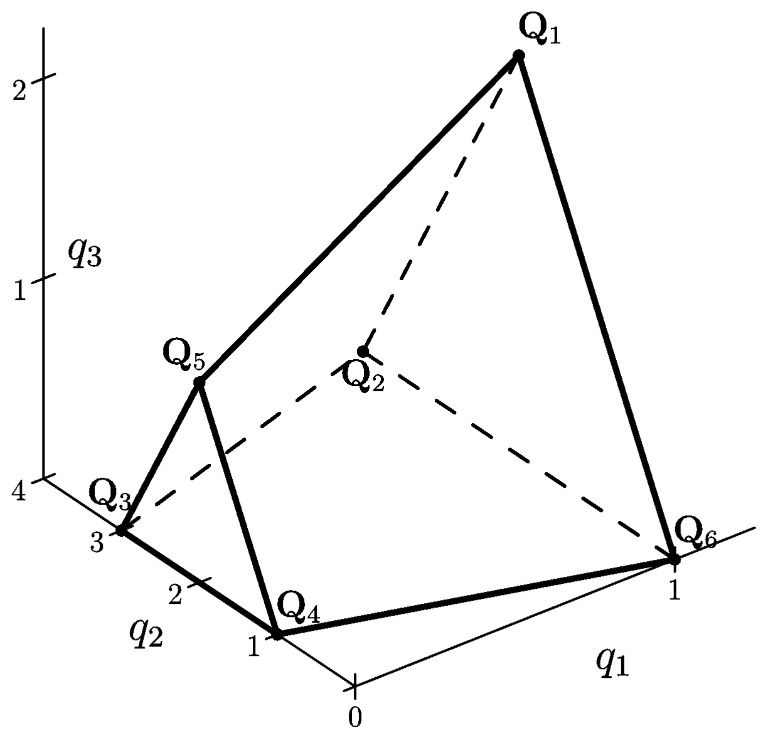


Figure 6: Support and polyhedron for equation P_3.

4.2: Newton's polyhedron [Bruno, 1962; 2000]. Let's write the system (4.1) as

$$(\ln \dot{x}_i) = \sum a_{iQ} X^Q, \quad i = 1, \ldots, n, \tag{4.4}$$

and put $A_Q = (a_{1Q}, \ldots, a_{nQ})$.

The set

$$\mathbf{S} = \{Q : A_Q \neq 0\}$$

is called the *support* of the system (4.4). Its convex hull Γ (2.3) is its *Newton's polyhedron*. Its boundary $\partial\Gamma$ consists of generalized faces $\Gamma_j^{(d)}$ of dimensions d, $0 \leqslant d \leqslant n-1$, and with numbers j.

Each generalized face $\Gamma_j^{(d)}$ corresponds to:

- *boundary subset* $\mathbf{S}_j^{(d)} = \Gamma_j^{(d)} \cap \mathbf{S}$,
- *truncated system*

$$(\ln \dot{X}) = \hat{A}_j^{(d)}(X) = \sum A_Q X^Q \text{ over } Q \in \mathbf{S}_j^{(d)}, \tag{4.5}$$

- *normal cone* $\mathbf{U}_j^{(d)} \subset \mathbb{R}_*^n$ (2.4) and
- *tangent cone* $T_j^{(d)}$.

According to [Bruno, 2000, Chapt. 1, §2] let $d > 0$ and $\widetilde{Q}$ be the interior point of a face $\Gamma_j^{(d)}$, that is, $\widetilde{Q}$ does not lie in a face of smaller dimension. If $d = 0$, then

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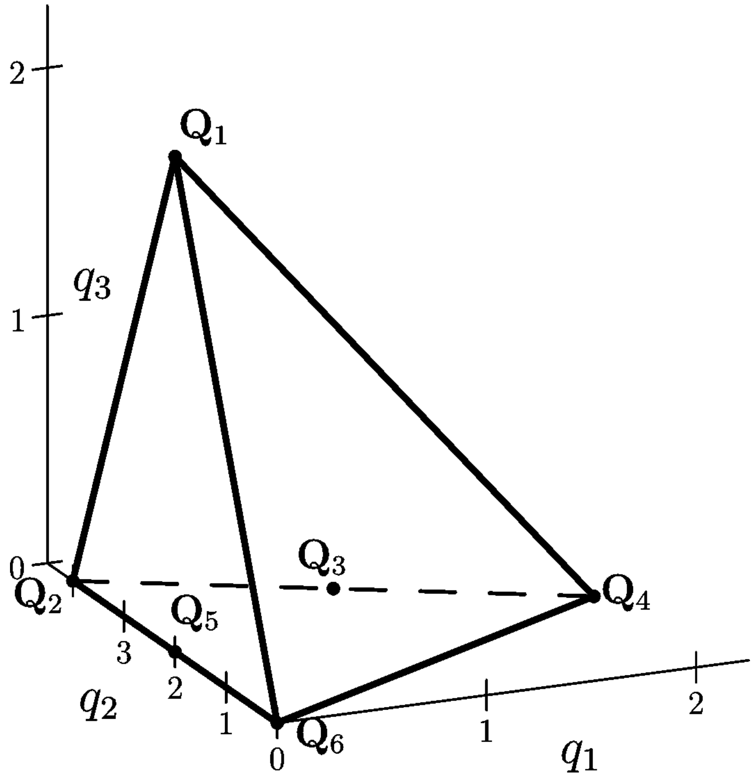

Figure 7: Support and polyhedron for equation P_4.

$\widetilde{Q} = \Gamma_j^{(0)}$. The conic hull of the set $\mathbf{S} - \widetilde{Q}$

$T_j^{(d)} = \left\{ Q = \mu_1 \left(Q_1 - \widetilde{Q} \right) + \cdots + \mu_k \left(Q_k - \widetilde{Q} \right), \mu_1, \ldots, \mu_k \geqslant 0, Q_1, \ldots, Q_k \in \mathbf{S} \right\}$

is called the *tangent cone* of the face $\Gamma_j^{(d)}$, $0 \leqslant d \leqslant n-1$, $T_j^{(d)} \subset \mathbb{R}^n$.

Theorem 4.3. *For each generalized face $\Gamma_j^{(d)}$, there exists power transformation*

$$\ln Y = (\ln X)\,\alpha$$

with the unimodular matrix α and change of time

$$d\tau = X^R dt,$$

$R \in \mathbb{Z}^n$, which reduce the system (4.4) to the form

$$d(\ln Y)/d\tau = B(Y), \tag{4.6}$$

where the system

$$d(\ln Y)/d\tau = \hat{B}_j^{(d)}(Y) \equiv \hat{B}_j^{(d)}(y_1, \ldots, y_d) = B(y_1, \ldots, y_d, 0, \ldots, 0), \tag{4.7}$$

corresponds to the truncated system (4.5).

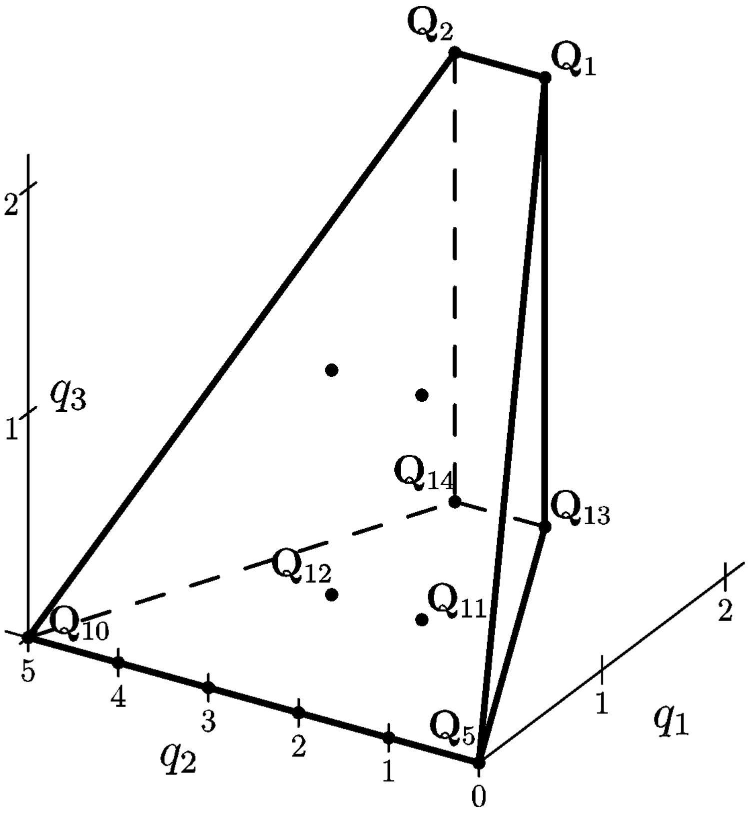

Figure 8: Support and polyhedron for equation P_5.

If the face $\Gamma_j^{(d)}$ had normal and tangent cones $\mathbf{U}_j^{(d)}$ and $T_j^{(d)}$, then the truncated system (4.7) has normal and tangent cones $\widetilde{\mathbf{U}}_j^{(d)}$ and $\widetilde{T}_j^{(d)}$, which are obtained from $\mathbf{U}_j^{(d)}$ and $T_j^{(d)}$ by conjugate linear transformations.

4.3. Generalized normal form [Bruno, 2022b]: Let the point

$$y_1^0, \ldots, y_d^0 \neq 0 \tag{4.8}$$

be singular for the truncated system (4.7). Near the point (4.8), the local coordinates are

$$\begin{aligned} z_i =& y_i - y_i^0, \quad i = 1, \ldots, d, \\ z_j =& y_j, \quad j = d+1, \ldots, n. \end{aligned}$$

Let at the point $Z = (z_1, \ldots, z_n) = 0$ the eigenvalues of the matrix of the linear part of the system (4.7) are $\widetilde{\Lambda} = \left(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n\right)$, where $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_d$ are the eigenvalues of the subsystem of the first d equations.

Theorem 4.4. *There exists an invertible formal change of coordinates*

$$z_i = \varphi_i(W), \quad i = 1, \ldots, n,$$

where $W = (w_1, \ldots, w_n)$ *which reduces the system* (4.6) *to the generalized normal form*

$$\dot{w}_i = w_i c_i(W) = w_i \sum c_{iQ} W^Q, \quad i = 1, \ldots, n, \tag{4.9}$$

where

$$\left\langle Q, \widetilde{\Lambda} \right\rangle = 0 \text{ and } Q \in \widetilde{T}_j^{(d)} \cap \mathbb{Z}^n. \tag{4.10}$$

Here $\varphi_i = w_i \sum \varphi_{iQ} W^Q$, $i = 1, \ldots, n$, *where* $Q \in \widetilde{T}_j^{(d)} \cap \mathbb{Z}^n$.

The system (4.9), (4.10) is reduced to a system of lower order by the power transformation of Theorem 4.2 .

4.4. Analysis of singularities: Let $X = X^0$ be a singular point of the system tem (4.1). Two cases are possible:

Case 1. $\Lambda \neq 0$, then by Theorem 4.1 we reduce the system to a normal form, then by Theorem 4.2 we reduce the normal form to a subsystem of order $k < n$ without linear part and obtain the problem of studying its singular points.

Case 2. $\Lambda = 0$, then we compute the Newton polyhedron and separate truncated systems in which the normal cone $\mathbf{U}_j^{(d)}$ intersects the negative orthant of $P \leqslant 0$. Each of them is reduced to the form (4.6), (4.7) by the transformation of Theorem 4.3. For each singular point (4.8), we apply Theorem 4.4 and obtain a subsystem of smaller order.

Continuing this branching process, after a finite number of resolution of singularities we come to an explicitly solvable system from which we can understand the nature of solutions of the original system.

But Theorem 4.3 can be applied to the original system (4.1), i.e. to each of the generalized faces $\Gamma_j^{(d)}$ of its Newton polyhedron Γ. Then to each singular point (4.8) we apply Theorems 4.4, 4.2 and reduce the order of the system. Here also through a finite number of steps of the singularity resolution we come to an explicitly solvable system. This allows us to study the singularities of the original system in infinity. This is the basis of the integrability criterion in [Bruno, Enderal, 2009; Bruno, Enderal, Romanovski, 2017].

The normal form can be computed in the neighborhood of a periodic solution or invariant torus [Bruno, 1972, II, §11], [Bruno, 2022a].

See [Bruno, Batkhin, 2023] for similar computations for a system of partial differential equations.

4.5. Hamiltonian system: It has the form

$$\dot{x}_i = \partial H/\partial y_i, \quad \dot{y}_i = -\partial H/\partial x_i, \quad i = 1, \ldots, m, \tag{4.11}$$

and is defined by one Hamiltonian function $H(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_m)$. Here the normal form of the system (4.11) corresponds to the normal form of one Hamiltonian function. See details in [Bruno, Batkhin, 2021].

V. ONE PARTIAL DIFFERENTIAL EQUATION

5.1. Support [Bruno, 2000 Ch. 6-8]: Let $X = (x_1, \ldots, x_n) \in \mathbb{C}^n$ or $\mathbb{R}^n$ be independent variables and $y \in \mathbb{C}$ or $\mathbb{R}$ be a dependent one. Consider $Z = (z_1, \ldots, z_n, z_{n+1}) = (x_1, \ldots, x_n, y)$.

Differential monomial $a(Z)$ is called a product of an ordinary monomial

$$cZ^R = cz_1^{r_1} \cdots z_{n+1}^{r_{n+1}},$$

where $c = \text{const}$, and a finite number of derivatives of the following form

$$\frac{\partial^l y}{\partial x_1^{l_1} \cdots \partial^{l_n} x_n} \stackrel{\text{def}}{=} \frac{\partial^l y}{\partial X^L}, \ 0 \leqslant l_j \in \mathbb{Z}, \sum_{j=1}^{n} l_j = l, \ L = (l_1, \ldots, l_n).$$

Vector *power exponent* $Q(a) \in \mathbb{R}^{n+1}$ corresponds to the differential monomial $a(Z)$, it is constructed according to the following rules:

$$Q(c) = 0, \text{ if } c \neq 0, \quad Q\left(Z^R\right) = R, \quad Q\left(\partial^l y_j/\partial X^L\right) = (-L, 1).$$

The product of monomials corresponds to the sum of their vector power exponents:

$$Q(ab) = Q(a) + Q(b).$$

Differential sum is the sum of differential monomials

$$f(Z) = \sum a_k(Z). \tag{5.1}$$

If $f(Z)$ has no similar terms, then the set $\mathbf{S}(f) = \{Q(a_k)\}$ is called *support* of the sum (5.1).

5.2. Resonant monomials: Let the support $\mathbf{S}(f)$ of the differential sum (5.1) consists of one point $E_{n+1} = (0, \ldots, 0, 1)$. Then the substitution

$$y = cX^P, \quad P = (p_1, \ldots, p_n) \in \mathbb{R}^n \tag{5.2}$$

in the differential sum $f(Z)$ gives the monomial

$$c\omega_P(P)X^P$$

where $\omega_P(P)$ is a *polynomial* of P which coefficients depend on P.

Monomial (5.2) will be called *resonant* for $f(Z)$ if for it

$$\omega_P(P) = 0.$$

Let μ_k be the maximal order of the derivative over x_k in $f(Z)$, $k = 1, \ldots, n$. If in $P = (p_1, \ldots, p_n)$

$$p_k \geqslant \mu_k, \quad k = 1, \ldots, n, \tag{5.3}$$

then

$$f(Z) = c\chi(P)X^P,$$

where $\chi(P)$ is the characteristic polynomial of the sum of $f(Z)$ and its coefficients do not depend on P. But if the inequalities (5.3) are not satisfied, then $\omega_P(P) \neq \chi(P)$.

Example. Let $n = 2$, $f(Z) = x_1 \dfrac{\partial y}{\partial x_1} + x_2^2 \dfrac{\partial^2 y}{\partial x_2^2}$.

If $P = (1, 1)$, then $f(x_1, x_2, cx_1x_2) = cx_1x_2$.
If $P = (1, 2)$, then $f(x_1, x_2, cx_1x_2^2) = c\left(x_1x_2^2 + x_1 \cdot x_2^2 \cdot 2\right) = c \cdot 3x_1x_2^2$.

Generally here for $p_1 \geqslant 1$, $p_2 \geqslant 2$ we have $f(x_1, x_2, cx_1x_2) = c[p_1 + p_2(p_2 - 1)]x_1^{p_1}x_2^{p_2}$ and $\chi(P) = p_1 + p_2(p_2 - 1)$.

5.3. Normal form: For a differential sum $f(Z)$ we denote by $f_k(Z)$ the sum of all differential monomials of $f(Z)$ which have $n+1$ coordinate q_{n+1} of vector power exponents $Q = (q_1, \ldots, q_n, q_{n+1})$ equal to k: $q_{n+1} = k$. Denote $\mathbb{Z}_+^n = \{P : 0 \leqslant P \in \mathbb{Z}^n\}$.

Consider the PDE

$$f(Z) = 0. \tag{5.4}$$

Theorem 5.1. *Let*

$$f(Z) = \sum_{k=0}^{\infty} f_k(Z),$$

where all $\mathbf{S}(f_k) \subset \mathbb{Z}_+^n \times \{q_{n+1} = k\}$. *Suppose*

1. $f_0(Z) = \varphi(X)$ *is a power series from* X *without a free term,*
2. $f_1(Z) = a(Z)+b(Z)$, *where* $\mathbf{S}(a) = E_{n+1} = (0,\ldots,0,1)$, $\mathbf{S}(b) \subset \left(\mathbb{Z}_+^{n+1} \backslash 0\right) \times \{q_{n+1} = 1\}$.

Then there exists a substitution $y = \zeta + \ (X)$, *where* $\ (X)$ *is a power series from* X *without a free term, which transforms the equation* (5.4) *to the normal form*

$$g(X, \zeta) = 0, \tag{5.5}$$

where $g_0(X) = \sum c_P X^P$ *is a power series without a free term,* $P \in \mathbb{Z}_+^n$ *containing only resonant monomials* $c_P X^P$ *for sum* $a(Z)$.

Corollary 5.1.1. *If the sum* $a\,(P)$ *has no resonance monomials* cX^P *with* $P \in \mathbb{Z}_+^n$, $P \neq 0$, *then* $g_0(X) \equiv 0$ *and*

$$y = \psi(X)$$

is the formal solution to the equation (5.4).

If in equation (5.4) differential sum does not contain derivatives, then

$$a(Z) = \text{const} \cdot z_{n+1} = \text{const} \cdot y.$$

Hence $a(Z)$ has no resonant monomials and in the normal form (5.5) the series $g_)(X) \equiv 0$. So Theorem 5.1 gives the Implicit Function Theorem 2.1 without T.

If in Equation (5.4) $n = 1$, then Theorem 5.1 gives Theorem 3.2.

5.4. Polyhedron and truncated equations: Closure of a convex hull

$$\mathbf{\Gamma}(f) = \left\{Q = \sum \lambda_j Q_j, Q_j \in \mathbf{S}, \lambda_j \geqslant 0, \sum \lambda_j = 1\right\}$$

of the support $\mathbf{S}(f)$ is called the *polyhedron of sum* $f(Z)$. The boundary $\partial\mathbf{\Gamma}$ of the polyhedron $\mathbf{\Gamma}(f)$ consists of generalized faces $\mathbf{\Gamma}_j^{(d)}$, where $d = \dim \mathbf{\Gamma}_j^{(d)}$. Each face $\mathbf{\Gamma}_j^{(d)}$ corresponds to *normal cone*

$$\mathbf{U}_j^{(d)} = \left\{P \in \mathbb{R}_*^{n+1} : \langle P, Q'\rangle = \langle P, Q'\rangle > \langle P, Q'\rangle, \text{ where } Q, Q' \in \mathbf{\Gamma}_j^{(d)}, Q' \in \mathbf{\Gamma}\backslash\mathbf{\Gamma}_j^{(d)}\right\},$$

where the space $\mathbb{R}_*^{n+1}$ is conjugate to the space $\mathbb{R}^{n+1}$, $\langle\cdot,\cdot\rangle$ is the scalar product, and *truncated sum*

$$\hat{f}_j^{(d)}(Z) = \sum a_k(Z) \text{by } Q(a_k) \in \mathbf{\Gamma}_j^{(d)} \bigcap \mathbf{S}.$$

Consider the equation

$$f(Z) = 0, \tag{5.6}$$

where f is the differential sum. In the solution of equation (5.6)

$$y = \varphi(X), \tag{5.7}$$

where φ is a series on the powers of x_k and their logarithms, the series φ corresponds to its support, polyhedron, normal cones $\mathbf{u}_i$ and truncations. The logarithm $\ln x_i$ has a zero power exponent on x_i. The truncated solution $y = \hat{\varphi}$ corresponds to the normal cone

$$\mathbf{u} \subset \mathbb{R}_*^{n+1}.$$

Theorem 5.2. *If the normal cone* $\mathbf{u}$ *intersects with the normal cone* (5.2), *then the truncation* $y = \hat{\varphi}(X)$ *of the solution* (5.3) *satisfies the truncated equation*

$$\hat{f}_j^{(d)}(Z) = 0. \tag{5.8}$$

5.5. Power transformations: To simplify the truncated equation (5.8), it is convenient to use a power transformation. Let α be a square real nondegenerate block matrix of dimension $n+1$ of the form

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix},$$

where α_{11} and α_{22} are square matrices of dimensions n and 1, respectively. We denote $\ln Z = (\ln z_1, \ldots, \ln z_{n+1})$, and by the asterisk * we denote transposition.

Variable change.

$$\ln W = (\ln Z)\,\alpha \tag{5.9}$$

is called *the power transformation.*

Theorem 5.3 ([Bruno, 2000]). *The power transformation* (5.5) *reduces a differential monomial* $a(Z)$ *with a power exponent* $Q(a)$ *into a differential sum* $b(W)$ *with a power exponent* $Q(b)$*:*

$$R = Q(b) = Q(a)\alpha^{-1*}.$$

Corollary 5.3.1. *The power transformation* (5.9) *reduces the differential sum* (2.1) *with support* $\mathbf{S}(f)$ *to the differential sum* $g(W)$ *with support* $\mathbf{S}(g) = \mathbf{S}(f)\alpha^{-1*}$, *i.e.*

$$\mathbf{S}(f) = \mathbf{S}(g)\alpha^*$$

Theorem 5.4. *For the truncated equation*

$$\hat{f}_j^{(d)}(Z) = 0$$

there is a power transformation (5.9) *and monomial* Z^T *that translates the equation above into the equation*

$$g(W) = Z^T \hat{f}_j(Z) = 0,$$

where $g(W)$ *is a differential sum whose support has* $n + 1 - d$ *zero coordinates.*

5.6. Logarithmic transformation: Let z_j be one of the coordinates x_k or y. *Transformation*

$$\zeta_j = \ln z_j$$

is called *logarithmic.*

Theorem 5.5. *Let* $f(Z)$ *be a differential sum such that all its monomials have a* j*th component* q_j *of the vector exponent of degree* $Q = (q_1, \ldots, q_{m+n})$ *equal to zero, then the logarithmic transformation* (5.1) *reduces the differential sum* $f(Z)$ *into a differential sum from* $z_1, \ldots, \zeta_j, \ldots, z_n$.

5.7. Calculating asymptotic forms of solutions: A truncated equation $\hat{f}_j^{(n)}(Z) = 0$ is taken. If it cannot be solved, then a power transformation of the Theorem 5.4 and then a logarithmic transformation of the Theorem 5.5 should be performed. Then a simpler equation is obtained. In case it is not solvable again, the above procedure is repeated until we get a solvable equation. Having its solutions, we can return to the original coordinates by doing inverse coordinate transformations. So the solutions written in original coordinates are the asymptotic forms of solutions to the original equation (5.2).

In [Bruno, Batkhin, 2023] method of selecting truncated equations was applied to systems of PDE.

Traditional approach to PDE see in [Oleinik, Samokhin, 1999; Polyanin, Zhurov, 2021].

VI. APPLICATIONS

Here we provide a list of some applications in complicated problems of **(c)** Mathematics, **(d)** Mechanics, **(e)** Celestial Mechanics and **(f)** Hydromechanics.

(c) *In Mathematics*: together with my students I found all asymptotic expansions of five types of solutions to the six Painlevé equations (1906) [Bruno, 2018c; Bruno, Goruchkina, 2010] and also gave very effective method of determination of integrability of ODE system [Bruno, Enderal, 2009; Bruno, Enderal, Romanovski, 2017].

(d) *In Mechanics*: I computed with high precision influence of small mutation oscillations on velocity of precession of a gyroscope [Bruno, 1989] and also studied values of parameters of a centrifuge, ensuring stability of its rotation [Batkhin, Bruno, (et al.), 2012].

(e) *In Celestial Mechanics*: together with my students I studied periodic solutions of the Beletsky equation (1956) [Bruno, 2002; Bruno, Varin, 2004], describing motion of satellite around its mass center, moving along an elliptic orbit. I found new families of periodic solutions, which are important for passive orientation of the satellite [Bruno, 1989], including cases with big values of the eccentricity of the orbit, inducing a singularity. Besides, simultaneously with [Hénon, 1997], I found all regular and singular generating families of periodic solutions of the restricted three-body problem and studied bifurcations of generated families. It allowed to explain some singularities of motions of small bodies of the Solar System [Bruno, Varin, 2007]. In particular, I found orbits of periodic flies round planets with close approach to the Earth [Bruno, 1981].

(f) *In Hydromechanics*: I studied small surface waves on a water [Bruno, 2000, Chapter 5], a boundary layer on a needle [Bruno, Shadrina, 2007], where equations of a flow have a singularity, and an one-dimensional model of turbulence bursts [Bruno, Batkhin, 2023].

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