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Highlights

Recursive decision systems are dynamic systems that represent the asymptotic behavior of decision makers.

Rational choice on recursive decision under measure preserving transformation can lead to asymptotic behavior in a stationary environment.

The ergodic theorem provides sufficient conditions for recursive decision systems to possess ergodic properties, that is, for sample averages to converge to an invariant limit.

I. INTRODUCTION

People frequently modify their behavior and We do observe behavioral anomalies that deviate from the rational choice model use in economics, expected utility maximization combined with Bayes' rule. Nevertheless, if sample averages converge for a sufficiently large class of measurements, i.e., ergodic behavior or average behavior, is there a behavior related to arbitrary choice sets under the dynamical system on a set of full measure has either measure 0 or full measure? This paper show that rational choice on recursive decision under measure preserving transformation can lead to asymptotic behavior in a stationary environment. I mean by asymptotic behavior choice sequences that time averages converge to the space average.

The property of ergodicty for recursive decision systems with an invariant probability measure have been put to describe, formulate and analyze the average behavior of various economic sectors and dynamic economic theories. asymptotic behavior of the type under investigation here was shown to sample path properties of economic dynamics by Kamihigashi and Stachurski(2016), recursive equilibrium in dynamically incomplete markets by Cao(2020), ergodicty in economic decisions by Peters(2019), existence conditions for stationary ergodic Markov equilibrium by Ma(1993), the ergodic behavior of stochastic processes of economic equilibria by Blume(1979). Applications in dynamic macro model followed in the work of Benhabib and Day(1981), Day and Shafer(1987), Nishimura et al(1994), Araujo and Maldonado(2000), Huang(2001). With the advancement of artificial intelligence and algorithmic game theory, more issues central to recursive decision systems that model human recursive reasoning and involve interactions between multiple agents, this paper contributes to understanding the asymptotic behavior of human decision-making and interaction.

In this paper, rather than focusing on a specific model, I attempt to derive conditions under which recursive decision models will give asymptotic convergence. The two behavioral assumptions I require are that choice sets need not be convex and that the decision making process is characterized by learning. The recursive decision systems refers to the model configuration of Day and Kennedy(1970).In the proofs of my results I rely on the ergodic properties for dynamical systems given by Aaronson(1997), Gray(2009), Pollicott and Yuri(1998), and the implications of average behavior for rational choice are similar to those discussed by Hammond(1976), Blume(1979), Elton(1987), Canning(1992), Carlsson(2004).

Section 2 contains necessary assumptions and definitions. Section 3 proves some lemmas and the ergodic theorem for recursive decision systems. The final section is a brief discussion of the theorem.

II. ASSUMPTIONS AND DEFINITIONS

Let (Ω, B, μ) be a probability space that represents the space of the states of the world. Ω is the set of all possible states of the world, $\omega \in \Omega$ shall be call an information; B is a σ-algebra of events which, in the case where Ω is finite, is simply the set of subsets of Ω.Then an event is a subset of Ω, and μ is a measure of probability over the events of B. Let $f: \Omega \to R$ be a measurable mapping of (Ω, B, μ) into a space of outcomes which an observable for an economical quantity.

ASSUMPTION 1. $f \in L^1(\Omega, B, \mu)$ the space of all linear integrable functions.

A recursive decision system is a function T defined on a set Ω, a transformation $T: \Omega \to \Omega$ is the law of motion which prescribes that if the decision system is an state ω at time $n \in Z$, then it will evolve to state $T(\omega)$ after one unit of time, and is measure-preserving if $\mu\left(T^{-1}(B)\right) = \mu(B)$ for every $B \in B$. The successive iterates of the map are defined by induction $T^n := T \circ T^{n-1}$, the transformation associates with the initial sets of possible information depending on the studied time scale, and the orbit $\{T^n(\omega)\}$ is a sequence of the time evolution for a recursive decision system returns to all accessible states with equal probability.

ASSUMPTION 2. An orbit set is compact.

DEFINITION 1. The time average or sample average of f, if it exists, is defined by:

$$f^*(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(\omega)),\ \omega \in \Omega,\ n \in Z^+.$$

DEFINITION 2.The space average is defined, if it exists, as:

$$\overline{f}(\omega) = \int_{\Omega} f(\omega) d\mu.$$

this statement is often referred to as iterated expectation since it shows that the expectation of f can be found in the conditional expectation given a class of measurements and then integrate the conditional expectation.

Ergodicity property with respect to a measurement f is to describe the average behavior of $T^n(\omega)$ as $n \to \infty$.

DEFINITION 3. Given a *measure-preserving* recursive decision *system is called ergodic if for* every $B \in B$ with $T^{-1}(B) = B$ have that either $\mu(B)$=1 or $\mu(B)$=0.

DEFINITION 4. A *measure-preserving* recursive decision *system is said to be invariant or stationary if* $\mu\left(T^{-n}(B)\right) = \mu(B)$, all events B, all $n \in Z^{+}$, and let set I_T={ $B \in B \mid T^{-1}(B) = B$} denote the σ-algebra of T-invariant sets.

III. MAIN RESULTS

The following lemmas gives a characterization of ergodic measures.

LEMMA 1. T is ergodic with respect to μ iff whenever $f \in L^{1}(\Omega,B,\mu)$ satisfies $f = f \circ T$ then f is a constant function.

Proof. See Gray(2009).

LEMMA 2. Given any continuous transformation T: Ω→Ω there exists at least one T-erodic probability measure μ.

Proof. Let Φ denote the set of invariant probability measures on Ω, there is a weak-star topology such that a sequence $\mu_k \in \Omega$ converges to $\mu \in \Omega$ iff $f \in C^{0}(\Omega)$. Choose a dense set of functions $f_k \in C^{0}(\Omega)$, there exists $\Phi_k = \{\varphi \in \Phi_{k-1} \mid \int f_k d\varphi = sup_{\varphi \in \Phi_{k-1}} \{\int f_k d\mu\}\}$, $k \in Z^{+}$, and shows nested sequence $\Phi_k \subset \Phi_{k-1} \ldots \subset \Phi$ is closed and non-empty. Assume that $\mu \in \Phi$ is convex, it suffices to see that $sup_{\varphi \in \Phi}\{\int f_k d\varphi\} = \int f_k d\mu = \int f_k d\alpha\mu_1 + \int f_k d(1 - \alpha)\mu_2$, where $\mu_1, \mu_2 \in \Phi_k$.The proof show that probability measure μ is ergodic.

Lemma 2 ensures that for transformation T generates a well-defined probability measure on the set of orbits.

THEOREM. Let (Ω,B,μ) be a measure space for which $\mu(\Omega) < \infty$,If $f \in L^{1}(\Omega,B,\mu)$ and the measure μ is ergodic then for almost all$\omega \in \Omega$ recursive decision *systems have that the* time average converge to the space average.

$$\text{Proof. Set } F_N f(\omega) = \frac{1}{N}\sum_{n=0}^{N-1} f(T^{n}(\omega)). \text{ For } \varphi \in L^{1}(\Omega,B,\mu), \text{ set}$$

$$G_N\varphi = \max\{\sum_{m=0}^{p-1} \varphi \circ T^{m} : 1 \le p \le N\},$$

and $F_N\varphi \le \frac{1}{N} G_M\varphi$. Set

$$A = A(\varphi) = \{\omega \in \Omega \mid sup_N G_M\varphi(\omega) = \infty\} \in I_T.$$

For $\forall \omega \in F^{c}$, there is

$$sup\, A_N\varphi(\omega) \le 0.$$

$G_M\varphi$ is the increasing function sequence, and

$$G_N\varphi(T(\omega)) = \max\{\sum_{m=1}^{p-1} \varphi(T^{m}(\omega)) : 2 \le p \le N+1\}.$$

So

$$G_{N+1}\varphi(\omega) - G_N\varphi(T(\omega)) = \varphi(\omega) - \min\{0, G_N\varphi(T(\omega))\} \ge \varphi(\omega).$$

Therefore, on the set A, the sequence $G_{N+1}\varphi(\omega) - G_N\varphi(T(\omega))$ gradually decreases and converges to φ, and for Lebesgue's dominated convergence theorem,

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$$0 \leq \int_A (G_{N+1}\varphi - G_N\varphi)d\mu \leq \int_A (G_{N+1}\varphi - G_N\varphi \circ T)d\mu \to \int_A \varphi d\mu.$$

Set $\varepsilon > 0$, and $\varphi = f - f^* - \varepsilon$. For $A=A(\varphi) \in I_T$, there is $\int_A f d\mu = \int_A f^* d\mu$, so

$$\int_A \varphi d\mu = \int_A (f - f^* - \varepsilon)d\mu = -\varepsilon\mu(A) \leq 0.$$

Further $\int_A \varphi d\mu = 0$, and $\mu(A(\varphi))=0$. Therefore $sup\, A_N\varphi(\omega) \leq 0, \forall\omega\in\Omega$ a.e.. Since f^* is T-invariant, and $A_N\varphi = A_N f - f^* - \varepsilon$, then lead to $sup\, A_N f(\omega) \leq f^* + \varepsilon$. Finally, the same can be said about $-f$, $sup\, A_N f(\omega) \geq f^* - \varepsilon$.

IV. CONCLUSION

I providing sufficient conditions for recursive decision systems to possess ergodic properties from its stationary mean, a recursive decision system is asymptotically dominated by measure preserving transformation. In fact, arbitrary choice set contains multiple distance between measures, the theorem fails to explain how such distances affect their respective ergodic properties; on the other hand, the ideas can be extended to the possibility of asymptotic behavior for a wide variety of rational decision involving dynamic economic model.

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