The Solution of the 3 Dimensional Navier-Stokes Momentum Equations (A 3 Dimensional Integral Equation Approach)

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This paper provides the solution of the classical navier stokes mo mentum equations within the common three dimensional and euclidean space. The ”function variable” of the equations, w:= (p, v) taking values in R^4, con sists of a skalar field ”pressure” p and a vector field ”velocity” v, where both, p and v depend on the same four variables (x, y, z, t) (space and time). Moreover, the solution space L ⊂ {(v, p) =: w ∈ C(R^4, R^4) (the continuous differentiable functions from $R^4$to R^4) v ∈ C^2(R^4, R^3) (the twice continuous differentiable functions from $R^4$to R^3) }. The navier-stokes equation system, as we will show, is underspecified in the sense that infinitly many and ”arbitrary” solutions exist. We will show smoothness and existence of the general solution of the navier stokes equations. We deal with fluid dynamics and their boundary conditions within the incompressible and the compressible (the general) case.

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Classification: DDC Code: QA377

Language: English

Great Britain Journals Press

LJP Copyright ID: 392955
Print ISSN: 2631-8474
Online ISSN: 2631-8482

London Journal of Engineering Research

Volume 24 | Issue 7 | Compilation 1.0

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This paper provides the solution of the classical navier stokes mo mentum equations within the common three dimensional and euclidean space. The "function variable" of the equations, w := (p, v) taking values in R^4, con sists of a skalar field "pressure" p and a vector field "velocity" v, where both, p and v depend on the same four variables (x, y, z, t) (space and time). Moreover, the solution space L ⊂ {(v, p) =: w ∈ C(R^4, R^4) (the continuous differentiable functions from $R^4$to R^4) v ∈ C^2(R^4, R^3) (the twice continuous differentiable functions from $R^4$to R^3) }. The navier-stokes equation system, as we will show, is underspecified in the sense that infinitly many and "arbitrary" solutions exist. We will show smoothness and existence of the general solution of the navier stokes equations. We deal with fluid dynamics and their boundary conditions within the incompressible and the compressible (the general) case.

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I. INTRODUCTION

The general form of the well known navier stokes (momentum) equations (compare to [6]) in three dimensions for compressible fluids in space and time reads as follows:

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right)=-\nabla p+\mu\Delta\vec{v}+(\lambda+\mu)\nabla(\nabla\cdot\vec{v})+\vec{f} \tag{1}$$

where its elements are:

- The laplace operator Δ
- The gradient operator ∇
- The parameters of the equation: the "Volume-force" density vector field $\vec{f}: R^3 \to R$ (in Newton/m^3), the viscosity $(\lambda+\mu)$ and the density of the viscosity ρ. We denote the set of function pairs $w=(p,v)$ for which the navier stokes equation system holds by L.

We will show that the four dimensional solution field of this system of differential equations is underspecified and possesses a solution for almost any arbitrary $v \in C^2(\mathbb{R}^4,\mathbb{R}^3)$, the twice differentiable functions from $\mathbb{R}^4$ to $\mathbb{R}^3$. A comparison to the well known transport equation from which the navier stokes equation system is constructed of, or maybe to the known laplace and poisson equations which are already solved (explicitly in three dimensions, see [7]) or certainly to the one and two dimensional navier stokes equations systems leads unfortunately not to new methods for solving our equations. Those will certainly not be solvable without any notion of lebesgue integration. We will therefore try to invert the (functional) operators within the compressible momentum equations as still, a good notion of understanding is provided by using the three dimensional lebesgue integral (since the "household" of the chemical entropy of the equations must be fullfilled at each point in time within incompressible fluid dynamics). As well one could compare to the case of the stochastic navier stokes equations in one, two or three dimensions (see [2] and [5]). Moreover and specifically, to any two or three dimensional stochastic partial differential equation using a probabilistic measure, the set of Lebesgue integration zerosets in three (space) or four (space and time) dimensions does differ in general from the set of probabilistic zerosets of the probability measure of the stochastic integration which is used to construct the stochastic equations and which gives the probability densities of the possible and probable solution paths of the equations. (see literature of the theory of stochastic integration, stochastic differential/integral equations (SDE) and stochastic partial differential equations in more than one dimension (SPDE)). Indeed, our experience shows that the methods obtained from the mentioned related equations are not directly relevant to the three dimensional case (see also [4]). In this work, we show smoothness and existence of the solution of the navier stokes equations in three dimensions, (even though the time variable makes it a four dimensional equation) as required internationally from the CMI in Oxford and provide it in the form of the largest set of functions solving the equations, depending on the given parameters $(\vec{f}, \rho, \lambda+\mu)$. As a well known and so called "millenium riddle" (see also [8], Andrés Boldori, 2024, as well as [3]), its solution is also expected to be humanistic! But must we necessarily deal with fluid dynamics if we want to solve the 3-dimensional navier stokes equations from above? And if we do, does the solution of the general case, within compressible fluid dynamics, differ from the solution of the particular incompressible case? The ideas of this chapter do in any case not replace any mathematical proof.

II. NECESSARY SMOOTHNESS CONDITIONS OF A SOLUTION PATH (VECTOR FIELD) OF THE 3 DIMENSIONAL COM PRESSIBLE NAVIER STOKES MOMENTUM EQUATIONS.

2.1 Particular Special Cases of the Navier Stokes Equations

(see also: [1]) The navier stokes equations are by far not just an "artificial" system of equations having no meaning, but exist within many specified contexts of real nature environnements. They arise from physical laws and can be combined with other equations. One could observe that it is not possible to fail the navier stokes equations without failing any physical law in addition. As a mathematical equation, at least the following four particular special cases must be taken into account:

1. $v \equiv 0$ (identical) and $p \equiv 0$ fullfills the equations if and only if $f \equiv 0$.
2. $v \equiv 0$ fullfills the equations for: $p(x, y, z, t) = -\int_{-\infty}^{x} f_1(x_0, y, z, t)dx_0 + \int_{-\infty}^{y}(-f_2(x, y_0, z, t))dy_0 - \int_{-\infty}^{x}(d(\int_{-\infty}^{y}(-f_2(x_0, y_0, z, t))dy_0)/dx_0)dx_0 + \int_{-\infty}^{z}(-f_3(x, y, z_0, t))dz_0 - \int_{-\infty}^{x}(d(\int_{-\infty}^{z}(-f_3(x_0, y, z_0, t))dz_0)/dx_0)dx_0 + c(t)$ by "inserting" since of $-\vec{f} = \nabla p$. (Notice that the expression of p is no solution if we take indefinite integrals since of possibly differing integration constants)

3. The mathematical and overall theoretical case of : $\frac{1}{\rho}\int_0^t f(x,y,z,t_0)dt_0 = v(x,y,z,t)$ with $f(t) = \partial v(x,y,z,t)/\partial t = 0$, outside a finite nonemp-ty domain D (usually the volume of a viscosity). This case must be solved like the ordinary compressible momentum navier stokes equations. Then, inside D :

$$\rho\left((\vec{v}\cdot\nabla)\vec{v}\right) = -\nabla p + \mu\Delta\vec{v} + (\lambda+\mu)\nabla(\nabla\cdot\vec{v})$$

4. In this case, in addition to the momentum equations, newtons second law is holding: $F = m\cdot a(x,y,z,t) = m\cdot\partial v(x,y,z,t)/\partial t$. And the external force field f as a parameter of the equations is a part of the overall force field F. See sections 3.2 and 3.3 for compressible and incompressible fluid dynamics of the momentum equations and their solutions.

2.2 A Solution of the Navier Stokes Equation for Which V is Not in C³(R⁴, R³)

Taking "cm" and "s" (second) as "standard" units during the following theoretical experiment, a vertical pipe of diameter $d = 5$, length $l = 1000$ and the z-axis as main axis within a standard coordinate system (3-dimensional euclidean space), starting at $z = 950$ and ending at $z = -50$ which possesses a "bottom" of plastic or gum, is filled with water and contains a "wedge" of shape and location so that the water flow velocity v along it takes the form $v(x,y,z,t) = (0, v_2, v_3)(x,y,z,t)$:

$v_2(z) := z^3$ and $v_3^2 = \left|v(x,y,z,t)\right|^2 - z^6$, $0 < y < 1$, $-0.1 \le x \le 0.1$
$v_2(z) := -z^3$ and $v_3^2 = \left|v(x,y,z,t)\right|^2 - z^6, -1 \le y < 0$, $-0.1 \le x \le 0.1$

with v having a constant lenght along the wedge. $v_2(z)$ as a function of z is not 3 times differentiable at the edge of the wedge $(0,0,0,t)$. Neglecting the friction of the wedge and the water flowing out the pipe if opening its bottom, a velocity field is created by the acceleration (gravity) of the water within the pipe obtaining a constant velocity after the initial acceleration. The acceleration of the water within can as well be used to create a constant velocity along the wedge: The hight $h(t)$ of the water level within the pipe while opening its bottom in the form of a hole of variable diameter $d_0(t)$ is:

$$h(t) = (V_{pipe} - Volume_{out}(t))/d = (V_{pipe} - \sqrt{2g\cdot h(t)}\cdot Area_{hole}(d_0(t)))/d$$

$$h(t) = ((\sqrt{2g}/d)\cdot Area_{hole}(d_0(t)) - \sqrt{((\sqrt{2g}/d)\cdot Area_{hole}(d_0(t)))^2 - 4\cdot V_{pipe}/d})/(-2)$$

$$Area_{hole}(d_0(t)) = (d\cdot h(t) - V_{pipe})/\sqrt{2g\cdot h(t)}$$

III. EXISTENCE OF A SOLUTION OF THE 3 DIMENSIONAL NAVIER STOKES MOMENTUM EQUATION

3.1 Boundary Conditions

The system of compressible navier stokes equations is a differential equation in the variable p, which is easily solvable as soon as v is given. Even though the compressible navier stokes momentum equation (system) differs (slightly) from the imcompressible one, our attempt here to solve the equations does not include any of the classical initial value problems of differential equations. Not mathematically, but from physics, the theory of the navier stokes equations may be connected to the theory of fluid dynamics.

3.2 Compressible Fluid Dynamics

The momentum equation in three dimensions (equation (1)) can be written as follows:

$$-\rho\left(\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right)+\mu\Delta\vec{v}+(\lambda+\mu)\nabla(\nabla\cdot\vec{v})+\vec{f}=\nabla p\quad(=gr\vec{a}d(p))\tag{2}$$

where the equation:

$$gr\vec{a}d(p)=\vec{f}\tag{3}$$

can easily be solved since of:

$$p=c(y,z,t)+\int f_1dx=c(x,z,t)+\int f_2dy=c(x,y,t)+\int f_3dz\tag{4}$$

and the expression of section 2.1 (particular special case 2.) follows by setting $\vec{f}:=-\vec{f}$ within equation (3). If we went a step further ahead and now replaced $-\vec{f}$ from the expression of the particular special case of section 2.1 again by $-\vec{f}_{new}:=\vec{g}:=-\rho\left(\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right)+\mu\Delta\vec{v}+(\lambda+\mu)\nabla(\nabla\cdot\vec{v})+\vec{f}$, we'd get an expression for p:

$p(x,y,z,t)=\int_{-\infty}^{x}g_1(x_0,y,z,t)dx_0+\int_{-\infty}^{y}(g_2(x,y_0,z,t))dy_0-\int_{-\infty}^{x}(d(\int_{-\infty}^{y}(g_2(x_0,y_0,z,t))dy_0)/dx_0)dx_0+\int_{-\infty}^{z}(g_3(x,y,z_0,t))dz_0-\int_{-\infty}^{x}(d(\int_{-\infty}^{z}(g_3(x_0,y,z_0,t))dz_0)/dx_0)dx_0+c(t)$

The solution space L for the navier stokes equations with the definition of $\vec{g}$ from above, is:

$L:=\{\big(v(x,y,z,t),\int_{-\infty}^{x}g_1(x_0,y,z,t)dx_0+\int_{-\infty}^{y}(g_2(x,y_0,z,t))dy_0-\int_{-\infty}^{x}(d(\int_{-\infty}^{y}(g_2(x_0,y_0,z,t))dy_0)/dx_0)dx_0+\int_{-\infty}^{z}(g_3(x,y,z_0,t))dz_0-\int_{-\infty}^{x}(d(\int_{-\infty}^{z}(g_3(x_0,y,z_0,t))dz_0)/dx_0)dx_0+c(t)\big)\mid$ The second component of the pair does exist and $c(t):\mathbb{R}_{\geq 0}\to\mathbb{R}\}$

The arbitrary shift $c(t)$ applies to the entire vector field (to any v which is "integrable enough", the expression within the solution set is a solution of the three dimensional navier stokes momentum equation, for any arbitrary $c(t):\mathbb{R}_{\geq 0}\to\mathbb{R}$). The domain of definition of the velocity vector field v should not be the empty set.

3.3 Incompressible Fluid Dynamics

A Fluid in physics is said to be incompressible if its velocity vector field $v(x,y,z,t)$has zero divergence everywhere in the sense of differential geometry: $div(v)\equiv 0$. The solution space of the incompressible navier stokes momentum equations must simply be restricted to functions and vector fields having divergence zero everywhere:

$L_{incompressible}:=\{\big(v(x,y,z,t),\int_{-\infty}^{x}g_1(x_0,y,z,t)dx_0+\int_{-\infty}^{y}(g_2(x,y_0,z,t))dy_0-\int_{-\infty}^{x}(d(\int_{-\infty}^{y}(g_2(x_0,y_0,z,t))dy_0)/dx_0)dx_0+\int_{-\infty}^{z}(g_3(x,y,z_0,t))dz_0-\int_{-\infty}^{x}(d(\int_{-\infty}^{z}(g_3(x_0,y,z_0,t))dz_0)/dx_0)dx_0+c(t)\big)\mid$ The second component of the pair does exist, $c(t):\mathbb{R}_{\geq 0}\to\mathbb{R}$ and $div(v)\equiv 0\}$

again by setting $\vec{g}$:

$$\vec{g} :\equiv -\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) + \mu\Delta\vec{v} + (\lambda+\mu)\nabla(\nabla\cdot\vec{v}) + \vec{f}$$

3.4 Conclusion

For now we have shown smoothness and existence for the compressible and incompressible navier stokes momentum equation system in three dimensions.

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